

THE WHALE PROBLEM

Consider the model

$$a_{n+1} = a_n + k(M - a_n)(a_n - m),$$

with $M, m, k > 0$, where a_n represents the number of individuals in a whale population after n years.

- (a) Discuss the model. What is the meaning of each of the model parameters?
- (b) Let $M = 5000$, $m = 100$, and $k = 0.0001$. Find the fixed points of the model. Determine the stability of the fixed points using linear stability analysis, and check the results using graphical (cobwebbing) analysis.
- (c) Implement the model on a spreadsheet, using the parameter values specified in part (b). Try several values for the initial condition a_0 . Are your simulations consistent with your stability analysis from part (b)? As part of your answer, submit a graph of a_n versus n for some well-chosen initial conditions, and briefly describe your results in terms of the whale population.
- (d) The model has two serious shortcomings. What are they (hint: what happens when $a_0 < m$, and when $a_0 \gg M$)? Think of a possible way to fix the model so as to overcome these shortcomings. You do *not* need to write down the equation of an improved model; it is sufficient to describe your ideas with words and/or sketches of graphs (hint: it helps to study the behaviour of the given model with a sketch of the graph of a_{n+1} versus a_n).

Solution:

- (a) The equation describes the population of whales next year in terms of the population this year. The last term describes the change in the population, which is dependent on three parameters, k, M, m , with $m < M$.
 - If $a_n = m$ or $a_n = M$, then $k(M - a_n)(a_n - m) = 0$, so the population will remain the same size.
 - If $0 < a_n < m$, then $k(M - a_n)(a_n - m) < 0$, so the population will decline (and eventually go extinct).
 - If $m < a_n < M$, then $k(M - a_n)(a_n - m) > 0$, so the population will grow.
 - If $a_n > M$, then $k(M - a_n)(a_n - m) < 0$, so the population will decline.

The parameter m represents the size of the population below which it has no reproductive success.

The parameter M represents the carrying capacity of the population.

Both m and M are fixed points of the model.

The parameter k measures how strongly the population reacts to deviations from m and M . The smaller the value of k , the smaller the changes will be, and vice versa.

(b) With the given model parameters, we have

$$a_{n+1} = a_n + 0.0001(5000 - a_n)(a_n - 100).$$

For fixed points, we require

$$a = a + 0.0001(5000 - a)(a - 100),$$

that is,

$$0.0001(5000 - a)(a - 100) = 0,$$

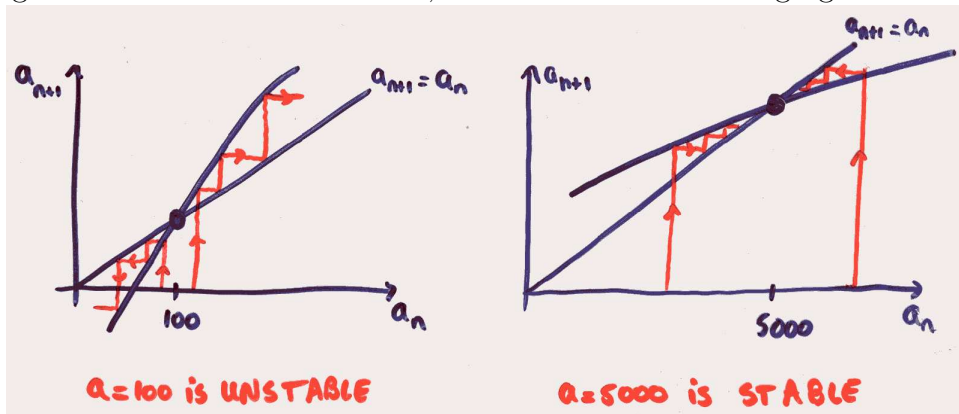
so that the fixed points of the model are $a = 100$ and $a = 5000$.

To determine the stability of the fixed points, we define $f(x)$ to be the right-hand side of the model, take its derivative, and evaluate the derivative at the fixed points to find the following:

$$\begin{aligned} f(x) &= x + 0.0001(5000 - x)(x - 100) \\ &= x + 0.0001(5000x - 500,000 - x^2 + 100x) \\ &= x + 0.0001(5100x - 500,000 - x^2) \\ f'(x) &= 1 + 0.0001(5100 - 2x) \\ f'(100) &= 1 + 0.0001(5100 - 200) = 1.49 > 1 \\ f'(5000) &= 1 + 0.0001(5100 - 10,000) = 0.51 < 1 \end{aligned}$$

That is, $a = 100$ is an UNSTABLE fixed point, and $a = 5000$ is a STABLE fixed point.

Cobwebbing leads to the same conclusion, as shown in the following figure:



(c) See whaleSimulation.xls for model implementation. Discussion points are as follows:

- Orbits starting with $a_0 = 100$ or $a_0 = 5000$ don't go anywhere, that is, $a = 100$ and $a = 5000$ are fixed points, as shown above.

- Orbits starting near $a_0 = 100$ move away from 100, and orbits starting near $a_0 = 5000$ move towards 5000, which is consistent with the results of the stability analysis performed above.
 - Any whale population with less than 100 individuals goes extinct. In fact, the model predicts a negative number of whales, which is a bit of a problem (see below).
 - Any whale population with 100 to 5000 individuals will gradually increase until the population stabilizes at 5000.
 - A population with more than 5000 individuals initially (but not too large, see below) is above carrying capacity, and will slowly decline, again stabilizing at 5000.
- (d) See whaleSimulation.xls for numerical experiments, starting with $a_0 = 50$ and $a_0 = 20,000$.
- When $a_0 < m$, the population declines, eventually becoming negative.
 - Similarly, there is a problem with $a_0 \gg M$. In this case, the change term in the model is negative, and large, so that $a_1 < m$, and the population will continue to decline forever, again reaching negative population numbers.

Negative population numbers make no sense biologically.

At the root of the problem lies the fact that the right-hand side of the given model can be negative. One could stipulate that we stop the iteration of the model as soon as the population level reaches zero. But we can do much better, and that is to adjust the right-hand-side of the model so that it no longer is negative for any value of a_n . We make an adjustment near the origin, as well as for large a_n , as shown in the following figure:

