

this section we present two examples of such constrained problems and explore a powerful method for finding the extreme values known as the method of *Lagrange multipliers*.

EXAMPLE 1 An Oil Transfer Company

Consider a situation in which we are hired as consultants for a small oil transfer company. The management desires a policy of minimum cost due to restricted tank storage space.

Problem Identification *Minimize the costs associated with dispensing and holding the oil to maintain sufficient oil to satisfy demand while meeting the restricted tank storage space constraint.*

Assumptions Many factors determine the total cost of transferring oil. For our model we include the following variables: holding costs of the oil in the storage tank, withdrawal rate of the oil from the tank per unit time, the cost of the oil, and the size of the tank.

Model Formulation We define the following variables for two types of oil (so $i = 1$ or 2):

- x_i = the amount of oil type i available
- a_i = the cost of oil type i
- b_i = withdrawal rate per unit time of oil type i
- h_i = holding (storage) costs per unit time for oil type i
- t_i = space in cubic feet to store one unit of oil type i
- T = the total amount of storage space available

Historical records have been studied and a formula has been derived that describes the system costs in terms of our variables. Our objective is to minimize the sum of the cost variables:

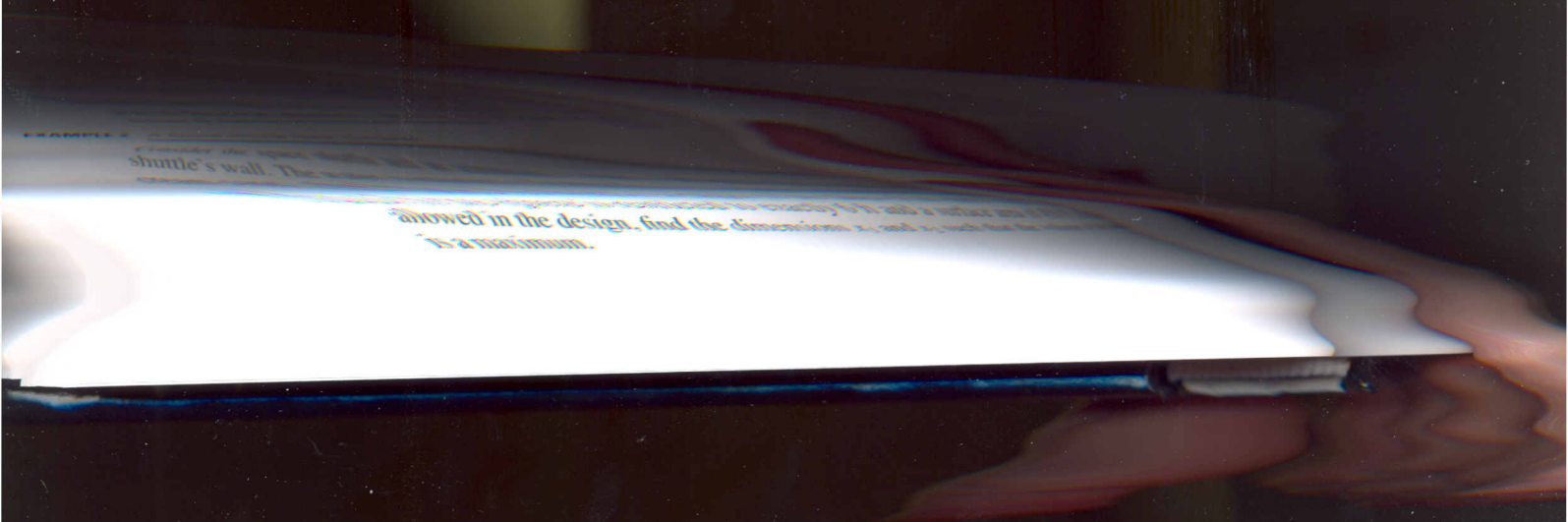
$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= \left(\frac{a_1 b_1}{x_1} + \frac{h_1 x_1}{2} \right) + \left(\frac{a_2 b_2}{x_2} + \frac{h_2 x_2}{2} \right) \\ \text{such that } g(x_1, x_2) &= t_1 x_1 + t_2 x_2 = T \text{ (space constraint)} \end{aligned} \quad (13.5)$$

We are provided the following data:

Oil type	a_i (\$)	b_i	h_i (\$)	t_i (ft ³)
1	9	3	0.50	2
2	4	5	0.20	4

We measure the storage tank and find only 24 ft³ of available space. After substitution of the data into Equations (13.5), the formulation for our problem is

$$\text{Minimize } f(x_1, x_2) = 27/x_1 + 0.25x_1 + 20/x_2 + 0.10x_2$$



shuttle's wall. The

allowed in the design, find the dimensions

is a maximum.

such that

$$2x_1 + 4x_2 = 24$$

Model Solution The method commonly used for solving nonlinear optimization problems with equality constraints is known as the method of **Lagrange multipliers**. The method involves introducing a new variable λ (called a Lagrange multiplier) and setting up the function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda [g(x_1, x_2) - T]$$

For our problem this function is

$$L(x_1, x_2, \lambda) = 27/x_1 + 0.25x_1 + 20/x_2 + 0.10x_2 + \lambda(2x_1 + 4x_2 - 24) \quad (13.6)$$

The solution methodology is to take the partial derivatives of Equation (13.6) with respect to the variables x_1 , x_2 , and λ and set them equal to zero; that is,

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{-27}{x_1^2} + 0.25 + 2\lambda = 0 \\ \frac{\partial L}{\partial x_2} &= \frac{-20}{x_2^2} + 0.10 + 4\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 2x_1 + 4x_2 - 24 = 0 \end{aligned}$$

Using a computer algebra system, we find the solution to be $x_1 = 5.0968$, $x_2 = 3.4516$, $\lambda = 0.3947$, and $f(x_1, x_2) = \$12.71$. By perturbing the values of x_1 and x_2 slightly in either direction (but satisfying the constraint), we find the value of $f(x_1, x_2)$ increases. Thus, we conclude the solution is a minimum.

Model Sensitivity The variable λ has special significance and is called a **shadow price**. The value of λ represents the amount the objective function would change for a 1-unit increase in the constraint represented by λ . Therefore, in this problem, the value of $\lambda = 0.3947$ means that if the storage space constraint is changed from 24 to 25 ft³, the value of the objective function would change from \$12.71 to approximately $12.71 + (1)(0.3947)$ or \$13.10. The economic interpretation is that another cubic foot of storage tank capacity increases the dispensing and holding costs by about \$0.40.

EXAMPLE 2 A Space Shuttle Water Container

Consider the space shuttle and an astronaut's water container that is stored within the shuttle's wall. The water container is formed as a sphere surmounted by a cone (like an ice cream cone), the base of which is equal to the radius of the sphere (Figure 13.11). If the radius of the sphere is restricted to exactly 6 ft and a surface area of 450 ft² is all that is allowed in the design, find the dimensions x_1 and x_2 such that the volume of the container is a maximum.

■ Figure 13
Space shuttle
container