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QUALITATIVE ANALYSIS OF SEMILINEAR CATTANEO EQUATIONS

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The linear Cattaneo equation appears in heat transport theory to describe heat wave propagation with finite speed. It can also be seen as a generalization of a correlated random walk. If the system admits nonconservative forces (or reactions), then a nonlinear Cattaneo system is obtained. Here we consider asymptotic behavior of solutions of the nonlinear Cattaneo system. Following Brayton and Miranker we define a Lyapunov function to show global existence of solutions and to show that each ω-limit set is contained in the set of all stationary solutions.

1. Introduction

A *semilinear Cattaneo system* is a system of the form

$$\begin{aligned} u_t + \nabla \cdot v &= f(u)\,, \\ \tau v_t + D\nabla u + v &= 0\,, \end{aligned} \tag{1.1}$$

where $u(t,x) \in \mathbb{R}$ and $v(t,x) \in \mathbb{R}^n$ are functions of time $t \geq 0$ and space $x \in \Omega \subset \mathbb{R}^n$. The diffusion constant D and the time constant τ are positive and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function with properties to be specified later. There are two interpretations of this equation. First, it appears to describe heat transport with finite speeds (Cattaneo[3]). Then u is a temperature distribution and v is the heat flow. It can also be seen as a generalization of a correlated random walk (Hadeler[9]). Then u is a particle density and v a particle flow.

If heat transport is considered in a closed system, then energy conservation is assumed. Let $\theta(x,t) \in \mathbb{R}$ describe the temperature distribution at time t in a spatial domain $\Omega \subset \mathbb{R}^n$ and let $q(x,t) \in \mathbb{R}^n$ denote the heat flow, then conservation of energy requires the first equation of (1.1) in the linear case ($f = 0$)

$$\theta_t + \nabla \cdot q = 0\,. \tag{1.2}$$

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In Fourier's law it is assumed that the heat flow is proportional to the negative gradient of the temperature

$$q = -k\nabla\theta\,. \tag{1.3}$$

Equations (1.2) and (1.3) lead directly to the heat equation $\theta_t = k\Delta\theta$. But the heat equation gives rise to infinite speed of propagation of heat. To avoid this unphysical property, Cattaneo[3] assumed that the flow adapts to the negative gradient of the temperature by

$$\tau q_t + q = -k\nabla\theta\,, \tag{1.4}$$

which is the second equation of (1.1) in terms of heat flow. Maxwell[22] derived the same equation in 1867 but he directly cast out the time derivative because it "... *may be neglected, as the rate of conduction will rapidly establish itself.*" For $\tau = 0$, Cattaneo's law (1.4) becomes Fourier's law (1.3). Cattaneo's law (1.4) and the conservation law (1.2) form a system of type (1.1). From (1.4) and (1.2) one can derive a telegraph equation for θ

$$\theta_{tt} + \frac{1}{\tau}\theta_t = \frac{k}{\tau}\Delta\theta\,. \tag{1.5}$$

This equation is known as equation for *second sound*, where $\sqrt{k/\tau}$ is the wave speed of second sound (Joseph and Preziosi[17]). Cattaneo's law has been modified in several ways to describe heat transport with finite speed in media with memory. To get more realistic adaptations to physical realizations, the law of Cattaneo is modified by many authors. The models of Jeffrey type (the name Jeffrey appears by analogy; Jeffrey considered stress and deformation), and those of Guyer and Krumhansl,[7] Lord and Sulman,[21] Gurtin and Pipkin[6] and Nunziato[23] are important. For more details see the review of Joseph and Preziosi[17] or Duan *et al.*[4] In the fundamental paper of Gurtin and Pipkin[6] it is assumed that the thermodynamic potentials (free energy, entropy, inertial energy) depend on the *history* of the heat and of the flow. This assumption leads to a constitutive equation for the heat flow

$$q(x,t) = -\int_0^\infty a(s)\nabla\theta(x,t-s)\,ds, \tag{1.6}$$

with some weight functional $a(s)$. If we choose $a(s) = k\delta_0(s)$, the δ-distribution (which is not allowed in the context of Gurtin and Pipkin, since they assume regularity), then Fourier's law (1.3) follows. If we choose $a(s) := ke^{-s/\tau}$ (which is allowed) then Cattaneo's law (1.4) follows. Thus Cattaneo's law says that the effect of the gradient of the temperature on the flow decays exponentially with rate $1/\tau$. If k is chosen such that $\int_0^\infty a(s)\,ds = 1$ then $a(s)$ is a probability density of a Poisson process. Hence *q is the expectation of the negative gradient of the temperature with respect to a Poisson process.*

Another way to obtain Cattaneo's law is a generalization of a one-dimensional correlated random walk (Hadeler[9]). Following Taylor,[25] Goldstein[5] and Kac[18] we assume that particles move along the line with constant speed γ and they change direction according to a Poisson process with rate μ. We split the particle density $u(t,x) = u^+(t,x)+u^-(t,x)$ into densities u^+, u^- for right and left moving particles. Then the correlated random walk is given by

$$u_t^+ + \gamma u_x^+ = \mu(u^- - u^+)$$

$$u_t^- - \gamma u_x^- = \mu(u^+ - u^-)\,.$$

Written in terms of the particle density u and the particle flow $v := u^+ - u^-$, this system reads

$$u_t + \gamma v_x = 0\,, \tag{1.7}$$

$$v_t + \gamma u_x = -2\mu v\,. \tag{1.8}$$

Again (1.7) is a conservation law similar to (1.2) and (1.8) connects the flow to the gradient of the particle density in the form of a Cattaneo law (1.4). Thus the linear Cattaneo system can be seen as a generalization of the correlated random walk model (1.7), (1.8) to several dimensions. For correlated random walk in one dimension, nonlinear models are derived and qualitative properties are studied in detail in Holmes,[16] Hadeler[8–10] and in Hillen.[12–15] An n-dimensional generalization of the correlated random walk Eqs. (1.7), (1.8) is the Cattaneo system (1.1) with $f = 0$. The parameters of the one-dimensional case are $\gamma^2/(2\mu) = D$ and $1/(2\mu) = \tau$. In both interpretations (heat flow and random walk), the parameter $1/\tau$ is the rate of a Poisson process. Throughout the paper we discuss the results in terms of particles and random walks. Obviously the results are also useful in the heat transport theory.

If the particles undergo reactions (or if the physical system is active) then the conservation law is modified into $u_t + \nabla \cdot v = f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuously differentiable. The nonlinear Cattaneo system (1.1) was introduced by Hadeler.[9] He described the connection to a damped wave equation. Assume that solutions of the nonlinear system (1.1) are two times differentiable, then a *reaction telegraph equation* follows:

$$\tau u_{tt} + (1 - \tau f'(u))u_t = D\Delta u + f(u)\,.$$

We consider the System of Cattaneo type (1.1) on a convex bounded domain $\Omega \subset \mathbb{R}^n$ with C^1-boundary $\partial\Omega$. Let η denote an outer normal at $\partial\Omega$. In the one-dimensional case, homogeneous Dirichlet and Neumann boundary conditions are introduced in Hadeler[9] and Hillen.[12] A generalization of these boundary conditions to the n-dimensional case leads to the following boundary conditions (Hadeler[9]):

- *Dirichlet conditions:* The flow in normal direction is proportional to the density

$$u = \sqrt{\frac{\tau}{D}}\eta \cdot v\,. \tag{1.9}$$

Though this boundary condition is of mixed type (Robin boundary condition) we call it Dirichlet boundary condition with respect to the physical meaning that no particle will enter the domain from outside and particles can leave the domain from inside.

- *Neumann condition:* There is no flow through the boundary

$$\eta \cdot v = 0\,. \tag{1.10}$$

In Sec. 2 we prove local existence of (weak) solutions of the boundary value problems (1.1), (1.9) and (1.1), (1.10) following a classical semigroup argument to make the paper self-contained. The main result is the following theorem which is proved in Sec. 3 in the case of Neumann boundary conditions and in Sec. 4 in the case of Dirichlet boundary conditions.

Theorem 1. *Assume*

(H1) $f \in C^1(\mathbb{R})$, *and if* $n > 2$ *then*

$$|f'(y)| \leq c(1 + |y|^{\beta-1}) \text{ with } \beta = \frac{n}{n-2} \text{ and } c > 0\,,$$

(H2) $\sup\limits_{y\in\mathbb{R}} f'(y) < \frac{1}{\tau}$,

(H3) $F(y) := \int_0^y f(u)\,du, \quad \lim\limits_{|y|\to\infty} F(y) = -\infty\,.$

Then for each of the problems (1.1), (1.9) *or* (1.1), (1.10) *the solutions exist globally. Moreover, each* ω*-limit set is contained in the set of all stationary solutions.*

To prove this result we construct a Lyapunov function on an appropriate Hilbert space. The two different boundary conditions have to be considered separately. Such Lyapunov functions were first introduced by Brayton and Miranker[2] for systems of hyperbolic equations for one spatial dimension. In Hillen[13] it is adapted to systems of one-dimensional correlated random walk for several types of particles. In Hadeler[9] a hint is given for the case of several space dimensions. We consider a variational problem such that the original system (1.1) appears as a gradient system. Then the energy functional can be modified such that a Lyapunov function is given.

The technical condition (H1) is necessary to show local existence in the L^2-setting (Hale[11]). The dissipative character of the equations is given by condition (H2). If τ is small enough, we can neglect the term τv_t in (1.1) and the Cattaneo system (1.1) becomes a reaction diffusion equation $u_t = D\Delta u + f(u)$. In this case we expect balance of density peaks and convergence to stationary solutions. Condition (H2) makes precise the meaning of "τ is small enough" compared to a given nonlinearity f. The third condition (H3) ensures that no blow up occurs and that solutions exist globally.

This result is a first step to prove the existence of a global attractor. If we use the additional assumption that the set of all stationary solutions is bounded, then it is not difficult to show that the interior of the level sets of the Lyapunov function are bounded absorbing sets. Moreover, a compactness property has to be shown, which will be done elsewhere. Global attractors for damped wave equations are also studied by Ball,[1] Hale,[11] Kapitanski,[19] Ladyžhenskaya,[20] Temam,[26] Webb[28] and many others (see the literature in Temam[26]). Most of them use modifications of the classical energy $\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$ which does not work in this case.

In the case of finitely many stationary solutions, a convergence result follows from Theorem 1:

Corollary 1. *Assume* (H1), (H2), (H3) *and that the set of all stationary solutions is finite and discrete. Then each solution converges to a stationary solution.*

2. Local Existence

To prove local existence for the linear problem we use a transformation $w := \gamma u$, with $\gamma := \sqrt{D/\tau}$. Then system (1.1) is equivalent to

$$\begin{aligned} w_t + \gamma \nabla \cdot v &= \tilde{f}(w)\,, \\ v_t + \gamma \nabla w + \frac{1}{\tau} v &= 0\,, \end{aligned} \tag{2.11}$$

with $\tilde{f}(w) := \gamma f(w/\gamma)$. The Dirichlet boundary condition for (2.11) is

$$w = \eta \cdot v\,, \tag{2.12}$$

whereas the Neumann boundary condition (1.10) is not modified. We introduce Hilbert spaces

$$\mathcal{L}^2 := (L^2(\Omega))^{n+1} \text{ and } \mathcal{H}^1 := (H^1(\Omega))^{n+1}\,.$$

Let $y := (w, v) \in \mathcal{L}^2$ and define operator matrices $G : \mathcal{D}(G) \to \mathcal{L}^2$ and $B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$G := \begin{pmatrix} 0 & -\partial_1 & \cdots & -\partial_n \\ -\partial_1 & 0 & & \\ \vdots & & \ddots & \\ -\partial_n & & & 0 \end{pmatrix}, \quad B := \frac{1}{\tau} \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix},$$

where ∂_j is the operator of the partial derivative in the direction of the jth coordinate, $1 \le j \le n$. The domain of G depends on the boundary condition:

$$\begin{aligned} \mathcal{D}(G) := \{(w, v) \in \mathcal{H}^1\colon\ & \eta v = w \text{ on } \partial\Omega \text{ for Dirichlet boundary conditions, and} \\ & \eta v = 0 \text{ on } \partial\Omega \text{ for Neumann boundary conditions}\}\,. \end{aligned} \tag{2.13}$$

Then system (2.11) with $\tilde{f} = 0$ can be written as

$$y_t = \gamma G y + B y\,. \tag{2.14}$$

Since B is a bounded operator on $\mathcal{L}^2$, we first investigate the operator G.

Theorem 2. *For Dirichlet and Neumann boundary conditions, the operator* $(G, \mathcal{D}(G))$ *is dissipative and generates a strongly continuous semigroup on* $\mathcal{L}^2$.

Proof. We consider the two boundary conditions separately.

1. Neumann boundary conditions (1.10):

Lemma 1. *The operator G is skew adjoint, i.e.* $G^* = -G$ *with* $\mathcal{D}(G^*) = \mathcal{D}(G)$.

Proof of Lemma 1. The tuple of functions $(r, s) \in \mathcal{H}^1$ belongs to $\mathcal{D}(G^*)$ if and only if

$$\langle (r,s), G(w,v)\rangle_{\mathcal{L}^2} = \langle G^*(r,s), (w,v)\rangle_{\mathcal{L}^2} \quad \forall\, (w,v) \in \mathcal{D}(G)\,.$$

With the integral formula of Gauss we have

$$\begin{aligned}\langle (r,s), G(w,v)\rangle &= \int_\Omega (-r\nabla \cdot v - s \cdot \nabla w)\, dx \\ &= \int_\Omega ((\nabla r)\cdot v + (\nabla \cdot s) w)\, dx - \int_{\partial\Omega} \eta \cdot (rv) + \eta \cdot (sw)\, dS\,. \quad (2.15)\end{aligned}$$

The first integral in (2.15) is $\langle -G(r,s), (w,v)\rangle_{\mathcal{L}^2}$. With the Neumann boundary condition, the second integral is $\int_{\partial\Omega} (\eta \cdot s) w\, dS$. It vanishes for all $w \in H^1$ if $\eta \cdot s = 0$ on $\partial\Omega$. Hence $G^* = -G$ and $\mathcal{D}(G^*) = \mathcal{D}(G)$. *square*

Skew adjoint operators are dissipative and their spectrum belongs to the imaginary axis (Theorem of Stone, see Pazy[24]). From the Lumer–Phillips Theorem (Pazy[24]) it follows that G generates a strongly continuous semigroup of contractions.

2. Dirichlet boundary conditions (2.12):
In this case G is not skew adjoint, since $\mathcal{D}(G^*) = \{(w,v) \in \mathcal{H}^1 : \eta \cdot v = -w\} \neq \mathcal{D}(G)$. Instead of G we consider $\Gamma := P^{-1}GP$ with $P := \operatorname{diag}(1, -1, \ldots, -1)$. The transformation P satisfies $P^2 = I$. The domain of Γ is

$$\mathcal{D}(\Gamma) := \{(w,v) \in \mathcal{H}^1 : \eta \cdot v = -w\}\,.$$

Then $P : \mathcal{D}(G) \to \mathcal{D}(\Gamma)$ is one-to-one. Hence $(\Gamma, \mathcal{D}(\Gamma))$ and $(G, \mathcal{D}(G))$ are representations of the same operator.

Lemma 2. *The operator* Γ *is skew adjoint.*

Proof of Lemma 2. Consider $(r,s) \in \mathcal{D}(\Gamma^*)$ and $(w,v) \in \mathcal{D}(\Gamma)$, then it is easy to verify that

$$\langle (r,s), \Gamma(w,v)\rangle = \langle -\Gamma(r,s), (w,v)\rangle + \int_{\partial\Omega} (-w)(r + \eta \cdot s)\, dS\,.$$

Hence $\eta \cdot s = -r$ for $(r,s) \in \mathcal{D}(\Gamma^*)$.

Then Γ is dissipative and generates a strongly continuous semigroup of contractions. Then $G = P\Gamma P^{-1}$ also generates a strongly continuous semigroup (Pazy[24]). $\square$

Since $B : \mathcal{D}(G) \to \mathcal{L}^2$ is bounded, Eq. (2.14) is solved.

Corollary 2. *With Dirichlet or with Neumann boundary conditions, the operator* $(\gamma G + B, \mathcal{D}(G))$ *is generator of a strongly continuous semigroup on* $\mathcal{L}^2$.

With (H1) a local existence result for the nonlinear Cattaneo-boundary value problems (1.1), (1.9) and (1.1), (1.10) follows (see Pazy[24] or Hale[11]). In the original (u, v) notation the domain of the infinitesimal generator is in the case of Dirichlet boundary conditions

$$\mathcal{D} := \{(u, v) \in \mathcal{H}^1 : \eta \cdot v = \gamma u \text{ on } \partial\Omega\}$$

and in the Neumann case

$$\mathcal{D} := \{(u, v) \in \mathcal{H}^1 : \eta \cdot v = 0 \text{ on } \partial\Omega\}.$$

Theorem 3. *For each initial data* $(u_0, v_0) \in \mathcal{D}$, *there is a* $T > 0$ *such that there exists a unique solution*

$$(u, v) \in C^1([0, T), \mathcal{L}^2) \cap C([0, T), \mathcal{H}^1)$$

of the nonlinear Cattaneo boundary value problem (1.1), (1.9) *or* (1.1), (1.10), *respectively.*

3. Proof of Theorem 1 for Neumann Boundary Conditions

For system (1.1) with homogeneous Neumann boundary conditions (1.10) on $\partial\Omega$ we consider a functional $P : \mathcal{D} \to \mathbb{R}$ defined by

$$P(y) := P(u, v) = \int_\Omega \left[DF(u) + \frac{1}{2}v^2 + D(\nabla u) \cdot v\right] dx\,.$$

For $(u, v) \in \mathcal{D}$ and $(\varphi, \xi) \in \mathcal{D}$ we form the first variation of P

$$\frac{d}{d\varepsilon}P(u + \varepsilon\varphi, v + \varepsilon\xi)\Big|_{\varepsilon=0} = \int_\Omega [Df(u)\varphi + v \cdot \xi + D(\nabla\varphi) \cdot v + D(\nabla u) \cdot \xi]\, dx\,.$$

Since v satisfies Neumann boundary conditions (1.10), it follows that

$$\int_{\partial\Omega} (\eta \cdot v)\varphi\, dS = 0\,, \tag{3.16}$$

hence we have

$$\int_\Omega D(\nabla\varphi) \cdot v\, dx = -\int_\Omega D(\nabla v) \cdot \varphi\, dx\,.$$

Thus the first variation can be written as

$$\frac{d}{d\varepsilon}P\bigg|_{\varepsilon=0} = \langle P_y(u,v),(\varphi,\xi)\rangle_{\mathcal{L}^2}\,, \tag{3.17}$$

where the *functional gradient* $P_y(u,v)$ is given by

$$P_y(u,v) = (Df(u) - D\nabla v, v + D\nabla u)\,. \tag{3.18}$$

The functional gradient P_y vanishes exactly at the stationary solutions of (1.1) and (1.10).

The term $(\nabla u)\cdot v$ of P can be of either sign thus P is neither bounded above nor below, i.e. P is not a Lyapunov function. We modify P by another functional which is non-negative and vanishes at the stationary solutions,

$$Q(y) := Q(u,v) = \frac{1}{2}\int_\Omega \left[D(f(u)-\nabla v)^2 + \frac{1}{\tau}(D\nabla u + v)^2\right] dx\,. \tag{3.19}$$

Then we consider the functional $L := Q - \lambda P : \mathcal{D} \to \mathbb{R}$, where $\lambda > 0$ will be chosen appropriately.

Lemma 3. *Assume* (H1), (H2) *and* (H3). *Then for each* λ *with* $\sup f'(u) < \lambda < 1/\tau$*, the functional* L *is bounded from below and* $\lim_{\|y\|_{\mathcal{D}}\to\infty} L(y) = +\infty$. *Moreover, if* $\{y_n\}_{n\in\mathbb{N}}$ *is a sequence in* $\mathcal{D}$ *such that* $\{L(y_n)\}$ *is uniformly bounded then* $\{y_n\}$ *is also uniformly bounded in* $\mathcal{D}$.

Proof. We consider $y \in \mathcal{D}$.

$$\begin{aligned}
L(y) &= \int_\Omega \bigg[\frac{1}{2}D(f(u)-\nabla v)^2 + \frac{1}{2\tau}(D\nabla u + v)^2 - \lambda D F(u) \\
&\quad -\lambda\frac{1}{2}v^2 - \lambda D\nabla u\cdot v\bigg]\, dx \\
&= \int_\Omega \bigg[\frac{1}{2}D(f(u)-\nabla v)^2 + \frac{1}{2\tau}(D\nabla u + v)^2 - \lambda D F(u) \\
&\quad -\lambda\left(-\frac{1}{2}D^2(\nabla u)^2 + \frac{1}{2}(D\nabla u + v)^2\right)\bigg]\, dx \\
&= \int_\Omega \bigg[-\lambda F(u) + \frac{1}{2}D(f(u)-\nabla v)^2 + \frac{1}{2}\left(\frac{1}{\tau}-\lambda\right)(D\nabla u + v)^2 \\
&\quad +\frac{\lambda}{2}D^2(\nabla u)^2\bigg]\, dx\,.
\end{aligned} \tag{3.20}$$

Since F is continuous and satisfies (H3), the functional $L(y)$ is bounded from below. To show that $\lim_{\|y\|_{\mathcal{D}}\to\infty} L(y) = +\infty$ we consider each term of the $\mathcal{D}$-norm of $y = (u,v)$ separately.

1. $\|u\|_{L^2} \to \infty$: Since F satisfies (H3), the first term in (3.20) diverges.
2. $\|\nabla u\|_{(L^2)^n} \to \infty$: The last term $(\nabla u)^2 \to \infty$.
3. $\|v\|_{(L^2)^n} \to \infty$: Since $\lambda < 1/\tau$, the factor of $(D\nabla u + v)^2$ is positive. If ∇u is bounded, then $L(y) \to \infty$. If ∇u grows such that $(D\nabla u + v)^2$ is bounded, then item 2 applies.
4. $\|\nabla v\|_{L^2} \to \infty$: If $f(u)$ stays bounded, then the second term in (3.20) leads to $L(y) \to \infty$. If the term $(f(u) - \nabla v)$ is bounded in L^2 then u is unbounded and item 1 applies.

If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$ such that $L(y_n) < L_0$ for all $n \in \mathbb{N}$, then from (3.20) it follows that

$$\|y_n\|_{\mathcal{D}} \le c\left(L_0 + \lambda \max_{u\in\mathbb{R}} F(u)\right),$$

where $\max F(u)$ is bounded since F is continuous and (H3) holds. □

We show that L decays along solutions. To evaluate $dL(y(t))/dt$ at a solution $y(.)$ we have to differentiate ∇u with respect to time. Since the component u is only in $C^1([0,T), L^2) \cap C([0,T), H^1)$ the time derivative of ∇u is not defined. Therefore we first consider solutions $y \in C^1([0,T), \mathcal{H}^1)$, $T < \infty$. Then $P(y(.))$ and $Q(y(.))$ are continuously differentiable with respect to time and we have

$$\begin{aligned}\frac{d}{dt}P(y(t)) &= \langle P_y(y(t)), y_t\rangle_{\mathcal{L}^2}\\ &= \int_\Omega [D(f(u) - \nabla v)u_t + (v + D\nabla u)\cdot v_t]\,dx\\ &= \int_\Omega [Du_t^2 - \tau v_t^2]\,dx\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}Q(y(t)) &= \int_\Omega \left[D(f(u) - \nabla v)(f'(u)u_t - \nabla v_t) + \frac{1}{\tau}(D\nabla u_t + v_t)\right] dx\\ &= \int_\Omega [Df'(u)u_t^2 - v_t^2]\,dx - D\int_\Omega [u_t\nabla v_t + v_t\cdot\nabla u_t]\,dx\,.\end{aligned}$$

Using Gauss theorem and the Neumann boundary condition (1.10), we get

$$\int_\Omega [u_t\nabla\cdot v_t + v_t\cdot\nabla u_t]\,dx = \int_{\partial\Omega} u_t(\eta\cdot v_t)\,dS = 0\,.$$

Hence

$$\frac{d}{dt}L(y(t)) = \int_\Omega [D(f'(u) - \lambda)u_t^2 + (\lambda\tau - 1)v_t^2]\,dx\,. \tag{3.21}$$

Since the right-hand side of (3.21) is continuous and well-defined on $\mathcal{D}$ we can define $dL(y(t))/dt$ for all solutions $y \in C^1([0,T), L^2) \cap C([0,T), H^1)$ by $dL(y(t))/dt :=$

$\lim_{n\to\infty} L(y_n(t))$, where y_n is any sequence in $C^1([0,T),\mathcal{H}^1)$ which approximates y.

Lemma 4. *Assume* (H1), (H2) *and* (H3) *and let* $y \in C^1([0,T),L^2) \cap C([0,T),H^1)$ *be a solution of* (1.1), (1.10). *Then*

1. *the solution exists for all* $t \geq 0$.
2. *If* $\lambda > 0$ *satisfies* $\sup f'(u) < \lambda < 1/\tau$ *then* L *is non-increasing along the solution. Moreover, there is a constant* $C > 0$ *such that*

$$\frac{d}{dt} L(y(t)) \leq -C\|P_y(y(t))\|^2_{\mathcal{L}^2} \,. \tag{3.22}$$

Proof. The bound (3.22) follows directly from (3.21). Indeed, by the choice of λ the factors $(f'(u) - \lambda)$ and $(\lambda\tau - 1)$ in (3.21) are negative. As a consequence, $L(y(t))$ is uniformly bounded in t

$$L(y(t)) \leq L(y(0)) =: L_0 \,.$$

From Lemma 5 it follows that $y(t)$ is uniformly bounded in $\mathcal{D}$ for $0 \leq t < T$ independent of T. Hence the solution exists for all $t \geq 0$. □

To complete the proof of Theorem 1 it remains to show that each solution converges in $\mathcal{D}$ to the set of all stationary solutions. This means that each sequence $(t_n) \to \infty$ has a subsequence $(s_n) \to \infty$ such that $y(s_n)$ converges to a stationary solution for $n \to \infty$. As shown in the proof of Lemma 3 the positive semi-orbit $\{y(t), 0 \leq t < \infty\}$ is uniformly bounded in $\mathcal{D}$. Since $\mathcal{D}$ is compactly embedded in $\mathcal{L}^2$ this semi-orbit is relatively compact in $\mathcal{L}^2$. For each sequence $(t_n) \to \infty$ there is a subsequence $(s_n) \to \infty$ such that $y(s_n) \to \bar{y} \in \mathcal{L}^2$ for $n \to \infty$. From (3.22) it follows that $P_y(y(s_n)) \to 0$ for $n \to \infty$ in $\mathcal{L}^2$. Then obviously $\bar{y}$ is a stationary solution of (1.1), (1.10). We solve formula (3.18) for the derivatives of u and v

$$\begin{pmatrix} \nabla v(s_n) \\ \nabla u(s_n) \end{pmatrix} = \begin{pmatrix} f(u(s_n)) \\ v(s_n)/D \end{pmatrix} - P_y(y(s_n)) \,.$$

The right-hand side converges in $\mathcal{L}^2$, hence $y(s_n) = (u(s_n), v(s_n))$ converges in $\mathcal{D}$ to $\bar{y}$. Then all accumulation points in $\mathcal{D}$ of the positive semi-orbit $\{y(t), 0 \leq t < \infty\}$ are stationary solutions and Theorem 1 is proved in the case of Neumann boundary conditions. □

4. Proof of Theorem 1 for Dirichlet Boundary Conditions

The functional $P : \mathcal{D} \to \mathbb{R}$ of the previous section cannot be used to define a Lyapunov function for Dirichlet boundary conditions (1.9) since the boundary integral (3.16) does not vanish. Following Brayton and Miranker[2] we introduce the boundary values of the solutions as independent functions, i.e. we define

$$V := \{(u, v, a, b) \in \mathcal{H}^1 \times L^2(\partial\Omega) \times L^2(\partial\Omega)^n ; u|_{\partial\Omega} = a, v|_{\partial\Omega} = b\} \,.$$

An element of V is denoted by $y := (u, v, a, b)$. On V we define a functional $R : V \to \mathbb{R}$

$$R(y) := \int_\Omega \left[DF(u) + \frac{1}{2}v^2 - Du\nabla v \right] dx + \frac{\sqrt{\tau D}}{2} \int_{\partial\Omega} (\eta \cdot v)^2 \, dS\,.$$

We consider the first variation of $R(y + \varepsilon\zeta)$ in the direction of an arbitrary $\zeta := (\varphi, \xi, \alpha, \beta) \in V$.

$$\frac{d}{d\varepsilon} R|_{\varepsilon=0} = \int_\Omega [Df(u)\varphi + v \cdot \xi - Du\nabla\xi - D(\nabla v)\varphi] \, dx + \sqrt{\tau D} \int_{\partial\Omega} (\eta \cdot v)(\eta \cdot \beta) \, dS\,.$$

On $\partial\Omega$ it is $\xi = \beta$. Then by partial integration it follows that

$$-\int_\Omega Du\nabla\xi \, dx = \int_\Omega D\nabla u \cdot \xi \, dx - \int_{\partial\Omega} Du(\eta \cdot \beta) \, dS\,.$$

Now the first variation can be written as

$$\frac{d}{d\varepsilon} R|_{\varepsilon=0} = \langle R_y(y), \zeta \rangle_{L^2(\Omega)^{n+1} \times L^2(\partial\Omega)^{n+1}}$$

with the *functional gradient* of R

$$R_y(y) := \left(D(f(u) - \nabla v), v + D\nabla u, 0, D\left(\left(\sqrt{\frac{\tau}{D}}\eta \cdot v - u \right) \eta \right) \right). \tag{4.23}$$

The functional gradient R_y vanishes exactly for stationary solutions of the Dirichlet problem (1.1), (1.9). The Dirichlet boundary condition appears as an *Euler boundary condition* for the variational problem of R (see Wan[27]).

Now we can proceed as in the Neumann case. We define a Lyapunov function on the space of solutions

$$V_D := \{ y \in V : y \text{ satisfies Dirichlet boundary conditions (1.9)} \}\,.$$

Again $Q : V_D \to \mathbb{R}$ is defined by (3.19) and $L := Q - \lambda R : V_D \to \mathbb{R}$ with an appropriate constant $\lambda > 0$. The functional L can be written as

$$\begin{aligned} L(y) = \int_\Omega \Big[& -\lambda DF(u) + \frac{1}{2}D(f(u) - \nabla v)^2 + \frac{1}{2}\left(\frac{1}{\tau} - \lambda\right)(D\nabla u + v)^2 \\ & + \frac{\lambda}{2}D^2(\nabla u)^2 \Big] dx + \lambda D \int_{\partial\Omega} \left(u - \frac{1}{2}\sqrt{\frac{\tau}{D}}\eta \cdot v \right)(\eta \cdot v) \, dS\,. \end{aligned}$$

Since y satisfies the Dirichlet boundary condition (1.9), the boundary integral of L reduces to

$$\frac{\lambda}{2}\sqrt{\tau D} \int_{\partial\Omega} (\eta \cdot v)^2 \, dS \geq 0\,.$$

The remaining terms in L are the same as in the Neumann case (3.20). Then analogous to Lemma 3 we have

Lemma 5. *Assume* (H1), (H2) *and* (H3). *Then for each* λ *with* $\sup f'(u) < \lambda < 1/\tau$ *the functional* L *is bounded from below and* $\lim_{\|y\|_{V_D}\to\infty} L(y) = +\infty$. *Moreover, if* $\{y_n\}_{n\in\mathbb{N}}$ *is a sequence in* V_D *such that* $\{L(y_n)\}$ *is uniformly bounded then* $\{y_n\}$ *is also uniformly bounded in* V.

To study the behavior of R, Q and L along solutions we again consider solutions $z(t) \in V_D$ with $z \in C^1([0,T), \mathcal{H}^1)$.

$$\frac{d}{dt}R(y(t)) = \int_\Omega [Du_t^2 - \tau v_t^2]dx\,,$$

$$\frac{d}{dt}Q(y(t)) = \int_\Omega [Df'(u)u_t^2 - v_t^2]\,dx - D\sqrt{D/\tau}\int_{\partial\Omega} u_t^2\,dS\,,$$

$$\frac{d}{dt}L(y(t)) = \int_\Omega [D(f'(u) - \lambda)u_t^2 + (\lambda\tau - 1)v_t^2] - D\sqrt{D/\tau}\int_{\partial\Omega} u_t^2\,dS\,.$$

Again the expressions on the right-hand sides are well-defined and continuous on the whole space V_D. Hence the time derivatives of R, Q and L can be extended to V_D. Similar to the previous section we have the following result.

Lemma 6. *Assume* (H1), (H2) *and* (H3) *and let* $y(t) \in V_D$ *be a solution of* (1.1), (1.9) *for* $0 \le t < T$. *Then*

1. *the solution exists for all* $t \ge 0$.
2. *If* $\lambda > 0$ *satisfies* $\sup f'(u) < \lambda < 1/\tau$, *then* L *is non-increasing along* $y(t)$. *Moreover, there is a constant* $C > 0$ *such that*

$$\frac{d}{dt}L(y(t)) \le -C\|R_y(y(t))\|^2_{L^2(\Omega)^{n+1}\times L^2(\partial\Omega)^{n+1}}\,. \tag{4.24}$$

To complete the proof of Theorem 1 in the case of Dirichlet boundary conditions we use the same arguments as at the end of the previous section.

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