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Exponential Utility Indifference Valuation: Correlation, Semimartingales, BSDEs, Convergence

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Abstract

Exponential utility indifference valuation assigns to contingent claims H due at time T a value for an investor with exponential utility preferences. The *indifference value* h_t for H at time $t \in [0, T]$ makes the investor indifferent, in terms of maximal expected utility, between not selling H and selling H for the amount h_t. This thesis studies the form of the implicitly defined h_t and properties of the process $(h_t)_{0 \le t \le T}$ in four chapters, whose keywords are "correlation", "semimartingales", "BSDEs" and "convergence".

Correlation. We consider a two-dimensional Brownian model where H depends on a nontradable asset stochastically correlated with the traded asset available for hedging. The use of martingale arguments yields a structurally explicit formula for h_t, even with a fairly general stochastic correlation ρ between the two Brownian motions. After a change of measure, h_t enjoys a monotonicity property in $|\rho|$. This is the reason why we can generalise the explicit formula for h_t known from the literature for constant ρ.

Semimartingales. Also in a general semimartingale model, we can derive a formula for h_t, although it is much less explicit than in the Brownian model. A second result in this general setting is a description of $(h_t)_{0 \le t \le T}$ as the unique solution (in a suitable class of processes) of a backward stochastic differential equation (BSDE). The key to both results is what we call the *fundamental entropy representation* of H, a decomposition of H into a hedged and an unhedged part depending on the investor's risk aversion.

BSDEs. In a multidimensional Brownian model, we study in more detail the type of BSDE related to $(h_t)_{0 \le t \le T}$, with the goal of deriving bounds for (h_t). We transform such BSDEs by changing the probability measure, shrinking the filtration, or symmetrising the underlying probability space. These transformations yield bounds for the solutions of the original BSDEs in terms of solutions to other BSDEs which are easier to solve.

Convergence. Revisiting the two-dimensional Brownian model with stochastic correlation, we derive an explicitly computable sequence that converges to h_t. This result complements the structurally explicit formula for h_t. It is based on a convergence theorem for quadratic BSDEs, which we prove in a general continuous filtration.

Kurzfassung

Die exponentielle Nutzenindifferenzbewertung liefert für zur Zeit T fällige bedingte Forderungen H einen Wert für einen Investor mit exponentiellen Nutzenpräferenzen. Der *Indifferenzwert* h_t von H zur Zeit $t \in [0, T]$ macht den Investor in Bezug auf den maximalen erwarteten Nutzen indifferent zwischen den beiden Möglichkeiten, H nicht zu verkaufen oder H für den Betrag h_t zu verkaufen. Diese Dissertation studiert die Form des implizit definierten h_t und Eigenschaften des Prozesses $(h_t)_{0 \leq t \leq T}$ in vier Kapiteln, deren Schlüsselwörter "Korrelation", "Semimartingale", "BSDEs" und "Konvergenz" sind.

Korrelation. Wir betrachten ein zweidimensionales Brownsches Modell, bei dem H von einer nicht handelbaren Anlage abhängt, die mit der gehandelten und zur Absicherung verfügbaren Anlage stochastisch korreliert ist. Martingalargumente führen zu einer strukturell expliziten Formel für h_t, sogar unter einer ziemlich allgemeinen stochastischen Korrelation ρ zwischen den beiden Brownschen Bewegungen. Nach einem Masswechsel besitzt h_t eine Monotonieeigenschaft in $|\rho|$. Dies ist der Grund, weshalb wir die für konstantes ρ aus der Literatur bekannte explizite Formel für h_t verallgemeinern können.

Semimartingale. Auch in einem allgemeinen Semimartingalmodell erhalten wir eine Formel für h_t, die aber viel weniger explizit ist als im Brownschen Modell. Ein zweites Resultat in diesem allgemeinen Rahmen ist eine Beschreibung von $(h_t)_{0 \leq t \leq T}$ als eindeutige Lösung (in einer geeigneten Klasse von Prozessen) einer stochastischen Rückwärtsdifferentialgleichung (BSDE). Der Schlüssel zu beiden Resultaten ist die sogenannte *fundamentale Entropie-Repräsentation* von H; das ist eine Zerlegung von H in einen abgesicherten und einen nicht abgesicherten Teil, die von der Risikoaversion des Investors abhängt.

BSDEs. In einem mehrdimensionalen Brownschen Modell studieren wir die zu $(h_t)_{0 \leq t \leq T}$ zugehörige Art von BSDEs mit dem Ziel, Schranken für (h_t) herzuleiten. Wir transformieren die BSDEs durch einen Wechsel des Wahrscheinlichkeitsmasses, eine Verkleinerung der Filtration oder eine Symmetrisierung des zugrunde liegenden Wahrscheinlichkeitsraumes. Diese Transformationen ergeben Abschätzungen für die Lösungen der ursprünglichen BSDEs

durch Lösungen zu anderen BSDEs, die einfacher zu lösen sind.

Konvergenz. Schliesslich kommen wir auf das zweidimensionale Brownsche Modell mit stochastischer Korrelation zurück. Dort leiten wir eine explizit berechenbare Folge her, die gegen h_t konvergiert. Dieses Resultat ergänzt die strukturell explizite Formel für h_t. Es beruht auf einem Konvergenzresultat für quadratische BSDEs, das wir in einer allgemeinen stetigen Filtration beweisen.

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Chapter 1

Introduction

This chapter gives a general definition of the indifference value, puts the results of the thesis into a broader context and explains the relations between the Chapters 2–5.

1.1 Definition of the indifference value

What is a fair value at time $t \in [0,T]$ for a contingent claim H due at time T? This question stands at the basis of the thesis. Before studying properties of such a fair value, we have to specify a valuation approach. In mathematical finance, it is common to assume that the agent valuing H is an investor who can trade in some financial assets. If the payoff H is replicable by trading in these assets in a self-financing way, the unique fair value must be the initial cost of a replicating strategy due to no-arbitrage considerations. But in practice, H is often not replicable so that there exists no unique fair value, and many different valuation approaches have been proposed.

Since the valuation may be different for another investor, it seems natural to incorporate the investor's risk preferences by attributing her a utility function $U : \mathbb{R} \to \mathbb{R}$, which is strictly increasing and strictly concave. The value $U(x)$ is interpreted as the investor's utility or happiness when she has total wealth $x \in \mathbb{R}$. Note that $U(x)$ is typically not in monetary units even if x is. We assume that a risk-free bank account yielding zero interest and risky assets with price process S are available on the financial market. In mathematical terms, S is a semimartingale and H a random variable on some filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{0 \le s \le T}, P\big)$. If the investor starts at time t with $\mathcal{F}_t$-measurable capital x_t, trades on $]t,T]$ in a self-financing way and has to pay out H at time T, then her maximal expected utility is

$$V_t^H(x_t) := \operatorname*{ess\,sup}_{\vartheta \in \mathcal{A}_t} E_P\left[U\left(x_t + \int_t^T \vartheta_s \,\mathrm{d}S_s - H\right)\middle|\mathcal{F}_t\right],$$

where $\mathcal{A}_t$ is the set of allowed strategies on $]t, T]$. $\big($If S is continuous (with $S_{0-} := S_0$), one can equally well consider strategies on the closed interval $[t, T]$, as we sometimes do later in this thesis.$\big)$ In this introduction, we do not specify $\mathcal{A}_t$ or the assumptions on S and H needed to exclude arbitrage and to have a well-defined problem. For a general setting, this is done in Section 3.2, while Chapters 2, 4 and 5 introduce model-specific assumptions. The *indifference (seller) value* $h_t(x_t)$ for H at time t is implicitly defined by

$$V_t^0(x_t) = V_t^H\big(x_t + h_t(x_t)\big). \tag{1.1}$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with initial capital $x_t + h_t(x_t)$ but paying an additional cash-flow H at maturity T.

Remark 1.1. 1) Although the notion of utility functions has been known for centuries, its application to value claims is not very old. Indifference valuation of a contingent claim in the presence of a financial market was introduced by Hodges and Neuberger [36] in 1989. They define the indifference value for $t = 0$ and trivial $\mathcal{F}_0$. The dynamic formulation for any $t \in [0, T]$ was first done in Markovian models, where $V_t^H(x_t)$ and $h_t(x_t)$ can be represented as functions of the state variables; see the references in Section 2.4.2. The explicit dynamic formulation in the above way using $\mathcal{F}_t$-conditional expectations can be found in Mania and Schweizer [44]. A less explicit dynamic formulation has already been used by Becherer [4], with a different (but in fact equivalent) definition for $t \in (0, T]$.

2) Similarly to h_t, one can define the indifference buyer value $h_t^{\text{buyer}}(x_t)$ for H by

$$V_t^0(x_t) = V_t^{-H}\big(x_t - h_t^{\text{buyer}}(x_t)\big).$$

Because $h_t^{\text{buyer}}(x_t)$ equals minus the seller value of $-H$, we consider in Chapters 1–3 only the seller value. In Chapters 4 and 5, however, we use the buyer value, because this avoids additional minus signs in some calculations. $\diamondsuit$

In all what follows, we restrict our considerations to an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. The main reason is the mathematical practicability, since we then have $V_t^H(x_t) = \exp(-\gamma x_t) V_t^H(0)$ for bounded $\mathcal{F}_t$-measurable x_t, and thus $h_t(x_t) = h_t$ does not depend on x_t. This wealth independence of h_t facilitates the mathematical problem, and it may also be desirable in applications. We can write

$$h_t = \frac{1}{\gamma} \log \frac{V_t^H(0)}{V_t^0(0)} \tag{1.2}$$

so that the implicit equation (1.1) is reduced to the two optimisation problems of determining $V_t^H(0)$ and $V_t^0(0)$. In all chapters, we prove results for $V_t^H(0)$, and then derive properties of h_t via (1.2).

1.2 Three different strands in the literature

Regarding the results and model assumptions, the vast literature on exponential utility indifference valuation can roughly be divided into the following three strands:

1) Explicit formulas for h_t (for all $t \in [0, T]$ or only for $t = 0$) in some specific models.

2) Characterisations of the process $(h_t)_{0 \leq t \leq T}$ via backward stochastic differential equations (BSDEs) in some settings, or via partial differential equations (PDEs) in some Markovian models.

3) Dual representations of h_0 $\big($or $V_0^H(x_0)\big)$ in general frameworks.

Focusing on the most important literature relevant for this thesis, we give a brief description of each group. An overview of various aspects of indifference valuation with a long literature list is provided by the recently published book [13] edited by Carmona.

1) Explicit formulas for h_t

Explicit formulas for h_t have been established in only a few specific models, mainly in discrete-time or Markovian diffusion models. Because the former are not in the scope of this thesis, we only mention as an example the binomial model in Musiela and Zariphopoulou [48]. Markovian diffusion models are extensively studied, and Section 2.4.2 provides an overview with some references. In the simplest case, S is a geometric Brownian motion depending on some Brownian motion W, and the claim H is a function of Z_T, where Z is a geometric Brownian motion driven by a Brownian motion Y which has constant instantaneous correlation ρ with W. A bit more generally, the coefficients in the dynamics $\frac{\mathrm{d}Z_t}{Z_t}$ and $\frac{\mathrm{d}S_t}{S_t}$ are not constant but depend on time and the current level of Z. These models are often called *nontradable asset models* since H depends on a nontradable asset Z and hedging is only possible in S, which is imperfectly correlated with Z. In such Markovian models, the usual approach is to first write the Hamilton-Jacobi-Bellman nonlinear PDE for the value function associated to V^H. This PDE is then linearised by a power transformation with a constant exponent called the *distortion power* so that an explicit formula for h_t can be established; for more details see Section 2.4.2. That method works only if one has a Markovian model and if ρ is constant.

In an alternative approach, Tehranchi [56] first proves a Hölder-type inequality which he then applies to solve the portfolio optimisation problem. The distortion power there arises as an exponent from the Hölder-type inequality. Tehranchi finds an explicit expression for h_t at time $t = 0$ if ρ is constant. While this method needs no Markovian assumption and can treat claims which are general (bounded) functionals of the Brownian motion Y, it is still restricted to situations with constant correlation. Independently from Tehranchi and still with constant ρ, Brendle and Carmona derive the same formula for h_0 in the technical report [11], whose results are also presented in Carmona [14] in a similar way. Their basic idea is to represent a transform of H by the terminal value of some stochastic exponential. While a representation of this type (but under a different measure) also appears in our Chapter 2, our approach and presentation largely differ from Carmona [14] and allow for a stochastic correlation ρ.

Nontradable asset models are also used in applications like the valuation of executive stock options; see for example Cvitanić [15] or Grasselli and Henderson [28]. Typically, a manager who receives options H on the share Z of her company is not allowed to trade in Z. However, she might be able to trade in a correlated stock or market index S.

2) PDEs and BSDEs for $(h_t)_{0\leq t\leq T}$

If in the above nontradable asset model the claim H also depends on the traded asset S, it is in general no longer possible to derive an explicit formula for h_t. Under Markovian assumptions, one still has — at least formally — the Hamilton-Jacobi-Bellman PDE, but this cannot be linearised. Therefore, $(h_t)_{0\leq t\leq T}$ can only be characterised via a nonlinear PDE which is in general not explicitly solvable. The proofs of such PDE characterisations usually consist of two steps. One first shows that if there exists a unique solution of the PDE, then the solution can be identified with $(h_t)_{0\leq t\leq T}$. In the more technical second step, one proves existence and uniqueness of solutions of the PDE. While showing this is a notoriously difficult task, the concept of viscosity solutions provides a possible way to circumvent this problem by introducing a weaker notion of solution; but then the identification with (h_t) often becomes more delicate. The PDE characterisations of $(h_t)_{0\leq t\leq T}$ can be used to derive bounds for h_t; see for example Proposition 3.1 and Theorem 3.2 of Sircar and Zariphopoulou [54]. Such bounds actually hold in much greater generality; they follow as special cases from our Theorems 3.12 and

3.16, stated in a general semimartingale model.

In situations more general than diffusion models, Becherer [5] and Mania and Schweizer [44] prove BSDE characterisations for $(h_t)_{0\le t\le T}$. While [44] assumes that $\mathbb{F}$ is continuous, i.e., all local martingales are continuous, the framework in [5] has a continuous S driven by Brownian motions and $\mathbb{F}$ is generated by these and a random measure allowing the modeling of nonpredictable events. Analogously to the PDE results, BSDE characterisations are commonly derived in two steps. One first shows that the candidate solution of the BSDE can be identified with $(h_t)_{0\le t\le T}$ by applying the martingale optimality principle, which is similar to the Hamilton-Jacobi-Bellman PDE, but more general because it needs no Markovian assumptions. Using BSDE theory, the second step establishes existence of solutions of the BSDE and shows uniqueness, which is often based on comparison results for BSDEs. Apart from Becherer [5] and Mania and Schweizer [44], we mention Hu et al. [37] in the group of BSDE-related literature. They give a BSDE characterisation of $\big(V_t^H(0)\big)_{0\le t\le T}$, but not of $(h_t)_{0\le t\le T}$, in a multidimensional Brownian model when the investor's trading strategies must obey constraints described by closed sets. This result is generalised by Morlais [46] to a setting with a continuous filtration $\mathbb{F}$.

3) Dual representation of h_0 $\big($or $V_0^H(x_0)\big)$

A third group of papers obtains duality results in general frameworks. Roughly speaking, duality results say that the optimisation problem related to $V_0^H(x_0)$ is equivalent to minimising some functional over all equivalent sigma-martingale measures, i.e., all probability measures which are equivalent to P and under which S is a sigma-martingale. Such a result was first proved by Delbaen et al. [16] for a locally bounded S, and then extended by Biagini and Frittelli [8] to a general S (in the absence of H; but under integrability conditions, H can be incorporated by a straightforward transformation).

Closely related to these results is the well-known result that the minimal entropy sigma-martingale measure Q^E, which minimises the relative entropy $E_P\big[\frac{\mathrm{d}Q}{\mathrm{d}P}\log\frac{\mathrm{d}Q}{\mathrm{d}P}\big]$ over all equivalent sigma-martingale measures Q, has the form $\frac{\mathrm{d}Q^E}{\mathrm{d}P} = c\exp\big(\int_0^T \zeta_s\,\mathrm{d}S_s\big)$ for a positive constant c and an S-integrable process ζ such that $\int\zeta\,\mathrm{d}S$ is a Q-martingale for every equivalent sigma-martingale measure Q with finite relative entropy. This result was first shown by Kabanov and Stricker [38] for a locally bounded S, and later extended by Biagini and Frittelli [9] to a general S.

1.3 Results of the thesis

The contributions of this thesis are summarised in the following three points:

- Extending the nontradable asset model to allow for stochastic correlation ρ, we prove a structurally explicit formula[1] for h_t (Chapter 2).
- We build bridges between the three different strands of work mentioned in the previous section. Based on the general duality results, we show that $(h_t)_{0\leq t\leq T}$ is the unique solution (in a suitable class of processes) of a BSDE and we derive an interpolation formula for h_t in a general semimartingale framework (Chapter 3).
- Restricting to Brownian settings, we show new results for the indifference value h_t by proving BSDE results and using the BSDE characterisation of $(h_t)_{0\leq t\leq T}$ (Chapters 4 and 5).

This thesis treats various aspects of exponential utility indifference valuation in four chapters, which are strongly based on the papers [21–24]. To guarantee that the chapters can be read independently from each other, we have deliberately allowed some duplication of terms and ideas. We have assigned to the four chapters the keywords "correlation", "semimartingales", "BSDEs" and "convergence", which are also reflected in the title of this thesis. The structure of the thesis is as follows.

Chapter 2 (essentially [23]): Correlation. We first consider an incomplete market driven by two Brownian motions W and Y with stochastic instantaneous correlation ρ. The traded asset S is driven by W, while H is measurable with respect to the σ-field generated by Y. With the goal of deriving explicit results for h_t, we provide motivation for the introduction of an auxiliary abstract optimisation problem in a martingale framework. The main theoretical result of Chapter 2 is Theorem 2.2, which gives a structurally explicit formula for the value of this abstract problem. The application of Theorem 2.2 yields structurally explicit formulas for h_t in two application situations, where one is typical for stochastic volatility models and the other for executive stock option valuations. The application in the former situation extends the results of Tehranchi [56] to a fairly general stochastic correlation ρ between the Brownian motions W and Y. The explicit form of h_t is preserved at any time t, except that a parameter of the formula called the distortion power

[1] We define a *structurally explicit formula* as a formula which is explicit in principle, but, unlike an explicit formula, some parameters are not directly given in terms of the input parameters of the model. For brevity, we use in Chapter 2 the notion "explicit formula" also for a structurally explicit formula.

is only shown to exist but not explicitly determined. The reason why the generalisation to stochastic ρ is possible is due to the fact that after a change of measure, V_t^H enjoys a monotonicity property in $|\rho|$. We conclude Chapter 2 by briefly showing how the results can be generalised to a model with multidimensional W and Y.

Chapter 3 (essentially [24]): Semimartingales. Leaving the Brownian world, we work in a broad framework with a general semimartingale S. Based on the representation $\frac{\mathrm{d}Q^E}{\mathrm{d}P} = c\exp\big(\int_0^T \zeta_s \,\mathrm{d}S_s\big)$ of the minimal entropy sigma-martingale measure Q^E, we introduce a decomposition of H called the *fundamental entropy representation of* H $\big(FER(H)\big)$. We first show that the existence of $FER(H)$ is equivalent to a notion of no-arbitrage and then that $FER(H)$ is closely related to the indifference value process $(h_t)_{0\le t\le T}$. This relation between $FER(H)$ and $(h_t)_{0\le t\le T}$ can be exploited in two ways. Firstly, it yields in Theorem 3.12 an interpolation formula for h_t, which generalises the structurally explicit formula for h_t known from Chapter 2 in a Brownian setting. Although the formula is here much less explicit than in a Brownian model, it has the same structure and provides deeper insights into indifference valuation. Secondly, we show in Theorem 3.16 that $(h_t)_{0\le t\le T}$ is the first component of the unique solution (in a suitable class of processes) of a general BSDE. Under additional assumptions, the other components of the solution are BMO-martingales under Q^E. This generalises results by Becherer [5] and Mania and Schweizer [44]. Compared to the article [24], Chapter 3 contains some additional material which is presented in Appendices A and B.

Similarly to the literature on BSDE characterisations of $(h_t)_{0\le t\le T}$, the thesis contains both the derivation of a BSDE for $(h_t)_{0\le t\le T}$ and results for BSDEs. While Chapter 3 clarifies the relation between $FER(H)$, BSDEs and $(h_t)_{0\le t\le T}$, Chapters 4 and 5 show how BSDE techniques can be used to prove results, for which alternative proofs seem to be difficult.

Chapter 4 (essentially [22]): BSDEs. Keeping the importance of BSDEs for indifference valuation in mind, we restrict our considerations to a multidimensional Brownian model. The goal is to find new bounds for h_t since the structurally explicit results in Chapter 2 yield bounds only when the nontradable claim H does not depend on the Brownian motion W driving the traded asset S. Moreover, the upper and lower bounds of Chapter 2 might have a big difference for multidimensional W and Y, depending on the structure of the instantaneous correlation matrix between W and Y. To derive new bounds, we study in detail Brownian BSDEs with a particular convex generator related to indifference valuation. In general, a BSDE is based on a probability space, a filtration and a probability measure. By changing each of these ingredients in a suitable way, we obtain Theorems

4.5, 4.7 and 4.11, which are the main results of Chapter 4. These results yield bounds for the solutions of BSDEs with a certain convex generator, and the applications show how these bounds can be fruitfully used in exponential utility indifference valuation.

Chapter 5 (essentially [21]): Convergence. We finally revisit the two-dimensional Brownian model with stochastic correlation ρ introduced in Chapter 2. As mentioned above, the distortion power, appearing in the structurally explicit formula for h_t, is not explicitly known. If ρ is time-dependent and/or stochastic, one only knows the range of values of the distortion power. To complement these structure results, we derive in Theorems 5.8 and 5.10 explicitly computable sequences which converge to h_t almost surely. The study is based on a convergence result for BSDEs with quadratic generators. Using BSDE techniques, we prove this result in a general continuous filtration. Another application of this convergence theorem shows that h_t enjoys a continuity property in ρ.

To facilitate reading, there is a list of notations at the end of the thesis.

Chapter 2

A Brownian model with stochastic correlation

In this chapter, we prove a structurally explicit formula[2] for the indifference value in a two-dimensional Brownian model with stochastic correlation.

2.1 Introduction

The model we present in Section 2.2 consists of a risk-free bank account and a stock S driven by a Brownian motion W. The contingent claim H to be valued depends on another Brownian motion Y, which has stochastic instantaneous correlation ρ with W. The valuation of H is done via exponential utility indifference valuation. In the literature, which we compare in Section 2.4 with our results, there are two main approaches to obtain explicit formulas for the value of the resulting optimisation problem. In a Markovian setting, Henderson [31], Henderson and Hobson [33, 34] and Musiela and Zariphopoulou [47], among others, start with the Hamilton-Jacobi-Bellman nonlinear partial differential equation (PDE) for the value function of the underlying stochastic control problem. This PDE is then linearised by a power transformation with a constant exponent called *distortion power*. This method works only if one has a Markovian model and if ρ is constant. In an alternative approach, Tehranchi [56] first proves a Hölder-type inequality which he then applies to solve the portfolio optimisation problem. The distortion power there arises as an exponent from the Hölder-type inequality. Tehranchi finds an explicit expression for the indifference value at time 0 if ρ is constant. While this method needs no Markovian assumption and can

[2] For brevity, we use later in this chapter the notion "explicit formula" also for a structurally explicit formula; see the footnote 1 on page 6.

treat claims which are general (bounded) functionals of the process Y, it is still restricted to situations with constant correlation.

Since (exponential) utility indifference valuation hinges on (exponential) utility maximisation with a random endowment, we start by tackling the latter. With the goal of deriving explicit results in our Brownian setting, in Section 2.2.2 we provide the motivation for the introduction of an auxiliary abstract optimisation problem in a martingale framework. Our main theoretical result is Theorem 2.2 in Section 2.3; it gives an explicit formula for the value of this abstract problem. The proof uses martingale arguments to give upper and lower bounds on that value, in terms of bounds on ρ. Crucially, these bounds have the same structure, which enables us to derive a closed-form expression by interpolation. In particular, this allows us to handle a *random* correlation ρ.

Section 2.4 contains two applications of Theorem 2.2. In the first, case (I), we extend the model of Tehranchi [56] to a fairly general stochastic correlation; the typical example is a model with stochastic volatility which is correlated with the stock in a nondeterministic way. In the second, case (II), the asset driving the claim H is traded in principle, but nontradable for our investor. A typical example here is the valuation of (European) executive stock options. In both cases, we obtain closed-form expressions for the exponential utility indifference value of the claim H at all times $t \in [0, T]$. The key feature of our formulas is that *the explicit form of the indifference value is preserved at any time* t, except that the distortion power, which is shown to exist but not explicitly determined, may now be random and depend on the contingent claim H to be valued. To the best of our knowledge, this is the first explicit result on exponential utility indifference valuation in a setting with nonconstant and nondeterministic correlation. As another novelty, our general framework allows us to distinguish (via measurability conditions) between the settings of case (I) and case (II); this is impossible when ρ and the instantaneous Sharpe ratio λ of S are constant, as in most of the existing literature. Section 2.4.2 discusses this and other issues in more detail.

In Section 2.5, we provide both intuitive and rigorous explanations for our results. We show that the value of the abstract optimisation problem is monotonic in $|\rho|$. Because this value can be computed explicitly for constant ρ and is continuous in the ρ-argument, interpolation implies that the basic structure is preserved for a random ρ. This explains why we can obtain our nice and explicit results. However, the precise interpretation of the above monotonicity is delicate, since it only holds when the (ρ-dependent) probability measure $\hat{P}(\rho)$ appearing in the abstract problem is kept fixed. A counterexample shows that the value of the original optimisation problem under P may fail to be monotonic in $|\rho|$ if we allow $\hat{P}(\rho)$ to vary with ρ, and we explain how keeping $\hat{P}(\rho)$ fixed is linked to standard financial reasoning.

For concreteness and ease of exposition, all our results are given for two correlated Brownian motions W and Y. The final Section 2.6 briefly shows how everything can be generalised to a multidimensional Itô process setting.

2.2 Preliminaries

2.2.1 Model setup

We work on a finite time interval $[0,T]$ for a fixed $T > 0$ and a complete filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$. The filtration $\mathbb{G} = (\mathcal{G}_s)_{0\leq s\leq T}$ satisfies the usual conditions, has $\mathcal{G}_0$ trivial, and $Y = (Y_s)_{0\leq s\leq T}$ and $Y^{\perp}$ are two independent $(\mathbb{G}, P)$-Brownian motions. Unless otherwise mentioned, all processes and filtrations are indexed by $s \in [0,T]$, *and we fix* $t \in [0,T]$. For any process X, $\mathbb{F}^X = \big(\mathcal{F}_s^X\big)$ denotes the P-augmented filtration generated by X. For any filtration $\mathbb{F} \subseteq \mathbb{G}$, a process X is called $\mathbb{F}$-predictable if it is measurable with respect to the $\mathbb{F}$-predictable σ-field on $\Omega \times [0,T]$, completed by the nullsets of $P\otimes$(Lebesgue measure). To simplify computations, we use the notation $\mathcal{E}(N)_{s,y} := \exp\big(N_y - N_s - \frac{1}{2}(\langle N\rangle_y - \langle N\rangle_s)\big)$, $0 \leq s \leq y \leq T$ for a continuous $\mathbb{G}$-semimartingale N. Notions such as L^∞ or 'almost surely' (a.s.) always refer to P (or any probability measure equivalent to P).

The stochastic framework of our model consists of two Brownian motions W and Y with random instantaneous correlation ρ. To construct this, let $\rho = (\rho_s)$ be a $\mathbb{G}$-predictable process valued in $[-1,1]$ such that $|\rho|$ is bounded away from one (uniformly in ω and s) and define

$$W_s := \int_0^s \rho_y \,\mathrm{d}Y_y + \int_0^s \sqrt{1-\rho_y^2}\,\mathrm{d}Y_y^{\perp}, \quad 0 \leq s \leq T. \tag{2.1}$$

In our financial market, two assets are available for investing and going short: a risk-free *bank account* and a *stock* S. The instantaneous yield of the bank account is described by a deterministic spot interest rate function $r : [0,T] \to [0,\infty)$ which is bounded and Borel-measurable. For ease of notation, we directly pass to discounted quantities which means that we take $r \equiv 0$. (See Section 2.4 for more comments on this.) The (discounted) dynamics of the stock is given by

$$\mathrm{d}S_s = \mu_s S_s \,\mathrm{d}s + \sigma_s S_s \,\mathrm{d}W_s, \quad 0 \leq s \leq T, \ S_0 > 0, \tag{2.2}$$

where the drift μ and the volatility σ are $\mathbb{G}$-predictable processes. We assume for simplicity that μ is bounded and σ is bounded away from zero and infinity.

Hence the *instantaneous Sharpe ratio* $\lambda := \frac{\mu}{\sigma}$ is also bounded. We write

$$\frac{\mathrm{d}S_s}{S_s} = \mu_s \,\mathrm{d}s + \sigma_s \,\mathrm{d}W_s = \sigma_s \,\mathrm{d}\hat{W}_s, \quad 0 \le s \le T$$

and note that by Girsanov's theorem, the processes

$$\hat{W}_s := W_s + \int_0^s \lambda_y \,\mathrm{d}y \quad \text{and} \quad \hat{Y}_s := Y_s + \int_0^s \rho_y \lambda_y \,\mathrm{d}y, \quad 0 \le s \le T \tag{2.3}$$

are Brownian motions under the probability $\hat{P} \approx P$ on $(\Omega, \mathcal{G}_T)$ given by

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\Big(-\int \lambda \,\mathrm{d}W\Big)_{0,T}. \tag{2.4}$$

In the terminology of Föllmer and Schweizer [20], $\hat{P}$ is the *minimal martingale measure* for S.

Let H be a bounded $\mathcal{G}_T$-measurable random variable, interpreted as a contingent claim or payoff due at time T. To value H, we assume that our investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. He starts at time $t \in [0,T]$ with initial capital x_t and runs a self-financing strategy $\pi = (\pi_s)_{t \le s \le T}$ so that his wealth at time $s \in [t,T]$ is

$$X_s^{x_t,\pi} = x_t + \int_t^s \frac{\pi_y}{S_y} \,\mathrm{d}S_y = x_t + \int_t^s \pi_y \sigma_y \,\mathrm{d}\hat{W}_y,$$

where π represents the amount invested in S. The set $\mathcal{A}_t$ of *admissible strategies* on $[t,T]$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_s)_{t \le s \le T}$ which satisfy $\int_t^T \pi_s^2 \,\mathrm{d}s < \infty$ a.s. and are such that

$$\bigg(\exp\Big(-\gamma \int_t^s \pi_y \sigma_y \,\mathrm{d}\hat{W}_y\Big)\bigg)_{t \le s \le T} \text{ is of class } (D) \text{ on } (\Omega, \mathcal{G}_T, \mathbb{G}, P), \tag{2.5}$$

abbreviated by 'of P-class (D)'. We define V^H (and analogously V^0) for $t \in [0,T]$ and x_t bounded $\mathcal{G}_t$-measurable by

$$\begin{aligned} V_t^H(x_t) &:= \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t} E_P\big[U(X_T^{x_t,\pi} - H)\big|\mathcal{G}_t\big] \\ &= -\mathrm{e}^{-\gamma x_t} \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} E_P\bigg[\exp\Big(-\gamma \int_t^T \pi_s \sigma_s \,\mathrm{d}\hat{W}_s + \gamma H\Big)\bigg|\mathcal{G}_t\bigg], \end{aligned} \tag{2.6}$$

using that (bounded) $\mathcal{G}_t$-measurable factors can be pulled out. Thus $V_t^H(x_t)$ is the maximal expected utility the investor can achieve by starting at time t

with initial capital x_t, using some admissible strategy π, and paying out H at time T.

Viewed over time t, $V^H(0)$ defined (up to a minus sign) by the essential infimum in (2.6) is the *dynamic value process* for the stochastic control problem associated to exponential utility maximisation. One can show by standard arguments that $V^H(0)$ has an RCLL version and then study its dynamic properties as a process; see for instance [5], [44] or [46]. However, our goal in this chapter is rather to provide explicit or structural formulas for $V_t^H(0)$ with a fixed t.

Remark 2.1. Condition (2.5) is technically useful, but also has the following desirable implication. From an economic point of view, one should only allow strategies which are close in some sense to investments with finite credit lines, as Schachermayer [51] emphasises after his Definition 1.3. In our model, any $\pi \in \mathcal{A}_t$ can be approximated in the following way. Consider a sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{G}$-stopping times increasing to T stationarily and define a self-financing strategy $\pi^{(n)} = \pi \mathbb{1}_{]\!]t,\tau_n]\!]}$ by trading according to π until τ_n and then putting all the capital into the bank account. This gives a terminal portfolio value $x_t + \int_t^{\tau_n} \pi_s \sigma_s \, \mathrm{d}\hat{W}_s$, leading to the individual utility $-\exp\big(-\gamma x_t - \gamma \int_t^{\tau_n} \pi_s \sigma_s \, \mathrm{d}\hat{W}_s\big)$, which converges in $L^1(P)$ to the utility of the final value $X_T^{x_t,\pi}$ of the strategy π due to (2.5). If we specifically choose

$$\tau_n := \inf\big\{s \in [t,T] \;\big|\; X_s^{x_t,\pi} - x_t \le -n\big\} \wedge T, \quad n \in \mathbb{N},$$

each of the approximating $\pi^{(n)}$ represents an investment with finite credit line. A similar approximation is used in Proposition 2.4 to find an upper bound for V^H, and the same class of strategies has been used in Hu et al. [37]. $\Diamond$

The time t *indifference (seller) value* $h_t(x_t)$ for H is implicitly defined by

$$V_t^0(x_t) = V_t^H\big(x_t + h_t(x_t)\big).$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with initial capital $x_t + h_t(x_t)$ but paying out H at T. Our final goal is to find an explicit formula for $h_t(x_t)$. By (2.6),

$$h_t = h_t(x_t) = \frac{1}{\gamma} \log \frac{V_t^H(0)}{V_t^0(0)} \tag{2.7}$$

does not depend on x_t. This also shows that we are done once we have $V_t^H(0)$ explicitly, and so our focus henceforth lies on the optimisation problem (2.6).

2.2.2 Motivation

Our goal is to find an explicit expression for

$$-V_t^H(0) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} E_P\left[\exp\left(-\gamma \int_t^T \pi_s\sigma_s \, \mathrm{d}\hat{W}_s + \gamma H\right)\middle|\mathcal{G}_t\right]. \tag{2.8}$$

Section 2.3 studies and solves an abstract martingale version of this problem, and we first explain how that formulation naturally arises out of (2.8). Since we only want to provide motivation, we ignore here all technical issues like integrability etc.

Suppose first that $H \equiv 0$ and S is a (local) P-martingale; equivalently, $\mu = \lambda = 0$ and $\hat{W}$ is a P-Brownian motion. Then the stochastic integral in (2.8) is a P-martingale, we minimise the expectation of a convex function of this, and so Jensen's inequality immediately tells us that the optimiser is $\pi^\star \equiv 0$ and that $V_t^H(0) = -1$.

In the general case where S is a P-semimartingale, the idea is now to reduce (2.8) to the martingale case by writing

$$-V_t^H(0) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} E_{P'}\left[\frac{Z'_t}{Z'_T}\exp\left(-\gamma \int_t^T \pi_s\sigma_s \, \mathrm{d}\hat{W}_s + \gamma H\right)\middle|\mathcal{G}_t\right], \tag{2.9}$$

where Z' is the P-density process of some fixed measure P' (not depending on π) under which S or $\hat{W}$ is a local martingale. To choose a good P', one might be tempted by the duality results of [16] to take the minimal entropy martingale measure Q^E, because its density Z_T^E is up to a constant the exponential of a stochastic integral of S. However, this is not true for the density Z_t^E on $\mathcal{G}_t$, and it is in general also very difficult to find Q^E explicitly in any given model. Because we want explicit formulas, we need Z'_t/Z'_T as explicitly as possible. Now any equivalent local martingale measure P' has a P-density process of the form $Z' = \mathcal{E}\big(-\int \lambda \, \mathrm{d}W\big)\mathcal{E}(N)$ for some local P-martingale N orthogonal to W, and inserting this expression for Z' in (2.9) gives after a straightforward calculation

$$\begin{aligned} &E_P\left[\exp\left(-\gamma \int_t^T \pi_s\sigma_s \, \mathrm{d}\hat{W}_s + \gamma H\right)\middle|\mathcal{G}_t\right] \\ &= E_{\hat{P}}\left[\exp\left(-\int_t^T (\gamma\pi_s\sigma_s - \lambda_s) \, \mathrm{d}\hat{W}_s + \gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \, \mathrm{d}s\right)\middle|\mathcal{G}_t\right]. \end{aligned} \tag{2.10}$$

The minimal martingale measure $\hat{P}$ from (2.4) appears naturally in this way, and it has the enormous benefit that its density $\hat{Z} = \mathcal{E}\big(-\int \lambda \, \mathrm{d}W\big)$ is completely explicit. Combining (2.10) with (2.8) gives

$$-V_t^H(0) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} E_{\hat{P}}\left[\exp\left(-\int_t^T (\gamma\pi_s\sigma_s - \lambda_s) \, \mathrm{d}\hat{W}_s + \overline{H}\right)\middle|\mathcal{G}_t\right], \tag{2.11}$$

and we can recognise this as a “martingale version” of (2.8) with an artificial random endowment

$$\overline{H} := \gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \,\mathrm{d}s. \tag{2.12}$$

Note that in the genuine semimartingale case $\lambda \not\equiv 0$, the quantity $\overline{H}$ appears even if the claim H is zero. Hence there is no simplification from assuming $H \equiv 0$, and so we do not discuss this case separately.

2.3 The main (but abstract) result

2.3.1 An explicit formula for an optimisation problem

This section contains the main mathematical contribution of this chapter. We derive an explicit formula for the value of the optimisation problem

$$-\hat{V}_t^{\hat{H}} := \operatorname*{ess\,inf}_{\hat{\pi}\,\in\,\hat{\mathcal{A}}_t} E_{\hat{P}}\left[\exp\left(-\int_t^T \hat{\pi}_s \,\mathrm{d}\hat{W}_s + \hat{H}\right)\middle|\mathcal{G}_t\right] =: \operatorname*{ess\,inf}_{\hat{\pi}\,\in\,\hat{\mathcal{A}}_t} \hat{\varphi}_t(\hat{\pi}), \tag{2.13}$$

where $\hat{\mathcal{A}}_t$ consists of all $\mathbb{G}$-predictable $\hat{\pi} = (\hat{\pi}_s)_{t\leq s\leq T}$ satisfying $\int_t^T \hat{\pi}_s^2 \,\mathrm{d}s < \infty$ a.s. and such that $\left(\exp\left(-\int_t^s \hat{\pi}_y \,\mathrm{d}\hat{W}_y\right)\right)_{t\leq s\leq T}$ is of $\hat{P}$-class (D). Here $\hat{H}$ is a bounded $\hat{\mathcal{H}}_T$-measurable random variable, where $\hat{\mathbb{H}} = \left(\hat{\mathcal{H}}_s\right) \subseteq \mathbb{G}$ is a filtration such that the $\hat{P}$-Brownian motion $\hat{Y}$ from (2.3) has the representation property in $\hat{\mathbb{H}}$. This means that any $\left(\hat{\mathbb{H}}, \hat{P}\right)$-martingale L is of the form $L = L_0 + \int \zeta \,\mathrm{d}\hat{Y}$ for an $\hat{\mathbb{H}}$-predictable ζ with $\int_0^T \zeta_s^2 \,\mathrm{d}s < \infty$ a.s. The assumption $\hat{H} \in L^\infty\left(\hat{\mathcal{H}}_T\right)$ is slightly weaker than $\hat{H} \in L^\infty\left(\mathcal{F}_T^{\hat{Y}}\right)$, and the two different applications in Section 2.4 will make it clear why this is useful. It is worth pointing out that all the subsequent arguments only involve the filtration $\hat{\mathbb{H}}$; this is the reason why we can formulate our model with a general filtration $\mathbb{G} \supseteq \mathbb{F}^{Y,Y^\perp}$ such that Y and $Y^\perp$ are $(\mathbb{G}, P)$-Brownian motions.

While the idea of considering a problem like (2.13) has been motivated in Section 2.2.2 from (2.8), it is not clear at this stage how $\hat{H}$ and especially $\hat{\mathbb{H}}$ arise. This will become clearer in Section 2.4 from the applications. However, we already point out that $\hat{H}$ and the artificial claim $\overline{H} = \gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \,\mathrm{d}s$ from (2.12) can well be different.

Theorem 2.2. *Under the above assumptions, set*

$$\underline{\delta}_t := \inf_{s\,\in\,[t,T]} \frac{1}{\left\|1-\rho_s^2\right\|_{L^\infty}} \quad \textit{and} \quad \overline{\delta}_t := \sup_{s\,\in\,[t,T]} \left\|\frac{1}{1-\rho_s^2}\right\|_{L^\infty}. \tag{2.14}$$

Then there exists a $\mathcal{G}_t$-measurable random variable $\delta_t^{\hat{H}}$ with values in $\big[\underline{\delta}_t, \overline{\delta}_t\big]$ such that

$$-\hat{V}_t^{\hat{H}}(\omega) = \Big(E_{\hat{P}}\Big[\exp\big(\hat{H}\big)^{\frac{1}{\delta}}\Big|\hat{\mathcal{H}}_t\Big](\omega)\Big)^{\delta}\bigg|_{\delta=\delta_t^{\hat{H}}(\omega)} \tag{2.15}$$

for almost all $\omega \in \Omega$.

The right-hand side of (2.15) is understood as follows: We compute for fixed δ (a version of) the $\big(\hat{\mathcal{H}}_t, \hat{P}\big)$-conditional expectation of $\exp\big(\hat{H}\big)^{\frac{1}{\delta}}$, evaluate that (version) in the given ω and then insert for δ the value $\delta_t^{\hat{H}}(\omega)$.

Before we actually prove Theorem 2.2, we provide here an outline of the proof. The key idea is to find a family of processes $Z^{(\hat{\pi})}$ with

$$Z_T^{(\hat{\pi})} = \exp\bigg(-\int_t^T \hat{\pi}_s \,\mathrm{d}\hat{W}_s + \hat{H}\bigg) \tag{2.16}$$

and such that $Z^{(\hat{\pi})}$ is a $\big(\mathbb{G}, \hat{P}\big)$-submartingale for every $\hat{\pi} \in \hat{\mathcal{A}}_t$, and a $\big(\mathbb{G}, \hat{P}\big)$-martingale for some $\hat{\pi} = \hat{\pi}^\star \in \hat{\mathcal{A}}_t$. If we can do this, the same argument as in Section 2.2.2 easily shows that the essential infimum in (2.13) is attained for $\hat{\pi}^\star$.

To find such a family $Z^{(\hat{\pi})}$, we need a good representation for $\mathrm{e}^{\hat{H}}$, and the multiplicative form of (2.16) might suggest that we write $\mathrm{e}^{\hat{H}}$ as the final value of some stochastic exponential martingale. But unless we believe that $\hat{\pi}^\star \equiv 0$ happens to be optimal, $\mathrm{e}^{\hat{H}} = Z_T^{(0)}$ should be the final value of a $\big(\mathbb{G}, \hat{P}\big)$-submartingale rather than a $\big(\mathbb{G}, \hat{P}\big)$-martingale. Again in view of the multiplicative structure, the simplest way to transform a positive martingale into a submartingale is to raise it to a power bigger than one. Fixing a constant $\delta \geq 1$ to be specified later and using $\hat{H} \in L^\infty\big(\hat{\mathcal{H}}_T\big)$, we thus write

$$\exp\big(\hat{H}\big) = \exp\big(\hat{H}/\delta\big)^{\delta} = \big(c_t \mathcal{E}(L)_{t,T}\big)^{\delta}, \quad c_t := E_{\hat{P}}\big[\exp\big(\hat{H}/\delta\big)\big|\hat{\mathcal{H}}_t\big] \tag{2.17}$$

for a $BMO\big(\hat{\mathbb{H}}, \hat{P}\big)$-martingale L. $\big($More precisely, the positive $\big(\hat{\mathbb{H}}, \hat{P}\big)$-martingale with final value $\exp\big(\hat{H}/\delta\big)$ is uniformly bounded away from zero and infinity, and thus its stochastic logarithm L is in BMO.$\big)$ By the representation property of $\hat{Y}$ in $\hat{\mathbb{H}}$, L is of the form

$$L = \int \zeta \,\mathrm{d}\hat{Y} \quad \text{for an } \hat{\mathbb{H}}\text{-predictable } \zeta \text{ with } E_{\hat{P}}\bigg[\int_0^T \zeta_s^2 \,\mathrm{d}s\bigg] < \infty. \tag{2.18}$$

So L is a $BMO\big(\mathbb{G}, \hat{P}\big)$-martingale, too, and combining (2.16) and (2.17) gives

$$\begin{aligned} Z_T^{(\hat{\pi})} &= c_t^{\delta}\big(\mathcal{E}(L)_{t,T}\big)^{\delta} \exp\bigg(-\int_t^T \hat{\pi}_s \,\mathrm{d}\hat{W}_s\bigg) \\ &= c_t^{\delta}\mathcal{E}(\delta L)_{t,T}\, \mathcal{E}\bigg(-\int \hat{\pi}\,\mathrm{d}\hat{W}\bigg)_{t,T} \exp\bigg(\frac{1}{2}\int_t^T \big((\delta^2-\delta)\zeta_s^2 + \hat{\pi}_s^2\big)\,\mathrm{d}s\bigg). \end{aligned}$$

Using Yor's formula, (2.17) and $\mathrm{d}\langle\hat{Y},\hat{W}\rangle_s = \rho_s\,\mathrm{d}s$ yields

$$\mathcal{E}(\delta L)\,\mathcal{E}\Big(-\int\hat{\pi}\,\mathrm{d}\hat{W}\Big) = \mathcal{E}\Big(\delta L - \int\hat{\pi}\,\mathrm{d}\hat{W} - \Big\langle\delta L,\int\hat{\pi}\,\mathrm{d}\hat{W}\Big\rangle\Big)$$
$$= M^{(\hat{\pi})}\exp\Big(-\int\delta\zeta_s\hat{\pi}_s\rho_s\,\mathrm{d}s\Big)$$

with the local $(\mathbb{G},\hat{P})$-martingale

$$M^{(\hat{\pi})} := \mathcal{E}\Big(\delta L - \int\hat{\pi}\,\mathrm{d}\hat{W}\Big) = \mathcal{E}\Big(\int\delta\zeta\,\mathrm{d}\hat{Y} - \int\hat{\pi}\,\mathrm{d}\hat{W}\Big), \tag{2.19}$$

and putting everything together and completing squares leads us to define

$$Z_s^{(\hat{\pi})} := c_t^\delta M_{t,s}^{(\hat{\pi})}\exp\bigg(\frac{1}{2}\int_t^s\Big((\hat{\pi}_y-\delta\zeta_y\rho_y)^2+\zeta_y^2\delta\big(\delta(1-\rho_y^2)-1\big)\Big)\,\mathrm{d}y\bigg) \tag{2.20}$$

for $t\le s\le T$. This gives (2.16) by construction, and if ρ is constant, choosing $\delta := \frac{1}{1-\rho^2}$ ensures that the integrand in (2.20) is always nonnegative and vanishes for $\hat{\pi}^\star = \delta\zeta\rho$. Hence $Z^{(\hat{\pi})}$ is then on $[t,T]$ a local $(\mathbb{G},\hat{P})$-submartingale for every $\hat{\pi}$ and a local $(\mathbb{G},\hat{P})$-martingale for $\hat{\pi}^\star$. Apart from integrability issues, we thus have achieved our goal in that case.

In general, ρ is not constant. Then we choose one δ for the submartingale property of $Z^{(\hat{\pi})}$ for all $\hat{\pi}$, and another δ for the martingale property of $Z^{(\hat{\pi}^\star)}$. This gives an upper and a lower bound for $\hat{V}_t^{\hat{H}}$, and Theorem 2.2 is obtained by interpolation. The detailed proof is given in the next section.

Remark 2.3. The attentive reader may have noticed that we only give results on the *value* of the optimisation problem, and may argue that for hedging or investing purposes, one would also like to know the optimal *strategy* explicitly. While this is a valid point, it is a well-known unfortunate fact that this problem is notoriously difficult even in quite specific (e.g., Markovian) settings. We hope to address this question in future, as it goes beyond the scope of the present work. $\diamondsuit$

2.3.2 Proof of Theorem 2.2

The argument for Theorem 2.2 follows the outline given in Section 2.3.1. We suppose throughout that the assumptions of Theorem 2.2 hold and first derive an upper bound for $\hat{V}_t^{\hat{H}}$. Recall $\hat{\varphi}_t(\hat{\pi})$ from (2.13) and $\overline{\delta}_t$ from (2.14).

Proposition 2.4. *For all $\hat{\pi}\in\hat{\mathcal{A}}_t$, we have*

$$\hat{\varphi}_t(\hat{\pi}) \ge\ E_{\hat{P}}\big[\exp\big(\hat{H}/\overline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\overline{\delta}_t}\quad a.s.$$

Proof. We use the reasonings and notations from Section 2.3.1 with $\delta := \overline{\delta}_t$.

1) Suppose first that $\int_t^T \hat{\pi}_s^2 \,\mathrm{d}s$ is uniformly bounded. Then $\int \hat{\pi}\,\mathrm{d}\hat{W}$ is a $BMO\big(\mathbb{G}, \hat{P}\big)$-martingale like $L = \int \zeta\,\mathrm{d}\hat{Y}$ from (2.18), and hence by Theorem 2.3 of Kazamaki [40], $M^{(\hat{\pi})} = \mathcal{E}\big(\delta L - \int \hat{\pi}\,\mathrm{d}\hat{W}\big)$ from (2.19) is a $\big(\mathbb{G}, \hat{P}\big)$-martingale. The choice $\delta = \overline{\delta}_t$ implies that the integrand in (2.20) is nonnegative and thus $Z^{(\hat{\pi})}$ is a $\big(\mathbb{G}, \hat{P}\big)$-submartingale; in fact, integrability follows via (2.16) because $Z_T^{(\hat{\pi})} = \exp\big(-\int_t^T \hat{\pi}_s\,\mathrm{d}\hat{W}_s + \hat{H}\big)$ is in $L^1\big(\hat{P}\big)$ since $\int \hat{\pi}\,\mathrm{d}\hat{W}$ is in BMO. Thus (2.13), (2.16), (2.20) and (2.17) yield

$$\hat{\varphi}_t(\hat{\pi}) = E_{\hat{P}}\big[Z_T^{(\hat{\pi})}\big|\mathcal{G}_t\big] \geq Z_t^{(\hat{\pi})} = E_{\hat{P}}\big[\exp\big(\hat{H}/\overline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\overline{\delta}_t} \quad \text{a.s.}$$

2) In general, we define a localising sequence by

$$\tau_n := \inf\left\{s \in [t,T] \text{ such that } \int_t^s \hat{\pi}_y^2\,\mathrm{d}y \geq n\right\} \wedge T, \quad n \in \mathbb{N}$$

and set $\hat{\pi}^{(n)} := \hat{\pi}\mathbb{1}_{]\!]t,\tau_n]\!]} \in \hat{\mathcal{A}}_t$. Applying step 1) to $\hat{\pi}^{(n)}$ then gives

$$\hat{\varphi}_t\big(\hat{\pi}^{(n)}\big) \geq E_{\hat{P}}\big[\exp\big(\hat{H}/\overline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\overline{\delta}_t} \quad \text{a.s.} \tag{2.21}$$

Because $\Big(\exp\big(-\int_t^s \hat{\pi}_y\,\mathrm{d}\hat{W}_y\big)\Big)_{t\leq s\leq T}$ is of $\hat{P}$-class (D) and $\hat{H}$ is bounded, the sequence $\Big(Z_T^{(\hat{\pi}^{(n)})}\Big)_{n\in\mathbb{N}} = \Big(\exp\big(-\int_t^T \hat{\pi}_s^{(n)}\,\mathrm{d}\hat{W}_s + \hat{H}\big)\Big)_{n\in\mathbb{N}}$ is $\hat{P}$-uniformly integrable and converges a.s. to $Z_T^{(\hat{\pi})}$. Hence the conditioned random variables $\hat{\varphi}_t\big(\hat{\pi}^{(n)}\big) = E_{\hat{P}}\Big[Z_T^{(\hat{\pi}^{(n)})}\Big|\mathcal{G}_t\Big]$, $n \in \mathbb{N}$, converge to $\hat{\varphi}_t(\hat{\pi})$ in $L^1\big(\hat{P}\big)$ and therefore also a.s. along a subsequence. This concludes the proof in view of (2.21). □

The next result entails a lower bound for $\hat{V}_t^{\hat{H}}$. Recall $\underline{\delta}_t$ from (2.14).

Proposition 2.5. *Define* $\hat{\pi}^\star = (\hat{\pi}_s^\star)_{t\leq s\leq T}$ *by*

$$\hat{\pi}_s^\star := \Big(\rho_s\underline{\delta}_t + \sqrt{\rho_s^2\underline{\delta}_t^2 + \underline{\delta}_t - \underline{\delta}_t^2}\Big)\zeta_s, \quad t \leq s \leq T, \tag{2.22}$$

where ζ *is now determined as in (2.17), (2.18) with* $\delta := \underline{\delta}_t$*. Then we have*

$$\hat{\varphi}_t(\hat{\pi}^\star) = E_{\hat{P}}\big[\exp\big(\hat{H}/\underline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\underline{\delta}_t} \quad \textit{a.s.}, \quad \textit{and} \quad \hat{\pi}^\star \in \hat{\mathcal{A}}_t. \tag{2.23}$$

To be more precise, $\hat{\pi}_s^\star$ is for any $s \in [t,T]$ defined by (2.22) on the set $\tilde{\Omega}_s = \Big\{\omega \in \Omega \,\Big|\, \frac{1}{1-\rho_s^2(\omega)} \geq \underline{\delta}_t\Big\}$, which has $\big(P$- and $\hat{P}$-$\big)$probability one. For $\omega \notin \tilde{\Omega}_s$, we set $\hat{\pi}_s^\star(\omega) := 0$. By the definition of $\underline{\delta}_t$, the expression under the square root in (2.22) is nonnegative, and $\hat{\pi}^\star$ is $\mathbb{G}$-predictable.

Proof. We use the notations of Section 2.3.1 with $\delta := \underline{\delta}_t$, and first show the equality in (2.23). Because $\hat{\pi}^\star$ in (2.22) is chosen to make the integrand in (2.20) vanish, we get from (2.20)

$$Z_T^{(\hat{\pi}^\star)} = c_t^{\underline{\delta}_t} M_{t,T}^{(\hat{\pi}^\star)} = E_{\hat{P}}\big[\exp\big(\hat{H}/\underline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\underline{\delta}_t} \mathcal{E}\big(N^{(\hat{\pi}^\star)}\big)_{t,T} \tag{2.24}$$

with $N^{(\hat{\pi}^\star)} := \int \underline{\delta}_t \zeta \, \mathrm{d}\hat{Y} - \int \hat{\pi}^\star \, \mathrm{d}\hat{W}$. An easy computation using (2.22) yields

$$\big\langle N^{(\hat{\pi}^\star)}\big\rangle_s - \big\langle N^{(\hat{\pi}^\star)}\big\rangle_t = \underline{\delta}_t \int_t^s \zeta_y^2 \, \mathrm{d}y = \underline{\delta}_t\big(\langle L\rangle_s - \langle L\rangle_t\big), \quad t \le s \le T \tag{2.25}$$

and so $N^{(\hat{\pi}^\star)}$ is like $\int \zeta \, \mathrm{d}\hat{Y} = L$ a $BMO\big(\mathbb{G}, \hat{P}\big)$-martingale; see below (2.18). We conclude by Theorem 2.3 of Kazamaki [40] that $M^{(\hat{\pi}^\star)} = \mathcal{E}\big(N^{(\hat{\pi}^\star)}\big)$ is a $\big(\mathbb{G}, \hat{P}\big)$-martingale, and so (2.13), (2.16) and (2.24) yield the equality in (2.23).

To prove $\hat{\pi}^\star \in \hat{\mathcal{A}}_t$, we first note that $E_{\hat{P}}\Big[\int_t^T \zeta_s^2 \, \mathrm{d}s\Big] < \infty$ from (2.18) implies $E_{\hat{P}}\Big[\int_t^T |\hat{\pi}_s^\star|^2 \, \mathrm{d}s\Big] < \infty$ by (2.22). To show that $\big(\exp\big(-\int_t^s \hat{\pi}_y^\star \, \mathrm{d}\hat{W}_y\big)\big)_{t \le s \le T}$ is of $\hat{P}$-class (D), we observe that (2.18) and (2.25) yield

$$\exp\bigg(-\int_t^s \hat{\pi}_y^\star \, \mathrm{d}\hat{W}_y\bigg) = \exp\bigg(N_s^{(\hat{\pi}^\star)} - N_t^{(\hat{\pi}^\star)} - \underline{\delta}_t \int_t^s \zeta_y \, \mathrm{d}\hat{Y}_y\bigg) = M_{t,s}^{(\hat{\pi}^\star)} \, \mathcal{E}(L)_{t,s}^{-\underline{\delta}_t}$$

for $s \in [t, T]$, and the process $\mathcal{E}(L)^{-\underline{\delta}_t}$ is bounded because (2.17) gives

$$\mathcal{E}(L)_{t,s} = \frac{E_{\hat{P}}\big[\exp\big(\hat{H}/\underline{\delta}_t\big)\big|\hat{\mathcal{H}}_s\big]}{E_{\hat{P}}\big[\exp\big(\hat{H}/\underline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]}, \quad t \le s \le T$$

and $\hat{H}$ is bounded. Moreover, $M^{(\hat{\pi}^\star)}$ as a $\big(\mathbb{G}, \hat{P}\big)$-martingale is of $\hat{P}$-class (D), and hence so is $\Big(\exp\big(-\int_t^s \hat{\pi}_y^\star \, \mathrm{d}\hat{W}_y\big)\Big)_{t \le s \le T}$. Thus $\hat{\pi}^\star$ is in $\hat{\mathcal{A}}_t$. $\square$

Remark 2.6. The choice of $\hat{\pi}^\star$ in (2.22) deserves a comment. As we have seen in the proof of Proposition 2.5, it ensures that the integrand

$$\big(\underline{\delta}_t \rho_s \zeta_s - \hat{\pi}_s^\star\big)^2 + \zeta_s^2 \underline{\delta}_t \big(\underline{\delta}_t(1 - \rho_s^2) - 1\big) \tag{2.26}$$

in (2.20) (with $\delta = \underline{\delta}_t$ and $\hat{\pi} = \hat{\pi}^\star$) vanishes identically. But for fixed $\omega \in \Omega$ and $s \in [t, T]$, (2.26) is a quadratic function in $\hat{\pi}_s^\star(\omega)$, and requiring it to be zero for each s does not determine the process $\hat{\pi}^\star$ uniquely. In fact, Proposition 2.5 remains true if $\hat{\pi}^\star$ is replaced by $\hat{\pi}^\eta$ with

$$\hat{\pi}_s^\eta := \Big(\rho_s \underline{\delta}_t + \eta_s \sqrt{\rho_s^2 \underline{\delta}_t^2 + \underline{\delta}_t - \underline{\delta}_t^2}\Big)\zeta_s, \quad t \le s \le T$$

for any $\mathbb{G}$-predictable process η on $[t,T]$ with values in $\{-1,1\}$.

Suppose now we replace $\hat{\pi}^\star$ by $\hat{\pi}^{\star\star}$ with $\hat{\pi}^{\star\star}_s := \underline{\delta}_t \rho_s \zeta_s$, $s \in [t,T]$, which minimises (2.26) pointwise and makes it nonpositive. Then we get

$$\hat{\varphi}_t(\hat{\pi}^{\star\star}) = E_{\hat{P}}\Big[Z_T^{(\hat{\pi}^{\star\star})}\Big|\mathcal{G}_t\Big] \le E_{\hat{P}}\Big[E_{\hat{P}}\big[\exp\big(\hat{H}/\underline{\delta}_t\big)\big|\hat{\mathcal{H}}_t\big]^{\underline{\delta}_t} M_{t,T}^{(\hat{\pi}^{\star\star})}\Big|\mathcal{G}_t\Big] = \hat{\varphi}_t(\hat{\pi}^\star),$$

using that $M^{(\hat{\pi}^{\star\star})}$ is like $M^{(\hat{\pi}^\star)}$ a $\big(\mathbb{G},\hat{P}\big)$-martingale. Similar arguments as for $\hat{\pi}^\star$ also yield $\hat{\pi}^{\star\star} \in \hat{\mathcal{A}}_t$, and we even obtain $\hat{\varphi}_t(\hat{\pi}^{\star\star}) < \hat{\varphi}_t(\hat{\pi}^\star)$ on a set $A \in \mathcal{G}_t$ with $\hat{P}[A] > 0$ if $\int_t^T \zeta_s^2\big(\underline{\delta}_t(1-\rho_s^2)-1\big)\,\mathrm{d}s$ is non-zero with positive probability. This shows that the lower bound

$$-\hat{\varphi}_t(\hat{\pi}^\star) \le -\operatorname*{ess\,inf}_{\hat{\pi}\,\in\,\hat{\mathcal{A}}_t} \hat{\varphi}_t(\hat{\pi}) = \hat{V}_t^{\hat{H}} \quad \text{a.s.} \tag{2.27}$$

entailed by Proposition 2.5 need not be sharp. Nevertheless, we work with $\hat{\pi}^\star$ and not with $\hat{\pi}^{\star\star}$, because $\hat{\varphi}_t(\hat{\pi}^\star)$ has the nice representation (2.23) which allows us to obtain an explicit expression for $\hat{V}_t^{\hat{H}}$; see the interpolation argument below. The sharper bound given via $\hat{\pi}^{\star\star}$ is not explicit enough to give this result. We remark that if ρ is constant, $\hat{\pi}^{\star\star}$ and $\hat{\pi}^\star$ coincide and (2.27) holds with equality; compare Propositions 2.4 and 2.5. $\diamondsuit$

As announced, we now prove Theorem 2.2 by an interpolation argument.

Proof of Theorem 2.2. Define $f(\cdot,\cdot) : \big[\underline{\delta}_t, \overline{\delta}_t\big] \times \Omega \to \mathbb{R}$ by

$$f(\delta,\omega) := \Big(E_{\hat{P}}\Big[\exp\big(\hat{H}\big)^{\frac{1}{\delta}}\Big|\hat{\mathcal{H}}_t\Big](\omega)\Big)^{\delta}, \quad (\delta,\omega) \in \big[\underline{\delta}_t, \overline{\delta}_t\big] \times \Omega.$$

Because $\hat{H}$ is bounded, dominated convergence and Jensen's inequality imply that f admits a version which is continuous and nonincreasing in δ for each fixed $\omega \in \Omega$. We use this version in the sequel. From Propositions 2.4 and 2.5 we already know that

$$f\big(\overline{\delta}_t,\omega\big) \le -\hat{V}_t^{\hat{H}}(\omega) \le f\big(\underline{\delta}_t,\omega\big) \quad \text{for a.a. } \omega \in \Omega.$$

By the intermediate value theorem, the set

$$\mathit{\Delta}(\omega) := \Big\{\delta \in \big[\underline{\delta}_t, \overline{\delta}_t\big] \;\Big|\; f(\delta,\omega) = -\hat{V}_t^{\hat{H}}(\omega)\Big\}$$

is thus nonempty for almost all $\omega \in \Omega$. Define $\delta_t^{\hat{H}} : \Omega \to \big[\underline{\delta}_t, \overline{\delta}_t\big]$ by

$$\delta_t^{\hat{H}}(\omega) := \sup \mathit{\Delta}(\omega), \quad \omega \in \Omega, \tag{2.28}$$

setting $\delta_t^{\hat{H}}(\cdot) := \big(\underline{\delta}_t + \overline{\delta}_t\big)/2$ on the nullset $\{\omega \in \Omega \,|\, \Delta(\omega) = \emptyset\}$. By the continuity of f in its first argument, $\Delta(\omega)$ is closed in $\mathbb{R}$ for all $\omega \in \Omega$, and we obtain for almost all $\omega \in \Omega$ that

$$f\big(\delta_t^{\hat{H}}(\omega), \omega\big) = -\hat{V}_t^{\hat{H}}(\omega). \tag{2.29}$$

It remains to prove that the mapping $\omega \mapsto \delta_t^{\hat{H}}(\omega)$ is $\mathcal{G}_t$-measurable. Because f is nonincreasing and due to (2.28) and (2.29), we have for any $a \in \big[\underline{\delta}_t, \overline{\delta}_t\big]$ that up to a null set,

$$\begin{aligned}\big\{\omega \in \Omega \,\big|\, \delta_t^{\hat{H}}(\omega) < a\big\} &= \Big\{\omega \in \Omega \,\Big|\, f\big(\delta_t^{\hat{H}}(\omega), \omega\big) > f(a, \omega)\Big\} \\ &= \big\{\omega \in \Omega \,\big|\, -\hat{V}_t^{\hat{H}}(\omega) > f(a, \omega)\big\} \\ &= \bigcup_{q \in \mathbb{Q}} \big\{\omega \,\big|\, -\hat{V}_t^{\hat{H}}(\omega) > q\big\} \cap \big\{\omega \,\big|\, q > f(a, \omega)\big\}.\end{aligned}$$

The last set is in $\mathcal{G}_t$ since $\hat{V}_t^{\hat{H}}$ and $f(a, \cdot)$ for a fixed $a \in \big[\underline{\delta}_t, \overline{\delta}_t\big]$ are $\mathcal{G}_t$-measurable. Since $\mathcal{G}_t$ is complete, we have $\big\{\omega \in \Omega \,\big|\, \delta_t^{\hat{H}}(\omega) < a\big\} \in \mathcal{G}_t$ for every $a \in \mathbb{R}$, and so $\delta_t^{\hat{H}}(\cdot)$ is $\mathcal{G}_t$-measurable. This ends the proof. $\square$

2.4 Applications in two settings

2.4.1 Explicit formulas for the indifference value

Our goal in this section is to find explicit formulas for $V_t^H(0)$ in (2.6) or (2.11) in two different settings. This will be achieved by applying Theorem 2.2 and will also yield explicit results for the indifference value h_t via (2.7). We recall W and $\hat{Y}$ from (2.1) and (2.3) and write for brevity

$$\mathbb{W} = (\mathcal{W}_s) \text{ for } \mathbb{F}^W, \quad \mathbb{Y} = (\mathcal{Y}_s) \text{ for } \mathbb{F}^Y, \quad \hat{\mathbb{Y}} = \big(\hat{\mathcal{Y}}_s\big) \text{ for } \mathbb{F}^{\hat{Y}}.$$

If $\rho\lambda$ is $\mathbb{Y}$-predictable, then $\hat{Y}$ from (2.3) is $\mathbb{Y}$-adapted and hence $\hat{\mathbb{Y}} \subseteq \mathbb{Y}$. In general, however, none of the above three filtrations contains any other.

Theorem 2.2 gives us the freedom to specify the artificial endowment $\hat{H}$, but also the task of finding a filtration $\hat{\mathbb{H}}$ such that $\hat{H}$ is $\hat{\mathcal{H}}_T$-measurable and $\hat{Y}$ has the representation property in $\hat{\mathbb{H}}$. Comparing (2.11) with (2.13) suggests to choose $\hat{H} = \overline{H} = \gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \, \mathrm{d}s$. In a first application, we do this, and moreover we set $\hat{\mathbb{H}} = \mathbb{Y}$ and assume that H is $\mathcal{Y}_T$-measurable and λ is $\mathbb{Y}$-predictable, to ensure that $\hat{H}$ is $\hat{\mathcal{H}}_T$-measurable. We shall later see in the proof of Theorem 2.9 that we also need to assume that ρ is $\mathbb{Y}$-predictable to guarantee that $\hat{Y}$ has the representation property in $\mathbb{Y}$.

For our second application, we choose $\hat{H} = \gamma H$ and assume that $\frac{1}{2}\int_t^T \lambda_s^2 \,\mathrm{d}s$ is replicable by trading in S. This is satisfied if λ is $\mathbb{W}$-predictable, as we shall see in the proof of Theorem 2.10. In this case, we moreover set $\hat{\mathbb{H}} = \hat{\mathbb{Y}}$ and assume that H is $\hat{\mathcal{Y}}_T$-measurable.

Cases. *In more detail, we consider one of the following two situations:*

(I) $H \in L^\infty(\mathcal{Y}_T)$*,* λ *is* $\mathbb{Y}$*-predictable, and* ρ *is* $\mathbb{Y}$*-predictable;*

(II) $H \in L^\infty\big(\hat{\mathcal{Y}}_T\big)$*,* λ *is* $\mathbb{F}^{S,\hat{Y}}$*-predictable, and* λ *is* $\mathbb{W}$*-predictable.*

The assumption in case (II) that λ is $\mathbb{F}^{S,\hat{Y}}$-predictable is quite natural since S and $\hat{Y}$ are the quantities observable for our investor. Moreover, it guarantees by Lemma 2.8 below that $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,\hat{Y}}$, i.e., the two basic driving Brownian motions Y and $Y^\perp$ are observable from S and $\hat{Y}$. In particular, if we take $\mathbb{G} = \mathbb{F}^{Y,Y^\perp}$, the a priori condition that λ is $\mathbb{F}^{S,\hat{Y}}$-predictable turns out to be innocent a posteriori.

To motivate our model choice, we discuss for each case a typical example.

Case (I): Here one should think of a *stochastic volatility* model, where μ and σ are $\mathbb{Y}$-predictable and the contingent claim H depends only on σ (e.g., a variance swap). The stock S is driven by the Brownian motion W, whereas its drift and volatility depend on a second factor Y. Our approach allows us to consider the situation where the correlation between W and Y is not constant, but more realistically a functional of Y.

In this setting, H is naturally $\mathcal{Y}_T$-measurable and $\lambda = \frac{\mu}{\sigma}$ is $\mathbb{Y}$-predictable like μ and σ. The only genuine condition is that ρ should be $\mathbb{Y}$-predictable, which we technically need to guarantee that $\hat{Y}$ has the representation property not only in $\hat{\mathbb{Y}}$, but also in $\mathbb{Y}$.

Case (II): A good application here comes from *executive stock options.* Consider a manager who receives call options on the stock $\big($driven by $\hat{Y}\big)$ of her company as part of her performance-related compensation. The manager must not trade the company stock and all its derivatives because of legal restrictions. However, she might be able to trade other, correlated stocks. So S is here a market index, a representative portfolio of other companies in the same line of business, or the stock of a leading company in the same line of business, which serves as a benchmark. We assume that the only source of incompleteness is the fact that the manager is not allowed to directly trade the stock of her company. In particular, we suppose that the market formed by the bank account and S is complete by assuming that μ and σ are both $\mathbb{W}$- and $\mathbb{F}^S$-predictable. Then $\mathbb{W} = \mathbb{F}^S$, i.e., the uncertainty ($\mathbb{W}$) about S equals the information $\big(\mathbb{F}^S\big)$ available from S. This follows from (2.2) because σ

is bounded away from zero. We then provide a fair value for the executive options in such a situation.

In this setting, $\lambda = \frac{\mu}{\sigma}$ is $\mathbb{W}$- and $\mathbb{F}^S$-predictable like μ and σ. The only genuine condition here is that H is $\hat{\mathcal{Y}}_T$-measurable, and the next remark explains why this is natural. Equivalently, that remark clarifies why we view the nontradable asset here as driven by $\hat{Y}$ and not by Y.

In both cases, the measurability assumptions make precise the underlying idea: The payoff H is driven by Y $\big($or $\hat{Y}\big)$, whereas hedging can only be done in S which is imperfectly correlated with Y $\big($or $\hat{Y}\big)$. The examples also illustrate two reasons why direct hedging in the stochastic process underlying H may be impossible; either its driver is not traded at all (e.g., a volatility or a consumer price index), or it is traded in principle but not tradable by our investor, due to legal, liquidity, practicability, cost or other reasons.

Remark 2.7. To see why $\hat{\mathcal{Y}}_T$-measurability of H is reasonable in case (II), recall that H is a claim on some asset Z, and write $\mathrm{d}Y = \rho\,\mathrm{d}W + \sqrt{1-\rho^2}\,\mathrm{d}W^\perp$ for a $(\mathbb{G}, P)$-Brownian motion $W^\perp$ P-independent of W. The asset change $\mathrm{d}Z$ is driven by two factors: the market development $\frac{\mathrm{d}S}{S}$ of the benchmark S, and company specific risks $\mathrm{d}W^\perp$. To determine the genuine driver of Z, we weight the two factors by the correlation process ρ, but first make them comparable by "normalising" $\frac{\mathrm{d}S}{S}$, which means that we use $\frac{1}{\sigma}\frac{\mathrm{d}S}{S} = \mathrm{d}\hat{W}$ instead of $\frac{\mathrm{d}S}{S}$. Thus Z is driven by

$$\rho\,\mathrm{d}\hat{W} + \sqrt{1-\rho^2}\,\mathrm{d}W^\perp = \rho\lambda\,\mathrm{d}s + \rho\,\mathrm{d}W + \sqrt{1-\rho^2}\,\mathrm{d}W^\perp = \rho\lambda\,\mathrm{d}s + \mathrm{d}Y = \mathrm{d}\hat{Y},$$

using (2.3). Hence assuming the Z-dependent claim H to be $\hat{\mathcal{Y}}_T$-measurable is more natural than having it $\mathcal{Y}_T$-measurable. Note that the filtrations $\hat{\mathbb{Y}}$ and $\mathbb{Y}$ differ in general, but coincide if ρ and λ are deterministic. $\diamondsuit$

Let us now briefly look at the information available to our investor. We always assume that the tradable stock S is observable. In addition, we assume in both cases (I) and (II) that the driver for the uncertainty behind H $\big($i.e., Y or $\hat{Y}$, respectively$\big)$ is also observable. The following result shows that the observable filtration then contains the filtration $\mathbb{F}^{Y,Y^\perp}$ of the underlying Brownian motions, and this justifies why we always use $\mathbb{G} \supseteq \mathbb{F}^{Y,Y^\perp}$ to describe the information on which our strategies $\pi \in \mathcal{A}_t$ must be based.

Lemma 2.8. *In case (I), $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,Y}$, and in case (II), $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,\hat{Y}}$.*

Proof. Note that the argument in each case uses only the middle condition on λ. For brevity, we write $Z \in \mathbb{F}^X$ to mean that Z is $\mathbb{F}^X$-predictable.

Case (I): By (2.2), $\langle S\rangle = \int \sigma^2 S^2\,\mathrm{d}s$ and $\langle S\rangle \in \mathbb{F}^S$ since it is a continuous pathwise quadratic variation; so $\sigma S = +\sqrt{\sigma^2 S^2} \in \mathbb{F}^S$ and hence also $\sigma \in \mathbb{F}^S$. Next, $\lambda \in \mathbb{Y}$ by assumption, and so $\mu = \sigma\lambda \in \mathbb{F}^{S,Y}$. Because σ is bounded away from 0, we obtain $W = \int \frac{1}{\sigma S}\,\mathrm{d}S - \int \frac{\mu}{\sigma}\,\mathrm{d}s \in \mathbb{F}^{S,Y}$. As a consequence, $\rho \in \mathbb{F}^{S,Y}$ since it is the density of $\langle W, Y\rangle$ with respect to Lebesgue measure and $\langle W, Y\rangle \in \mathbb{F}^{S,Y}$, being a continuous pathwise quadratic covariation. Finally, $|\rho|$ is bounded away from 1; so solving (2.1) for $Y^\perp$ implies that $Y^\perp \in \mathbb{F}^{S,Y}$ and therefore $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,Y}$.

Case (II): Again, $\sigma \in \mathbb{F}^S$. Moreover, (2.2), (2.1) and the definition (2.3) of $\hat{Y}$ give $\langle S, \hat{Y}\rangle = \int \sigma S \rho\,\mathrm{d}s$ so that $\rho \in \mathbb{F}^{S,\hat{Y}}$. Because $\lambda \in \mathbb{F}^{S,\hat{Y}}$ by assumption, we get $\mu = \sigma\lambda \in \mathbb{F}^{S,\hat{Y}}$, and now we can argue like in case (I) to deduce that $Y^\perp \in \mathbb{F}^{S,\hat{Y}}$. Moreover, $Y = \hat{Y} - \int \rho\lambda\,\mathrm{d}s \in \mathbb{F}^{S,\hat{Y}}$ and hence $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,\hat{Y}}$. $\square$

The two following theorems give explicit formulas for the value V^H and the indifference value h in cases (I) and (II). To facilitate comparisons with the literature, we state them for a spot interest rate on the bank account given by a bounded deterministic Borel-measurable function $r : [0,T] \to [0,\infty)$. Our results and arguments given for $r \equiv 0$ easily extend to this case; allowing r to be stochastic, however, would be a different issue.

Theorem 2.9. *Consider the setting and the assumptions from Section 2.2.1 and recall $\underline{\delta}_t$, $\overline{\delta}_t$ from (2.14). In case (I), define $\hat{H} := \gamma H - \frac{1}{2}\int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2}\,\mathrm{d}s$ and $\hat{0} := -\frac{1}{2}\int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2}\,\mathrm{d}s$. Then there exist $\mathcal{G}_t$-measurable random variables $\delta_t^{\hat{H}}$, $\delta_t^{\hat{0}}$ with values in $[\underline{\delta}_t, \overline{\delta}_t]$ such that*

$$V_t^H(x_t)(\omega) = -\exp\Big(-\gamma x_t(\omega)\mathrm{e}^{\int_t^T r(s)\,\mathrm{d}s}\Big)\Big(E_{\hat{P}}\Big[\exp\big(\hat{H}\big)^{\frac{1}{\delta}}\Big|\mathcal{Y}_t\Big](\omega)\Big)^{\delta}\bigg|_{\delta=\delta_t^{\hat{H}}(\omega)} \tag{2.30}$$

and

$$h_t(\omega) = \frac{\mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}}{\gamma}\log\frac{\big(E_{\hat{P}}\big[\exp(\hat{H})^{1/\delta}\big|\mathcal{Y}_t\big](\omega)\big)^{\delta}}{\big(E_{\hat{P}}\big[\exp(\hat{0})^{1/\delta'}\big|\mathcal{Y}_t\big](\omega)\big)^{\delta'}}\Bigg|_{\delta'=\delta_t^{\hat{0}}(\omega),\,\delta=\delta_t^{\hat{H}}(\omega)} \tag{2.31}$$

for almost all $\omega \in \Omega$ and every bounded $\mathcal{G}_t$-measurable random variable x_t.

Theorem 2.10. *Consider the setting and the assumptions from Section 2.2.1 and recall $\underline{\delta}_t$, $\overline{\delta}_t$ from (2.14). In case (II), there exists a $\mathcal{G}_t$-measurable random variable $\delta_t^{\gamma H}$ with values in $[\underline{\delta}_t, \overline{\delta}_t]$ such that*

$$\begin{aligned} V_t^H(x_t)(\omega) = &-\exp\bigg(-\gamma x_t(\omega)\mathrm{e}^{\int_t^T r(s)\,\mathrm{d}s} - \frac{1}{2}E_{\hat{P}}\bigg[\int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2}\,\mathrm{d}s\bigg|\mathcal{W}_t\bigg]\bigg) \\ &\times\Big(E_{\hat{P}}\Big[\exp(\gamma H)^{\frac{1}{\delta}}\Big|\hat{\mathcal{Y}}_t\Big](\omega)\Big)^{\delta}\bigg|_{\delta=\delta_t^{\gamma H}(\omega)} \end{aligned}$$

and

$$h_t(\omega) = \frac{\mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}}{\gamma} \log\Big(E_{\hat{P}}\Big[\exp(\gamma H)^{\frac{1}{\delta}}\Big|\hat{\mathcal{Y}}_t\Big](\omega)\Big)^{\delta}\bigg|_{\delta=\delta_t^{\gamma H}(\omega)}$$

for almost all $\omega \in \Omega$ and every bounded $\mathcal{G}_t$-measurable random variable x_t.

To the best of our knowledge, results like Theorems 2.9 or 2.10 have not been available in the literature so far; all previous approaches leading to explicit formulas have only considered situations where the correlation ρ is deterministic and constant in time. One nice feature of all formulas in Theorems 2.9 and 2.10 is that the only unknowns are the distortion powers $\delta^{\hat{H}}$, $\delta^{\hat{0}}$ or $\delta^{\gamma H}$, and we have precise bounds for these in terms of bounds on the correlation ρ. In general, each such power is random (in a $\mathbb{G}$-adapted way) and depends on H via $\hat{H}$. Since we have assumed that $\mathcal{G}_0$ is trivial, $\delta_0^{\hat{H}}$ is deterministic, but may still depend on $\hat{H}$. However, if the correlation ρ is deterministic and constant in time, the functions $\underline{\delta}$ and $\overline{\delta}$ in (2.14) coincide and equal $\frac{1}{1-\rho^2}$, and then $\delta^{\hat{H}} = \frac{1}{1-\rho^2}$ becomes constant and independent of H or $\hat{H}$. This explains why the constant correlation case is easier to handle and understand.

We defer the proofs of Theorems 2.9 and 2.10 to Section 2.4.3, and first compare our results with the existing literature.

2.4.2 Comparison with the literature

Exponential utility indifference valuation in Brownian settings has been extensively studied, particularly in Markovian models. An overview with a long literature list is provided by Henderson and Hobson [35]. We present here some references and comment first on the different model assumptions and then on the methods and results.

Recall the model in (2.1) and (2.2). Henderson [31, 32], Henderson and Hobson [33,34], and Musiela and Zariphopoulou [47] all work in a Markovian framework where μ, σ, r and ρ are all constant. [31–34] have a nontraded asset Z satisfying, for some constants $a > 0$ and $b \in \mathbb{R}$,

$$\frac{\mathrm{d}Z_s}{Z_s} = b\,\mathrm{d}s + a\,\mathrm{d}Y_s, \quad 0 \leqslant s \leqslant T,\ Z_0 > 0, \tag{2.32}$$

and the contingent claim $H = H(Z_T)$ is a function of the terminal value Z_T alone. Like in (2.1), Y is a Brownian motion having correlation ρ with W. [47] contains a slightly more general diffusion setting where $a_s = a(Z_s, s)$ and $b_s = b(Z_s, s)$ may depend on the current level of Z and on time. Monoyios [45] studies a similar model where σ and $\lambda = \frac{\mu}{\sigma}$ are not constant, but σ equals Z

and $\lambda_s = \lambda(Z_s)$ is a function of the current level of Z. Grasselli and Hurd [29] and Stoikov and Zariphopoulou [55] consider claims which depend not only on Z_T, but also in a certain way on the trajectory of Z. In contrast to all the above Markovian models, Tehranchi [56] analyses a more general situation very similar to case (I); but his approach is still restricted to a constant correlation ρ.

To the best of our knowledge, the only article where ρ is not constant is by Benth and Karlsen [7] who study a Markovian setting with $\rho = \rho(Z_s)$ depending on the present level of the nontraded asset Z. They show that the minimal entropy martingale measure can be expressed in terms of the solution of a semilinear PDE for which they prove existence and uniqueness of a classical solution. However, they have no claim H and they also do not derive any general explicit formulas.

Remark 2.11. All Markovian models above with *constant* μ, σ, r, ρ, a, b satisfy the measurability conditions for both cases (I) and (II). It is therefore somewhat arbitrary whether one views them as stochastic volatility or rather as executive stock option models. (Indeed, only our general model makes this precise distinction really possible.) The subsequent generalisations in [56], [45], [29] and [55] all head towards our case (I), whereas models from case (II) have not yet been studied for nondeterministic λ or ρ. In that sense, it seems fair to say that our formulation with a clear distinction between cases (I) and (II) represents a significant generalisation of previously considered models. $\diamondsuit$

We now recall and comment on how explicit formulas for the indifference value h are derived in the literature. As in Section 2.2.1, one usually first derives an expression for the value V^H and then obtains a formula for h via (2.7). In a Markovian model, the usual approach is to condition on the current state of the nontraded asset Z in (2.32), i.e., to write

$$V_t^H(x_t) = v(x_t, z_t, t) := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t} E\big[U\big(X_T^{x_t,\pi} - H(Z_T)\big)\big|X_t^{x_t,\pi} = x_t, Z_t = z_t\big].$$

Henderson [31], Henderson and Hobson [33,34], and later Musiela and Zariphopoulou [47] first write the Hamilton-Jacobi-Bellman nonlinear PDE for the value function v. Exploiting the scaling properties of the exponential utility function U, they try an ansatz of the form

$$v(x, z, t) = U(x)F(z, t),$$

which results in a *nonlinear* PDE for F. A clever power transformation,

$$F(z, t) = f(z, t)^{\frac{1}{1-\rho^2}}, \tag{2.33}$$

reduces this to a *linear* and solvable PDE for f. This yields an explicit formula for v and thus also for h via (2.7).

The idea to convert a nonlinear to a linear PDE by a power transformation was introduced by Zariphopoulou [57] for optimal portfolio management problems with nontraded assets when the utility is of the separable CRRA type: the payoff $H(Z_T)$ of the claim is multiplied by a power of the investor's final portfolio value $X_T^{x_t,\pi}$, i.e., $\widetilde{U}\big(X_T^{x_t,\pi}, H(Z_T)\big) = H(Z_T)|X_T^{x_t,\pi}|^\gamma/\gamma$ with $0 \neq \gamma < 1$. The application of the power transformation (2.33) to exponential utility indifference valuation appeared first in Henderson [31], Henderson and Hobson [33,34], and later in Musiela and Zariphopoulou [47]. The exponent $\delta := \frac{1}{1-\rho^2}$ from (2.33) is called *distortion power*, a terminology due to Zariphopoulou [57], and the approach is also known as *distortion method*. Henderson [31] and Henderson and Hobson [33, 34] also derive an approximation (for a small number of claims) of the power utility indifference value, which they compare with the exponential indifference value. Henderson [32] examines the latter criterion and incentives for executive stock options in the Markovian model of [31,33,34]. Monoyios [45] derives a representation of the optimal measure for the dual problem by combining the distortion method with general duality results. He further considers the optimisation problem under power utility, but without random endowment. Grasselli and Hurd [29] and Stoikov and Zariphopoulou [55] present explicit formulas for the exponential utility indifference value of a path-dependent claim on the volatility. But as already mentioned, all these approaches work only in a Markovian model and if the instantaneous correlation ρ between W and Y is constant.

In an alternative approach, Tehranchi [56] obtains an explicit expression for $V_t^H(x_t)$ in (2.6) with $t = 0$. He first proves a Hölder-type inequality which he then applies to determine $V_0^H(x_0)$, and this also yields an explicit formula for the indifference value at time 0. His method has the advantage that it needs no Markovian assumption and can treat general (bounded) $\mathcal{Y}_T$-measurable claims; but it is still restricted to situations with constant correlation. The distortion power $\delta = \frac{1}{1-\rho^2}$ from (2.33) arises there as an exponent in the Hölder-type inequality.

In all the above approaches, δ plays an important role, and it is crucial that it is deterministic and constant in time. We also use in (2.17) a power transformation with a power δ which must be constant, whereas $\delta = \frac{1}{1-\rho^2}$ in the above methods depends on ρ. This explains why we use two different powers in our proof of Theorem 2.2: $\overline{\delta}_t$ gives in Proposition 2.4 an upper bound for $\hat{V}_t^{\hat{H}}$, and $\underline{\delta}_t$ a lower bound in Proposition 2.5. The deeper reason why we can deal with a random correlation ρ is then a monotonicity property, as will be explained in Section 2.5.

Remark 2.12. **1)** We can in Theorem 2.9 replace $\hat{P}$ by the restriction Q of $\hat{P}$ to $\mathcal{Y}_T$, since $\hat{H}$ and $\hat{0}$ are $\mathcal{Y}_T$-measurable; so for almost all $\omega \in \Omega$, we have

$$h_t(\omega) = \frac{\mathrm{e}^{-\int_t^T r(s)\,\mathrm{d}s}}{\gamma} \log \frac{\big(E_Q\big[\exp(\hat{H})^{1/\delta}\big|\mathcal{Y}_t\big](\omega)\big)^{\delta}}{\big(E_Q\big[\exp(\hat{0})^{1/\delta'}\big|\mathcal{Y}_t\big](\omega)\big)^{\delta'}}\Bigg|_{\delta'=\delta_t^{\hat{0}}(\omega),\,\delta=\delta_t^{\hat{H}}(\omega)}.$$

Since ρ and λ are $\mathbb{Y}$-predictable in case (I), (2.4) and (2.1) yield explicitly

$$\frac{\mathrm{d}Q}{\mathrm{d}P} := E_P\left[\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}\middle|\mathcal{Y}_T\right] = \mathcal{E}\Big(-\int \rho\lambda\,\mathrm{d}Y\Big)_{0,T}. \tag{2.34}$$

This formula is used by Tehranchi [56] to *define* Q in his setting with constant ρ. Similarly, we could in Theorem 2.10 replace $\hat{P}$ by the restriction $\hat{Q}$ of $\hat{P}$ to $\hat{\mathcal{Y}}_T$. However, this is less useful because $\hat{Q}$, unlike Q, has in general no explicit form.

2) Apart from exponential utility, Tehranchi [56] also explicitly determines V_0^H for constant ρ when the investor's utility is of the same separable form as in Zariphopoulou [57], i.e., $\widetilde{U}\big(X_T^{x_t,\pi}, H\big) = H|X_T^{x_t,\pi}|^{\gamma}/\gamma$ with $0 \neq \gamma < 1$, or $\widetilde{U}\big(X_T^{x_t,\pi}, H\big) = H \log X_T^{x_t,\pi}$. Those results could be extended with our techniques as for exponential utility to all times t and to random ρ. But we give no details since this provides no essential new insights and, above all, does not help for finding an indifference value, because the above utilities are not of the form $U\big(X_T^{x_t,\pi} + H\big)$ required for a natural formulation.

3) The original motivation for this chapter was that we were intrigued by the elegantly simple and yet general approach of Tehranchi [56]. Along the way, we then discovered that not all arguments in [56] seem completely rigorous; the proof there of Lemma 4.2 is not quite clear (measurability of integrands?), and we see no argument why the portfolios constructed in Propositions 3.3–3.5 satisfy the integrability requirements to lie in the respective classes $\mathcal{A}$ of admissible strategies. Moreover, the proofs of these propositions also contain an incorrect statement; in general, a Brownian motion W and a process of the form $W + \int \lambda\,\mathrm{d}s$ do not generate the same filtration or σ-field, even if λ is predictable with respect to the filtration generated by W. A counterexample is given by Dubins et al. [18]. Despite all this, the final results in [56] are essentially correct; one way to circumvent the last problem is contained in the proof of our Theorem 2.9. $\diamondsuit$

2.4.3 Proofs of Theorems 2.9 and 2.10

We first need the following general result which says that the class (D) property behaves under a change to an equivalent probability measure in the same way as martingales. This is very intuitive and probably folklore, but we have not found it anywhere.

Lemma 2.13. *Denote by Z' the P-density process of a probability measure P' equivalent to P, i.e., $Z'_s := E_P\big[\frac{\mathrm{d}P'}{\mathrm{d}P}\big|\mathcal{G}_s\big]$, $s \in [0,T]$. A $\mathbb{G}$-adapted RCLL process Λ is of P'-class (D) if and only if $\Lambda Z'$ is of P-class (D).*

Proof. By symmetry and Bayes' formula it is enough to prove the "only if" part. Take a $\mathbb{G}$-adapted RCLL process Λ of P'-class (D) and fix $\epsilon > 0$. We want to find $K > 0$ with $\sup_\tau E_P\big[|\Lambda_\tau|Z'_\tau \mathbb{1}_{\{|\Lambda_\tau|Z'_\tau > K\}}\big] \le \epsilon$, where the supremum is taken over all $\mathbb{G}$-stopping times τ. Using that $\mathrm{d}P' = Z'_\tau \,\mathrm{d}P$ on $\mathcal{G}_\tau$ gives

$$E_P\big[|\Lambda_\tau|Z'_\tau \mathbb{1}_{\{|\Lambda_\tau|Z'_\tau > K\}}\big] = E_{P'}\big[|\Lambda_\tau| \mathbb{1}_{\{|\Lambda_\tau|Z'_\tau > K\}}\big].$$

Since Λ is of P'-class (D), $m := 1 \vee \sup_\tau E_{P'}\big[|\Lambda_\tau|\big]$ is finite and there exists $d_1 > 0$, which does not depend on τ, such that

$$A \in \mathcal{G}_T \text{ with } P'[A] \le d_1 \text{ and } \tau \text{ a } \mathbb{G}\text{-stopping time} \implies E_{P'}\big[|\Lambda_\tau|\mathbb{1}_A\big] \le \epsilon. \tag{2.35}$$

Because $P' \ll P$ by assumption, there exists $d_2 > 0$ such that

$$A \in \mathcal{G}_T \text{ with } P[A] \le d_2 \implies P'[A] \le d_1. \tag{2.36}$$

Set $K := m/d_2$ and use Markov's inequality to obtain

$$P\big[|\Lambda_\tau|Z'_\tau > K\big] \le \frac{1}{K} E_P\big[|\Lambda_\tau|Z'_\tau\big] = \frac{1}{K} E_{P'}\big[|\Lambda_\tau|\big] \le \frac{m}{K} = d_2$$

for any $\mathbb{G}$-stopping time τ. By (2.35) and (2.36), $E_{P'}\big[|\Lambda_\tau|\mathbb{1}_{\{|\Lambda_\tau|Z'_\tau > K\}}\big] \le \epsilon$ uniformly over τ, which ends the proof. □

Now we can prove Theorems 2.9 and 2.10 by applying Theorem 2.2.

Proof of Theorem 2.9. (2.31) follows directly from (2.7) and (2.30). To prove (2.30), we apply Theorem 2.2 with $\hat{H} := \overline{H} = \gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \,\mathrm{d}s$ and $\hat{\mathbb{H}} := \mathbb{Y}$. Comparing (2.13) with (2.11) shows that it only remains to argue that

i) $\hat{Y}$ has the representation property in $\mathbb{Y}$, and

ii) $\pi \in \mathcal{A}_t \iff \gamma\pi\sigma - \lambda \in \hat{\mathcal{A}}_t$.

The latter follows from Lemma 2.8 which yields that $\exp\big(-\int \gamma\pi\sigma \,\mathrm{d}\hat{W}\big)$ is of P-class (D) if and only if $\exp\big(-\int(\gamma\pi\sigma - \lambda)\,\mathrm{d}\hat{W}\big)$ is of $\hat{P}$-class (D), because $\int \lambda^2 \,\mathrm{d}s$ is bounded. Property i) is deduced from Itô's representation theorem in the form of Lemma 1.6.7 of Karatzas and Shreve [39]. In more detail, consider the restriction Q of $\hat{P}$ to $\mathcal{Y}_T$, given as in (2.34) by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = E_P\bigg[\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}\bigg|\mathcal{Y}_T\bigg] = \mathcal{E}\bigg(-\int \rho\lambda \,\mathrm{d}Y\bigg)_{0,T}$$

because λ and ρ are $\mathbb{Y}$-predictable. Note that this uses the assumptions of case (I). Here $\hat{Y}$ is also a $(\mathbb{Y}, Q)$-Brownian motion, and Lemma 1.6.7 in [39] now yields that any $(\mathbb{Y}, Q)$-martingale L is of the form $L = L_0 + \int \zeta \, \mathrm{d}\hat{Y}$ for a $\mathbb{Y}$-predictable ζ with $\int_0^T \zeta_s^2 \, \mathrm{d}s < \infty$ a.s. This crucially needs that $\rho\lambda$ is $\mathbb{Y}$-predictable, to ensure that $\hat{Y} = Y + \int \rho\lambda \, \mathrm{d}s$ from (2.3) is $\mathbb{Y}$-adapted. □

Proof of Theorem 2.10. As in the proof of Theorem 2.9, we apply Theorem 2.2, but now with $\hat{H} := \gamma H$ and $\hat{\mathbb{H}} := \hat{\mathbb{Y}}$. Of course, the $(\hat{\mathbb{H}}, \hat{P})$-Brownian motion $\hat{Y}$ then has the representation property in $\hat{\mathbb{H}}$. To get rid of the term $\frac{1}{2}\int_t^T \lambda_s^2 \, \mathrm{d}s$ in $\overline{H}$ in (2.11), we use again Itô's representation theorem as in Lemma 1.6.7 of [39] and obtain a $\mathbb{W}$-predictable process $\eta = (\eta_s)_{t \le s \le T}$ with

$$\frac{1}{2}\int_t^T \lambda_s^2 \, \mathrm{d}s = \frac{1}{2} E_{\hat{P}}\left[\int_t^T \lambda_s^2 \, \mathrm{d}s \middle| \mathcal{W}_t\right] + \int_t^T \eta_s \, \mathrm{d}\hat{W}_s \quad \text{and} \quad E_{\hat{P}}\left[\int_t^T \eta_s^2 \, \mathrm{d}s\right] < \infty.$$

Here we use that λ is $\mathbb{W}$-predictable in case (II), where we recall that $\mathbb{W} = \mathbb{F}^W$. Finally, comparison of (2.11) and (2.13) with $\hat{H} = \gamma H$ shows that it remains to prove that $\pi \in \mathcal{A}_t$ if and only if $\gamma\pi\sigma - \lambda + \eta \in \hat{\mathcal{A}}_t$. But this follows as in the proof of Theorem 2.9 from Lemma 2.8, using that $\int \eta \, \mathrm{d}\hat{W}$ is like $\int \lambda^2 \, \mathrm{d}s$ uniformly bounded. □

2.5 On the monotonicity in the correlation

In this section, we explain both intuitively and mathematically why we can obtain results even for a random correlation ρ.

For a constant correlation ρ, the abstract optimisation problem (2.13) has by Theorem 2.2 (or from Tehranchi [56] for $t = 0$) an explicit value, namely (2.15) with $\delta = \frac{1}{1-\rho^2}$. This expression is continuous in ρ and increasing in $|\rho|$, *for fixed* $\hat{P}$, and the intuition is as follows. The endowment $\hat{H}$ is driven by $\hat{Y}$, whereas hedging can only be done in $\hat{W}$ which is imperfectly correlated with $\hat{Y}$. If the correlation between $\hat{W}$ and $\hat{Y}$ is increased, better hedging is possible; so the value of the optimisation problem (2.13) decreases. (Note that (2.13) gives us *minus* the maximal expected utility.)

If we can extend the above monotonicity to a general correlation, it is clear why we can get the explicit structure in Theorem 2.2. Indeed, if ρ is random but lies between two bounds, the corresponding optimisation problem must by monotonicity have an explicit expression with the same basic structure — and of course the interpolating distortion power may now be random and depend on $\hat{H}$.

Let us now introduce more precise notations by writing (2.13) as

$$\hat{V}_t^{\hat{H}}\big(\rho', \hat{P}\big) := -\operatorname*{ess\,inf}_{\hat{\pi} \in \hat{\mathcal{A}}_t(\rho')} E_{\hat{P}}\left[\exp\left(-\int_t^T \hat{\pi}_s \,\mathrm{d}\hat{W}_s(\rho') + \hat{H}\right)\middle|\mathcal{G}_t\right]$$

for a $\mathbb{G}$-predictable process ρ' denoting the instantaneous correlation between the $\big(\mathbb{G}, \hat{P}\big)$-Brownian motions $\hat{W}(\rho')$ and $\hat{Y}$; the set $\hat{\mathcal{A}}_t(\rho')$ depends on ρ' through the $\hat{P}$-class (D) condition on $\big(\exp\big(-\int_t^s \hat{\pi}_y \,\mathrm{d}\hat{W}_y(\rho')\big)\big)_{t \le s \le T}$. Note that if we change ρ', only $\hat{W}(\rho')$ and all expressions depending on it will change. This is reasonable; clearly $\hat{H}$ and $\hat{\mathbb{H}}$ should not be affected.

The above intuitive argument now says that if we keep $\hat{P}$ fixed and vary ρ', we get a monotonicity, which is made precise in the following result.

Proposition 2.14. *Let $\hat{P}$ be fixed and suppose that ρ' and ρ'' are $\mathbb{G}$-predictable processes such that $|\rho'| \le c_1 \le |\rho''| \le c_2 < 1$ on $\Omega \times [t, T]$ for some constants c_1 and c_2. Then $\hat{V}_t^{\hat{H}}\big(\rho', \hat{P}\big) \le \hat{V}_t^{\hat{H}}\big(\rho'', \hat{P}\big)$ a.s.*

Proof. This follows from applying twice Theorem 2.2, once for $\hat{V}_t^{\hat{H}}\big(\rho', \hat{P}\big)$ and once for $\hat{V}_t^{\hat{H}}\big(\rho'', \hat{P}\big)$, and then using Jensen's inequality. □

Remark 2.15. Proposition 2.14 says that $\rho' \mapsto \hat{V}_t^{\hat{H}}\big(\rho', \hat{P}\big)$ is monotonic for correlation processes ρ', ρ'' that can be separated by a constant, uniformly in ω and s. Below Proposition 3 of the paper [23], we remarked that we did not know if the weaker assumption $|\rho'| \le |\rho''|$ on $\Omega \times [t, T]$ is also sufficient to prove the same conclusion. This question is later answered in an affirmative way in Proposition 5.6 by using different methods. ◊

The above intuition and Proposition 2.14 make it tempting to think that also the value $V_t^H(0)$ in (2.9) is monotonic in $|\rho|$. However, this is not true in general; we give a counterexample in the next paragraph. The crucial point is that $\hat{P}$ itself depends on ρ because W does; this can be seen from (2.1) and (2.4). So the abstract optimisation problem (2.13) has the structure $-\hat{V}_t^{\hat{H}}\big(\rho', \hat{P}(\rho)\big)\big|_{\rho'=\rho}$, and proving as in Proposition 2.14 that $\rho' \mapsto \hat{V}_t^{\hat{H}}\big(\rho', \hat{P}(\rho)\big)$ is monotonic *for fixed* ρ need not imply the monotonicity of $\rho \mapsto \hat{V}_t^{\hat{H}}\big(\rho, \hat{P}(\rho)\big)$.

We now show by a counterexample that $\rho \mapsto \hat{V}_t^{\hat{H}}\big(\rho, \hat{P}(\rho)\big)$ and thus $\rho \mapsto V_t^H(0; \rho) := V_t^H(0)$ from (2.9) are indeed not monotonic in general. In view of Proposition 2.14, this can only fail in the non-martingale case $\lambda \ne 0$, since otherwise $\hat{P}(\rho) = P$ does not depend on ρ. We take ρ and λ both constant, $t = 0$, and set $\hat{\mathbb{H}} = \mathbb{Y}$ as in case (I). Then Theorem 2.2 implies

$$\begin{aligned}\hat{V}_0^{\hat{H}}\big(\rho, \hat{P}(\rho)\big) &= -\Big(E_{\hat{P}(\rho)}\Big[\exp\big(\hat{H}\big)^{1-\rho^2}\Big]\Big)^{\frac{1}{1-\rho^2}} \\ &= -\bigg(E_P\bigg[\exp\bigg(\hat{H}(1-\rho^2) - \lambda\rho Y_T - \frac{T\lambda^2\rho^2}{2}\bigg)\bigg]\bigg)^{\frac{1}{1-\rho^2}},\end{aligned}$$

where we have conditioned on $\mathcal{Y}_T$ under P and used that $\hat{H}$ is $\mathcal{Y}_T$-measurable. For $\hat{H} = -Y^n := \big((-Y_T) \wedge n\big) \vee (-n), n \in \mathbb{N}$, dominated convergence and an easy calculation yield

$$\begin{aligned}\lim_{n\to\infty} \hat{V}_0^{-Y^n}\big(\rho, \hat{P}(\rho)\big) &= -\bigg(E_P\bigg[\exp\bigg(-Y_T(1-\rho^2) - \lambda\rho Y_T - \frac{T\lambda^2\rho^2}{2}\bigg)\bigg]\bigg)^{\frac{1}{1-\rho^2}} \\ &= -\exp\bigg((-\rho^2 + 2\lambda\rho + 1)\frac{T}{2}\bigg) =: g(\rho). \qquad (2.37)\end{aligned}$$

The mapping $\rho \mapsto g(\rho)$ is clearly not monotonic in $|\rho|$ except in the martingale case $\lambda = 0$. Because of (2.37), the mapping $\rho \mapsto \hat{V}_0^{-Y^n}\big(\rho, \hat{P}(\rho)\big)$ for n big enough is not monotonic in $|\rho|$ either. If we now consider case (I) with $\gamma = 1$ and $H = -Y^n$, the proof of Theorem 2.9 implies that

$$V_0^{-Y^n}(0;\rho) := V_0^{-Y^n}(0) = \mathrm{e}^{-\lambda^2 T/2}\,\hat{V}_0^{-Y^n}\big(\rho, \hat{P}(\rho)\big)$$

so that $\rho \mapsto V_0^{-Y^n}(0;\rho)$ for n big enough is not monotonic in $|\rho|$ either. This completes our counterexample.

Remark 2.16. One can directly show that $\hat{V}_0^{-Y_T}\big(\rho, \hat{P}(\rho)\big) = g(\rho)$ if one adapts the definition of $\hat{\mathcal{A}}_t(\rho)$. For such an *unbounded* $\hat{H}$, one stipulates that $\big(\exp\big(-\int_t^s \hat{\pi}_y \,\mathrm{d}\hat{W}_y(\rho) + \hat{H}\big)\big)_{t\le s\le T}$ instead of $\big(\exp\big(-\int_t^s \hat{\pi}_y \,\mathrm{d}\hat{W}_y(\rho)\big)\big)_{t\le s\le T}$ is of $\hat{P}$-class (D). The point is then that one can for this example explicitly determine $L = (\rho^2 - 1)(Y + \lambda\rho s)$ defined in (2.17), and L is obviously a $BMO\big(\mathbb{G}, \hat{P}(\rho)\big)$-martingale. ◊

The above counterexample shows that V^H is in general not monotonic in $|\rho|$. We now explain the intuition for this. In the martingale case $\lambda = 0$, the value $V_0^{-Y_T}(0;\rho) = \hat{V}_0^{-Y_T}\big(\rho, \hat{P}(\rho)\big) = -\exp\big((1-\rho^2)T/2\big)$ is clearly monotonic in $|\rho|$, and we have already seen why: Higher correlation permits better hedging, and so the investor runs less risk and has a higher expected utility. For the semimartingale case $\lambda \neq 0$, this effect is still there, but now also interacts with the correlation. Consider for instance the case where $\lambda > 0$ and $\rho > 0$. The optimal strategy $\hat{\pi}^\star$ for $\hat{V}_0^0\big(\rho, \hat{P}(\rho)\big)$ is zero and hence the optimal strategy for $V_0^0(0;\rho)$ is $\pi^\star = \frac{\lambda}{\gamma\sigma}$; compare the proof of Theorem 2.9. This strategy $\pi^\star$ makes a positive investment in the stock S. Adding $-H = Y_T$ leads to a total position with a higher risk, since the correlation ρ between Y and S is positive. To counteract this exposure, the investor will reduce his position in S and smooth out his terminal wealth. Hence he accepts in average a lower return on his portfolio in S to reduce the risk of his total position. So an increase in correlation yields a higher risk exposure for a fixed strategy; this is compensated by more conservative (smaller) investment in S, leading to a lower return and hence a decrease of the value $V_0^{-Y_T}(0;\rho)$. In total, $\rho \mapsto V_0^{-Y_T}(0;\rho)$ can therefore become decreasing in $|\rho|$ — despite the better hedging possibility. The above argument explains why this *can* happen, and (2.37) shows that it *does* happen for $0 < \rho < \lambda$.

Remark 2.17. In a Markovian framework with constant ρ and λ, the result of Proposition 2.14 has already been established by Henderson [32] who shows that the indifference value h $\big($or, equivalently in that setting, $V^H\big)$ is increasing in $|\rho|$. Henderson's analysis at first sight seems to contradict our nonmonotonic counterexample, and closer inspection reveals that it crucially depends on fixing some parameter called δ in [32] while varying ρ. But this exactly corresponds to our fixing $\hat{P}$ in Proposition 2.14 while varying ρ, and it has in both cases a very natural financial interpretation. In fact, the standard viewpoint in financial theory is that the instantaneous Sharpe ratio $\frac{a}{b}$ of the nontraded asset Z in (2.32) is not fixed exogenously, but related to λ via the correlation ρ. This tacit assumption is usually not spelt out explicitly in the finance literature, and the point of our counterexample is to illustrate that monotonicity may fail in its absence. $\diamondsuit$

2.6 The multidimensional case

In this section, we extend our main results to the case of more than two Brownian motions. Since most arguments are straightforward generalisations, we only sketch the main differences.

The probabilistic framework consists of an n-dimensional $(\mathbb{G}, P)$-Brownian motion Y and an m-dimensional $(\mathbb{G}, P)$-Brownian motion W, each having P-independent components. Instantaneous correlations are now given by a matrix $R = (\rho^{ij})_{\substack{i=1,\dots,n,\\ j=1,\dots,m}}$ with $\rho^{ij}_s := \frac{\mathrm{d}\langle Y^i, W^j\rangle_s}{\mathrm{d}s}$, and we choose R to be $\mathbb{G}$-predictable. It can be shown that the symmetric positive semidefinite matrix RR' has nonnegative eigenvalues which are all at most 1. We assume that all eigenvalues are bounded away from one uniformly on $\Omega \times [t,T]$, i.e.,

$$\text{there exists } c < 1 \text{ such that } \max \operatorname{spec}(RR') \leq c \quad \text{a.e. on } \Omega \times [t,T],$$

where $\operatorname{spec}(RR')$ denotes the spectrum (the set of eigenvalues) of RR'. Recall that $t \in [0,T]$ *is fixed*. There are m traded risky assets $S = (S_s^j)_{j=1,\dots,m}$ with dynamics

$$\mathrm{d}S_s^j = S_s^j \mu_s^j \,\mathrm{d}s + \sum_{k=1}^m S_s^j \sigma_s^{jk} \,\mathrm{d}W_s^k, \quad 0 \le s \le T,\ S_0^j > 0, \quad j = 1, \dots, m;$$

the drift vector $\mu = (\mu_s^j)_{j=1,\dots,m}$ and the volatility matrix $\sigma = (\sigma_s^{jk})_{j,k=1,\dots,m}$ are $\mathbb{G}$-predictable. We assume that σ is invertible, $\lambda := \sigma^{-1}\mu$ is bounded uniformly (in ω and s) and that there exists a constant C such that

$$C\beta'\beta \ge \beta'\sigma\sigma'\beta \ge \frac{1}{C}\beta'\beta \quad \text{on } \Omega \times [0,T] \text{ for all } \beta \in \mathbb{R}^m.$$

(In other words, σ is uniformly both bounded and elliptic.) The processes

$$\hat{W} := W + \int \lambda \,\mathrm{d}s \quad \text{and} \quad \hat{Y} := Y + \int R\lambda \,\mathrm{d}s$$

are Brownian motions under the minimal martingale measure $\hat{P}$ given by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\big(-\int \lambda \,\mathrm{d}W\big)_{0,T}$. All other definitions and model assumptions of Sections 2.2.1 and 2.3.1 can be easily translated to this setting and we do not detail this. The multidimensional version of Theorem 2.2 reads as follows.

Theorem 2.18. *Under the above assumptions, recall that* $\operatorname{spec}(R_s R_s')$ *denotes the spectrum (the set of eigenvalues) of* $R_s R_s'$, *and define* $\underline{\delta}_t$ *and* $\overline{\delta}_t$ *by*

$$\underline{\delta}_t := \inf_{s \in [t,T]} \frac{1}{\|1 - \min \operatorname{spec}(R_s R_s')\|_{L^\infty}}, \quad \overline{\delta}_t := \sup_{s \in [t,T]} \left\| \frac{1}{1 - \max \operatorname{spec}(R_s R_s')} \right\|_{L^\infty}. \tag{2.38}$$

Then there exists a $\mathcal{G}_t$*-measurable random variable* $\delta_t^{\hat{H}}$ *with values in* $\big[\underline{\delta}_t, \overline{\delta}_t\big]$ *such that*

$$-\hat{V}_t^{\hat{H}}(\omega) = \Big(E_{\hat{P}}\Big[\exp\big(\hat{H}\big)^{\frac{1}{\delta}}\Big|\hat{\mathcal{H}}_t\Big](\omega)\Big)^{\delta}\bigg|_{\delta=\delta_t^{\hat{H}}(\omega)} \tag{2.39}$$

for almost all $\omega \in \Omega$.

Outline of the proof. This goes similarly to Theorem 2.2 via analogues of Propositions 2.4 and 2.5, and we only point out where significant changes occur. The analogue to (2.20) is, for $t \le s \le T$,

$$Z_s^{(\hat{\pi})} := c_t^{\delta} M_{t,s}^{(\hat{\pi})} \exp\bigg(\frac{1}{2}\int_t^s \Big(|\hat{\pi}_y - \delta R_y'\zeta_y|^2 + \delta\zeta_y'\big(\delta(I - R_y R_y') - I\big)\zeta_y\Big)\,\mathrm{d}y\bigg) \tag{2.40}$$

with $M^{(\hat{\pi})} := \mathcal{E}\big(\int \delta\zeta \,\mathrm{d}\hat{Y} - \int \hat{\pi} \,\mathrm{d}\hat{W}\big)$ like in (2.19) and $c_t := E_{\hat{P}}\big[\exp\big(\hat{H}/\delta\big)\big|\hat{\mathcal{H}}_t\big]$ like in (2.17). In (2.40), I denotes the $(n \times n)$-identity matrix. As in the proof of Proposition 2.4, the key point is that the integrand in (2.40) with $\delta := \overline{\delta}_t$ is nonnegative for every $\hat{\pi} \in \hat{\mathcal{A}}_t$. To see this, one must prove that $\overline{\delta}_t(I - R_y R_y') - I$ is positive semidefinite or, equivalently, that all its eigenvalues are nonnegative. But if α is such an eigenvalue, then $1 - (\alpha + 1)\big/\overline{\delta}_t$ is an eigenvalue of $R_y R_y'$; this implies $1 - (\alpha + 1)\big/\overline{\delta}_t \leq 1 - 1\big/\overline{\delta}_t$ by (2.38), and hence $\alpha \geq 0$.

For the analogue of Proposition 2.5, one defines, for $t \leq s \leq T$,

$$\hat{\pi}^\star_s := \underline{\delta}_t R_s' \zeta_s + \sqrt{\underline{\delta}_t \zeta_s' \big(I - \underline{\delta}_t (I - R_s R_s')\big)\zeta_s}\,(1, 0, \dots, 0)', \tag{2.41}$$

where ζ is determined as in (2.17), (2.18) with $\delta := \underline{\delta}_t$ and $(1, 0, \dots, 0) \in \mathbb{R}^m$. Using (2.38) and a similar reasoning as above, one sees that the expression under the square root in (2.41) is nonnegative, and (2.40) simplifies to $Z^{(\hat{\pi}^\star)}_s = c_t^{\underline{\delta}_t} M^{(\hat{\pi}^\star)}_{t,s}$, $t \leq s \leq T$, for $\hat{\pi} = \hat{\pi}^\star$ and $\delta = \underline{\delta}_t$ like (2.24). As in the proof of Proposition 2.5, one can show that $M^{(\hat{\pi}^\star)}$ is a $\big(\mathbb{G}, \hat{P}\big)$-martingale and that $\hat{\pi}^\star \in \hat{\mathcal{A}}_t$.

Finally, (2.39) is proved from the analogues of Propositions 2.4 and 2.5 similarly as in the two-dimensional case. This concludes the proof outline. $\square$

Using Theorem 2.18, one can of course obtain results like Theorems 2.9 and 2.10 also in the multidimensional case. We refrain from giving details because the procedure goes essentially along the same lines as in Section 2.4. However, we emphasise that it is important to assume that the rank of the volatility matrix σ equals the dimension m of W. (In particular, we typically want at least m risky assets.) This condition, implied by the assumption that σ is invertible, is required to show that the sets $\hat{\mathcal{A}}_t$ and $\mathcal{A}_t$ fit together; compare ii) in the proofs of Theorems 2.9 and 2.10.

Chapter 3

A general semimartingale model

This chapter gives an interpolation formula and a BSDE description for the indifference value process in a general semimartingale model.

3.1 Introduction

Even in a concrete model, it is difficult to obtain a closed-form formula for the exponential utility indifference value of a contingent claim H. The majority of existing explicit results are for Brownian settings; see Chapter 2 and the references in Section 2.4.2. In more general situations, Becherer [5] and Mania and Schweizer [44] derive a backward stochastic differential equation (BSDE) for the indifference value process. While [44] assumes a continuous filtration, the framework in [5] has a continuous price process driven by Brownian motions and a filtration generated by these and a random measure allowing the modeling of nonpredictable events.

The main contribution of this chapter is to extend the above results to a setting where asset prices are given by a *general semimartingale*. We show that the exponential utility indifference value can still be written in a closed-form expression similar to that known for Brownian models, although the structure of this formula is here much less explicit. Independently from that, we establish a BSDE formulation for the dynamic indifference value process. Both of these results are based on a representation of the claim H and on the relationship between a notion of no-arbitrage, the form of the so-called minimal entropy martingale measure, and the indifference value.

As our starting point, we take the work of Biagini and Frittelli [8, 9]. Their results yield a representation of the minimal entropy martingale measure which we can use to derive a decomposition of a fixed payoff H in a similar way as in Becherer [4]. We call this decomposition, which is closely related to the minimal entropy martingale measure, the *fundamental entropy representation of H* $\big(FER(H)\big)$. It is central to all our results here, because we can express the indifference value for H as a difference of terms from $FER(H)$ and $FER(0)$. We derive from this a fairly explicit formula for the indifference value by an interpolation argument, and we also establish a BSDE representation for the indifference value process. Its proof is based on the idea that the two representations $FER(H)$ and $FER(0)$ can be merged to yield a single BSDE. This direct procedure allows us to work with a general semimartingale, whereas Becherer [5] as well as Mania and Schweizer [44] use more specific models because they first prove some results for more general classes of BSDEs and then apply these to derive the particular BSDE for the indifference value. The price to pay for working in our general setting is that we must restrict the class of solutions of the BSDE to get uniqueness. Under additional assumptions, the components of the solution to the BSDE for the indifference value are again BMO-martingales for the minimal entropy martingale measure; this applies in particular to the value process of the indifference hedging strategy.

This chapter is organised as follows. Section 3.2 lays out the model, motivates, and introduces the important notion of $FER(H)$. In Section 3.3, we prove that the existence of $FER(H)$ is essentially equivalent to an absence-of-arbitrage condition. Moreover, we develop a uniqueness result for $FER(H)$ and its relationship to the minimal entropy martingale measure. Section 3.4 establishes the link between the exponential indifference value of H and the two decompositions $FER(H)$ and $FER(0)$. By an interpolation argument, we derive a fairly explicit formula for the indifference value. In Section 3.5, we extend to a general filtration the BSDE representation of the indifference value by Becherer [5] and Mania and Schweizer [44]. We further provide conditions under which the components of the solution to the BSDE are BMO-martingales for the minimal entropy martingale measure. Section 3.6 gives an application to a Brownian model, and Appendices A and B contain additional results on the indifference value in specific settings. These appendices are not part of the article [24].

3.2 Motivation and definition of $FER(H)$

We start with a probability space $(\Omega, \mathcal{F}, P)$, a finite time interval $[0, T]$ for a fixed $T > 0$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness. For simplicity, we assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. For a positive process Z, we use the abbreviation $Z_{t,s} := Z_s/Z_t$, $0 \le t \le s \le T$.

In our financial market, there are d risky assets whose price process $S = (S_t)_{0\le t\le T}$ is an $\mathbb{R}^d$-valued semimartingale. In addition, there is a riskless asset, chosen as numeraire, whose price is constant at 1. Our investor's risk preferences are given by an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. We always consider a fixed contingent claim H which is a real-valued $\mathcal{F}$-measurable random variable satisfying $E_P\big[\exp(\gamma H)\big] < \infty$. Expressions depending on H are introduced with an index H so we can later use them also in the absence of the claim by setting $H = 0$. However, the dependence on γ is not explicitly mentioned. We define a probability measure P_H on $(\Omega, \mathcal{F})$ equivalent to P by $\frac{\mathrm{d}P_H}{\mathrm{d}P} := \exp(\gamma H)\big/E_P\big[\exp(\gamma H)\big]$. Note that $P_0 = P$. We denote by $L(S)$ the set of all $\mathbb{R}^d$-valued predictable S-integrable processes, so that $\int \vartheta \,\mathrm{d}S$ is a well-defined semimartingale for each ϑ in $L(S)$.

We always impose without further mention the following **standing assumption**, introduced by Biagini and Frittelli [8, 9] for $H = 0$. We assume that

$$\mathcal{W}_H \neq \emptyset \quad \text{and} \quad \mathcal{W}_0 \neq \emptyset, \tag{3.1}$$

where $\mathcal{W}_H$ is the set of *loss variables* w which satisfy the following two conditions:

1) $w \ge 1$ P-a.s., and for every $i = 1, \dots, d$, there exists some $\beta^i \in L(S^i)$ such that $P\big[\exists\, t \in [0,T] \text{ s.t. } \beta^i_t = 0\big] = 0$ and $\big|\int_0^t \beta^i_s \,\mathrm{d}S^i_s\big| \le w$ for all $t \in [0,T]$;

2) $E_{P_H}\big[\exp(cw)\big] < \infty$ for all $c > 0$.

Clearly, $\mathcal{W}_H = \mathcal{W}_0$ if H is bounded. Lemma 3.4 at the beginning of Section 3.3 gives a less restrictive condition on H for $\mathcal{W}_H = \mathcal{W}_0$. The standing assumption (3.1) is automatically fulfilled if S is locally bounded since then $1 \in \mathcal{W}_H \cap \mathcal{W}_0$ by Proposition 1 of Biagini and Frittelli [8], using $P_H \approx P$. But (3.1) is for example also satisfied if H is bounded and $S = S^1$ is a scalar compound Poisson process with Gaussian jumps. This follows from Section 3.2 in Biagini and Frittelli [8]. So the model with condition (3.1) is a genuine generalisation of the case of a locally bounded S.

To assign to H at time $t \in [0,T]$ a value based on our exponential utility function, we first fix an $\mathcal{F}_t$-measurable random variable x_t, interpreted as the investor's starting capital at time t. Then we define

$$V_t^H(x_t) := \operatorname*{ess\,sup}_{\vartheta \in \mathcal{A}_t^H} E_P\bigg[-\exp\bigg(-\gamma x_t - \gamma \int_t^T \vartheta_s \,\mathrm{d}S_s + \gamma H\bigg)\bigg|\mathcal{F}_t\bigg], \tag{3.2}$$

where the set $\mathcal{A}_t^H$ of H-*admissible strategies* on $]t,T]$ consists of all processes $\vartheta \mathbb{1}_{]\!]t,T]\!]}$ with $\vartheta \in L(S)$ and such that $\int \vartheta \, \mathrm{d}S$ is a Q-supermartingale for every $Q \in \mathbb{P}_H^{e,f}$; the set $\mathbb{P}_H^{e,f}$ is defined in the paragraph after the next. We recall that $x_t + \int_t^T \vartheta_s \, \mathrm{d}S_s$ is the investor's final wealth when starting with x_t and investing according to the self-financing strategy ϑ over $]t,T]$. Therefore, $V_t^H(x_t)$ is the maximal conditional expected utility the investor can achieve from the time t initial capital x_t by trading during $]t,T]$ and paying out H (or receiving $-H$) at the maturity T.

The time t *indifference (seller) value* $h_t(x_t)$ for H is implicitly defined by

$$V_t^0(x_t) = V_t^H\big(x_t + h_t(x_t)\big). \tag{3.3}$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with initial capital $x_t + h_t(x_t)$ but paying an additional cash-flow H at maturity T.

To define our strategies, we need the sets

$$\mathbb{P}_H^f := \big\{Q \ll P_H \,\big|\, I(Q|P_H) < \infty \text{ and } S \text{ is a } Q\text{-sigma-martingale}\big\},$$
$$\mathbb{P}_H^{e,f} := \big\{Q \approx P_H \,\big|\, I(Q|P_H) < \infty \text{ and } S \text{ is a } Q\text{-sigma-martingale}\big\},$$

where

$$I(Q|P_H) := \begin{cases} E_Q\Big[\log \frac{\mathrm{d}Q}{\mathrm{d}P_H}\Big] & \text{if } Q \ll P_H \\ +\infty & \text{otherwise} \end{cases}$$

denotes the relative entropy of Q with respect to P_H. Since P_H is equivalent to P, the sets $\mathbb{P}_H^f$ and $\mathbb{P}_H^{e,f}$ depend on H only through the condition $I(Q|P_H) < \infty$. By Proposition 3 and Remark 3 of Biagini and Frittelli [8], applied to P_H instead of P, there exists a unique $Q_H^E \in \mathbb{P}_H^f$ that minimises $I(Q|P_H)$ over all $Q \in \mathbb{P}_H^f$, provided of course that $\mathbb{P}_H^f \neq \emptyset$. We call Q_H^E the *minimal H-entropy measure*, or H-MEM for short. If $\mathbb{P}_H^{e,f} \neq \emptyset$, then Q_H^E is even equivalent to P_H, i.e., $Q_H^E \in \mathbb{P}_H^{e,f}$; see Remark 2 of Biagini and Frittelli [8]. Note that the proper terminology would be "minimal H-entropy sigma-martingale measure" or H-MEσMM, but this is too long.

We briefly recall the relation between Q_H^E, Q_0^E and the indifference value $h_0(x_0)$ at time 0 to motivate the definition of $FER(H)$, which we introduce later in this section. Assume $\mathbb{P}_H^{e,f} \neq \emptyset$ and $\mathbb{P}_0^{e,f} \neq \emptyset$. The P_H-density of Q_H^E and the P-density of Q_0^E have the form

$$\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H} = c^H \exp\bigg(\int_0^T \zeta_s^H \, \mathrm{d}S_s\bigg) \quad \text{and} \quad \frac{\mathrm{d}Q_0^E}{\mathrm{d}P_0} = c^0 \exp\bigg(\int_0^T \zeta_s^0 \, \mathrm{d}S_s\bigg) \tag{3.4}$$

for some positive constants c^H, c^0 and processes ζ^H, ζ^0 in $L(S)$ such that $\int \zeta^H \, \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$ and $\int \zeta^0 \, \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^f$, whence $\zeta^H \in \mathcal{A}_0^H$ and $\zeta^0 \in \mathcal{A}_0^0$. This was first shown by Kabanov and Stricker [38] in their Theorem 2.1 for a locally bounded S (and $H = 0$), and extended by Biagini and Frittelli [9] in their Theorem 1.4 to a general S for $H = 0$ (under the assumption $\mathcal{W}_0 \neq \emptyset$). By using this result also under P_H instead of P, we immediately obtain (3.4). It is now straightforward to calculate (and also well known — at least for locally bounded S) that for $x_0 \in \mathbb{R}$, we can write

$$
\begin{aligned}
V_0^H(x_0) &= \sup_{\vartheta \in \mathcal{A}_0^H} E_P\left[-\exp\left(-\gamma x_0 - \gamma \int_0^T \vartheta_s \,\mathrm{d}S_s + \gamma H\right)\right] \\
&= -\mathrm{e}^{-\gamma x_0} E_P[\exp(\gamma H)] \inf_{\vartheta \in \mathcal{A}_0^H} E_{P_H}\left[\exp\left(-\gamma \int_0^T \vartheta_s \,\mathrm{d}S_s\right)\right] \\
&= -\mathrm{e}^{-\gamma x_0} E_P[\exp(\gamma H)] \inf_{\vartheta \in \mathcal{A}_0^H} E_{Q_H^E}\left[\frac{1}{c^H} \exp\left(\int_0^T \left(-\gamma\vartheta_s - \zeta_s^H\right) \mathrm{d}S_s\right)\right] \\
&= -\frac{\mathrm{e}^{-\gamma x_0} E_P[\exp(\gamma H)]}{c^H} \qquad (3.5)
\end{aligned}
$$

and therefore

$$
h_0(x_0) = h_0 = \frac{1}{\gamma} \log \frac{c^0 E_P[\exp(\gamma H)]}{c^H}. \qquad (3.6)
$$

In Section 3.4, we study the relation between Q_H^E, Q_0^E and $V_t^H(x_t)$, h_t for arbitrary $t \in [0,T]$. From this we can derive, on the one hand, an interpolation formula for each h_t in Section 3.4 and, on the other hand, a BSDE characterisation of the process h in Section 3.5. To generalise the static relations (3.5), (3.6) to dynamic ones, we introduce a certain representation of H that we call *fundamental entropy representation of* H $\big(FER(H)\big)$. Its link to the minimal H-entropy measure is elaborated in the next section. We give two different versions of this representation. The idea is that the first definition only requires some minimal conditions, whereas the second strengthens the conditions to guarantee uniqueness of the representation and ensure the identification of the H-MEM; see Proposition 3.6.

Definition 3.1. We say that $FER(H)$ *exists* if there is a decomposition

$$
H = \frac{1}{\gamma} \log \mathcal{E}\big(N^H\big)_T + \int_0^T \eta_s^H \,\mathrm{d}S_s + k_0^H, \qquad (3.7)
$$

where

(i) N^H is a local P-martingale null at 0 such that $\mathcal{E}\big(N^H\big)$ is a positive P-martingale and S is a $P\big(N^H\big)$-sigma-martingale, where $P\big(N^H\big)$ is defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathcal{E}\big(N^H\big)_T$;

(ii) η^H is in $L(S)$ and such that $\int_0^T \eta^H_s \,\mathrm{d}S_s \in L^1\big(P\big(N^H\big)\big)$;

(iii) $k^H_0 \in \mathbb{R}$ is constant.

In this case, we say that $\big(N^H, \eta^H, k^H_0\big)$ is an $FER(H)$. If moreover

$$\begin{gathered} \int_0^T \eta^H_s \,\mathrm{d}S_s \in L^1(Q) \quad \text{and} \quad E_Q\bigg[\int_0^T \eta^H_s \,\mathrm{d}S_s\bigg] \le 0 \quad \text{for all } Q \in \mathbb{P}^f_H \\ \text{and } \int \eta^H \,\mathrm{d}S \text{ is a } P\big(N^H\big)\text{-martingale,} \end{gathered} \tag{3.8}$$

we say that $\big(N^H, \eta^H, k^H_0\big)$ is an $FER^\star(H)$. For any $FER(H)$ $\big(N^H, \eta^H, k^H_0\big)$, we set

$$k^H_t := k^H_0 + \frac{1}{\gamma}\log \mathcal{E}\big(N^H\big)_t + \int_0^t \eta^H_s \,\mathrm{d}S_s \quad \text{for } t \in [0,T] \tag{3.9}$$

and call $P\big(N^H\big)$ the probability measure *associated with* $\big(N^H, \eta^H, k^H_0\big)$.

Because $\mathcal{E}\big(N^H\big)$ is by (i) a positive P-martingale, the local P-martingale N^H has no negative jumps whose absolute value is 1 or more, and $P\big(N^H\big)$ is a probability measure equivalent to P. We consider two $FER(H)$ $\big(N^H, \eta^H, k^H_0\big)$ and $\big(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}^H_0\big)$ as equal if $\widetilde{N}^H$ and N^H are versions of each other (hence indistinguishable, since both are RCLL), $\int \widetilde{\eta}^H \,\mathrm{d}S$ is a version of $\int \eta^H \,\mathrm{d}S$, and $\widetilde{k}^H_0 = k^H_0$. For future use, we note that (3.7) and (3.9) combine to give

$$H = k^H_t + \frac{1}{\gamma}\log \mathcal{E}\big(N^H\big)_{t,T} + \int_t^T \eta^H_s \,\mathrm{d}S_s \quad \text{for } t \in [0,T]. \tag{3.10}$$

The next result shows that for continuous asset prices, we can write $FER(H)$ in a different (and perhaps more familiar) form. For its formulation, we need the following definition. We say that S satisfies the *structure condition (SC)* if

$$S^i = S^i_0 + M^i + \sum_{j=1}^d \int \lambda^j \,\mathrm{d}\langle M^i, M^j\rangle, \quad i = 1, \dots, d,$$

where M is a locally square-integrable local P-martingale null at 0 and λ is a predictable process such that the (final value of the) mean-variance tradeoff, $K_T = \sum_{i,j=1}^d \int_0^T \lambda^i_s \lambda^j_s \,\mathrm{d}\langle M^i, M^j\rangle_s = \langle \int \lambda \,\mathrm{d}M\rangle_T$, is almost surely finite.

Proposition 3.2. *Assume that S is continuous. Then a triple $\big(N^H, \eta^H, k^H_0\big)$ is an $FER(H)$ if and only if S satisfies (SC) and $\widetilde{N}^H = N^H + \int \lambda \,\mathrm{d}M$, $\widetilde{\eta}^H = \eta^H - \frac{1}{\gamma}\lambda$, $\widetilde{k}^H_0 = k^H_0$ satisfy*

$$H = \frac{1}{\gamma}\log \mathcal{E}\big(\widetilde{N}^H\big)_T + \int_0^T \widetilde{\eta}^H_s \,\mathrm{d}S_s + \frac{1}{2\gamma}\bigg\langle \int \lambda \,\mathrm{d}M \bigg\rangle_T + \widetilde{k}^H_0 \tag{3.11}$$

and

(i′) $\widetilde{N}^H$ *is a local* P*-martingale null at 0 and strongly* P*-orthogonal to each component of* M*, and* $\mathcal{E}\big(\widetilde{N}^H\big)\mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)$ *is a positive* P*-martingale;*

(ii′) $\widetilde{\eta}^H$ *is in* $L(S)$ *and such that* $\int_0^T \big(\widetilde{\eta}^H_s + \frac{1}{\gamma}\lambda_s\big)\,\mathrm{d}S_s$ *is* $P\big(N^H\big)$*-integrable, where* $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathcal{E}\big(\widetilde{N}^H\big)_T\mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)_T$*;*

(iii′) $\widetilde{k}^H_0 \in \mathbb{R}$ *is constant.*

Proof. Let first $\big(N^H, \eta^H, k^H_0\big)$ be an $FER(H)$. Its associated measure $P\big(N^H\big)$ is equivalent to P and S is a local $P\big(N^H\big)$-martingale since S is continuous. By Theorem 1 of Schweizer [52], S satisfies (SC) and we can write $N^H = \widetilde{N}^H - \int \lambda\,\mathrm{d}M$, where $\widetilde{N}^H$ is a local P-martingale null at 0 and strongly P-orthogonal to each component of M, and $\mathcal{E}\big(N^H\big) = \mathcal{E}\big(\widetilde{N}^H\big)\mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)$. The last equality uses that $\big[\widetilde{N}^H, \int \lambda\,\mathrm{d}M\big] = 0$ due to the continuity of M. Hence conditions (i)–(iii) of $FER(H)$ imply (i′)–(iii′), and (3.7) is equivalent to (3.11) by (SC) and the continuity of S.

Conversely, let $\big(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}^H_0\big)$ be as in the proposition. We claim that the triple $\big(\widetilde{N}^H - \int \lambda\,\mathrm{d}M, \widetilde{\eta}^H + \frac{1}{\gamma}\lambda, \widetilde{k}^H_0\big)$ is an $FER(H)$. Because M is a local P-martingale and $\mathcal{E}\big(N^H\big) = \mathcal{E}\big(\widetilde{N}^H\big)\mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)$ is the P-density process of $P\big(N^H\big)$, the process L defined by

$$L_t := M_t - \big\langle N^H, M\big\rangle_t, \quad t \in [0,T]$$

is a local $P\big(N^H\big)$-martingale by Girsanov's theorem; see for instance Theorem III.40 of Protter [49] and observe that $\big\langle \mathcal{E}\big(N^H\big), M\big\rangle = \int \mathcal{E}\big(N^H\big)_-\,\mathrm{d}\big\langle N^H, M\big\rangle$ exists since M is continuous like S. Because $\widetilde{N}^H$ is strongly P-orthogonal to each component of M and M is continuous, we have

$$\big\langle N^H, M^i\big\rangle = \left\langle \widetilde{N}^H - \int \lambda\,\mathrm{d}M, M^i\right\rangle = -\sum_{j=1}^d \int \lambda^j\,\mathrm{d}\langle M^j, M^i\rangle, \quad i = 1, \dots, d,$$

and so (SC) shows that $S = L + S_0$ is also a local $P\big(N^H\big)$-martingale. The other conditions of $FER(H)$ are easy to check. □

Remark 3.3. 1) Suppose that S is continuous and satisfies (SC). If the stochastic exponential $\mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)$ is a P-martingale, conditions (i′) and (ii′) in Proposition 3.2 can be written under the probability measure $\hat{P}$ defined by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\big(-\int \lambda\,\mathrm{d}M\big)_T$, which is called the *minimal local martingale measure* in the terminology of Föllmer and Schweizer [20]. This means that condition (i′) in Proposition 3.2 is equivalent to (i″) $\widetilde{N}^H$ is a local $\hat{P}$-martingale null at 0 and strongly $\hat{P}$-orthogonal to each component of S, and $\mathcal{E}\big(\widetilde{N}^H\big)$ is a positive $\hat{P}$-martingale,

and $P\big(N^H\big)$ can be defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}\hat{P}} := \mathcal{E}\big(\widetilde{N}^H\big)_T$. To prove the equivalence of (i′) and (i″), first assume that $\widetilde{N}^H$ is a local P-martingale null at 0 and strongly P-orthogonal to each M^i. Then

$$\left[\widetilde{N}^H, \int \lambda \,\mathrm{d}M\right] = \left\langle \widetilde{N}^H, \int \lambda \,\mathrm{d}M \right\rangle = 0$$

by the continuity of M, and hence $\widetilde{N}^H$ is also a local $\hat{P}$-martingale by Girsanov's theorem; see, for instance, Theorem III.40 of Protter [49]. The continuity of S, (SC) and the strong P-orthogonality of $\widetilde{N}^H$ to M entail

$$\left[\widetilde{N}^H, S^i\right] = \left\langle \widetilde{N}^H, M^i \right\rangle = 0, \quad i = 1, \ldots, d,$$

implying that $\widetilde{N}^H$ is strongly $\hat{P}$-orthogonal to each component of S. The proof of "(i″) $\Longrightarrow$ (i′)" goes analogously.

2) Assume that S is not necessarily continuous but locally bounded and satisfies (SC) with $\lambda^i \in L^2_{loc}\big(M^i\big)$, $i = 1, \ldots, d$, and let $\big(N^H, \eta^H, k_0^H\big)$ be an $FER(H)$. Then we can still write $N^H = \widetilde{N}^H - \int \lambda \,\mathrm{d}M$ for a local P-martingale $\widetilde{N}^H$ null at 0 and strongly P-orthogonal to each component of M, by using Girsanov's theorem, (SC) and the fact that $\mathcal{E}\big(N^H\big)$ defines an equivalent local martingale measure. However, we cannot separate $\mathcal{E}\big(\widetilde{N}^H - \int \lambda \,\mathrm{d}M\big)$ into two factors. $\diamondsuit$

3.3 No-arbitrage and existence of $FER(H)$

Theorem 3.5 below says that a certain notion of no-arbitrage is equivalent to the existence of $FER(H)$. It can be considered as an exponential analogue to the L^2-result of Theorem 3 in Bobrovnytska and Schweizer [10]. For a locally bounded S, the implication "$\Longrightarrow$" roughly corresponds to Proposition 2.2 of Becherer [4], who makes use of the idea to consider known results under P_H instead of P. This technique, which already appears in Delbaen et al. [16], will also be central for the proofs of our Theorem 3.5 and Proposition 3.6.

We start with a result that gives sufficient conditions for $\mathcal{W}_H \subseteq \mathcal{W}_0$ and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$ as well as for $\mathcal{W}_0 = \mathcal{W}_H$ and $\mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$. The relation between $\mathbb{P}_0^{e,f}$ and $\mathbb{P}_H^{e,f}$ will be used later, while $\mathcal{W}_0 = \mathcal{W}_H$ is helpful in applications to verify the condition (3.1).

Lemma 3.4. *If H satisfies*

$$E_P\big[\exp(-\varepsilon H)\big] < \infty \quad \textit{for some } \varepsilon > 0, \tag{3.12}$$

then $\mathcal{W}_H \subseteq \mathcal{W}_0$, $\mathbb{P}_0^f \subseteq \mathbb{P}_H^f$ and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$. If H satisfies

$$E_P\big[\exp\big((\gamma+\varepsilon)H\big)\big] < \infty \quad \textit{and} \quad E_P\big[\exp(-\varepsilon H)\big] < \infty \quad \textit{for some } \varepsilon > 0, \tag{3.13}$$

then $\mathcal{W}_0 = \mathcal{W}_H$, $\mathbb{P}_0^f = \mathbb{P}_H^f$ and $\mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$.

Proof. We first show $\mathcal{W}_H \subseteq \mathcal{W}_0$ under (3.12). For $c > 0$, Hölder's inequality yields

$$\begin{aligned}
&E_P\big[\exp(cw)\big] \\
&= E_P\left[\exp\left(cw + \frac{\varepsilon\gamma}{\epsilon+\gamma}H\right)\exp\left(-\frac{\varepsilon\gamma}{\varepsilon+\gamma}H\right)\right] \\
&\le \left(E_P\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cw + \gamma H\right)\right]\right)^{\frac{\varepsilon}{\epsilon+\gamma}}\Big(E_P\big[\exp(-\varepsilon H)\big]\Big)^{\frac{\gamma}{\varepsilon+\gamma}} \\
&= \left(E_{P_H}\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cw\right)\right]E_P\big[\exp(\gamma H)\big]\right)^{\frac{\varepsilon}{\varepsilon+\gamma}}\Big(E_P\big[\exp(-\varepsilon H)\big]\Big)^{\frac{\gamma}{\varepsilon+\gamma}}.
\end{aligned} \tag{3.14}$$

Because of $E_P\big[\exp(\gamma H)\big] < \infty$ and (3.12), this is finite if $w \in \mathcal{W}_H$, and then $w \in \mathcal{W}_0$.

To prove $\mathcal{W}_0 = \mathcal{W}_H$ under (3.13), we only need to show $\mathcal{W}_0 \subseteq \mathcal{W}_H$. For $c > 0$ and $w \in \mathcal{W}_0$, we obtain similarly to (3.14) that

$$E_{P_H}\big[\exp(cw)\big] \le \frac{\Big(E_P\big[\exp\big((\varepsilon+\gamma)H\big)\big]\Big)^{\frac{\gamma}{\varepsilon+\gamma}}}{E_P[\exp(\gamma H)]}\left(E_P\left[\exp\left(\frac{\varepsilon+\gamma}{\varepsilon}cw\right)\right]\right)^{\frac{\varepsilon}{\epsilon+\gamma}} < \infty$$

by (3.13), and hence $w \in \mathcal{W}_H$.

The remainder of the second part follows from Lemma A.1 in Becherer [4]. The proof of the rest of the first part is very similar. Indeed, (3.12) and the standing assumption that $E_P\big[\exp(\gamma H)\big] < \infty$ imply $E_P\big[\exp\big(\tilde{\varepsilon}|H|\big)\big] < \infty$, where $\tilde{\varepsilon} := \min(\varepsilon, \gamma)$. Lemma 3.5 of Delbaen et al. [16] yields

$$E_Q\big[\tilde{\varepsilon}|H|\big] \le I(Q|P) + \frac{1}{\mathrm{e}}E_P\big[\exp\big(\tilde{\varepsilon}|H|\big)\big] \quad \text{for } Q \ll P. \tag{3.15}$$

If $Q \in \mathbb{P}_0^f$, the right-hand side is finite, thus $E_Q\big[|H|\big] < \infty$, and we have

$$I(Q|P_H) = E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}P} - \log\frac{\mathrm{d}P_H}{\mathrm{d}P}\right] = I(Q|P) + \log E_P\big[\exp(\gamma H)\big] - \gamma E_Q[H],$$

which is finite. This shows $Q \in \mathbb{P}_H^f$, and $\mathbb{P}_0^{e,f} \subseteq \mathbb{P}_H^{e,f}$ follows analogously. $\square$

Theorem 3.5. *We have that*

$$\mathbb{P}^{e,f}_H \neq \emptyset \iff FER^\star(H) \text{ exists} \iff FER(H) \text{ exists.}$$

In particular, if $\mathbb{P}^{e,f}_0 \neq \emptyset$ *and* H *satisfies (3.12), then* $FER^\star(H)$ *exists.*

Proof. We first show that $\mathbb{P}^{e,f}_H \neq \emptyset$ yields the existence of $FER^\star(H)$. As already mentioned, $\mathbb{P}^{e,f}_H \neq \emptyset$ (and the standing assumption $\mathcal{W}_H \neq \emptyset$) imply by Proposition 3 and Remarks 2, 3 of Biagini and Frittelli [8], applied to P_H instead of P, existence and uniqueness of the H-MEM $Q^E_H \in \mathbb{P}^{e,f}_H$. Using $Q^E_H \approx P_H \approx P$, we can write

$$\frac{\mathrm{d}Q^E_H}{\mathrm{d}P} = \mathcal{E}\big(N^H\big)_T \tag{3.16}$$

for some local P-martingale N^H null at 0 such that $\mathcal{E}\big(N^H\big)$ is a positive P-martingale and S is a Q^E_H-sigma-martingale. Moreover, by Theorem 1.4 of Biagini and Frittelli [9], applied to P_H instead of P, we have as in (3.4)

$$\frac{\mathrm{d}Q^E_H}{\mathrm{d}P_H} = c^H \exp\bigg(\int_0^T \zeta^H_s \,\mathrm{d}S_s\bigg) \tag{3.17}$$

for a constant $c^H > 0$ and some ζ^H in $L(S)$ such that $\int \zeta^H \,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}^f_H$. Since $\frac{\mathrm{d}P_H}{\mathrm{d}P} = \exp(\gamma H)/E_P\big[\exp(\gamma H)\big]$, comparing (3.17) with (3.16) gives

$$\mathcal{E}\big(N^H\big)_T = c^H_1 \exp\bigg(\int_0^T \zeta^H_s \,\mathrm{d}S_s + \gamma H\bigg),$$

where $c^H_1 := c^H/E_P\big[\exp(\gamma H)\big]$ is a positive constant. We thus obtain

$$H = \frac{1}{\gamma}\log \mathcal{E}\big(N^H\big)_T - \frac{1}{\gamma}\int_0^T \zeta^H_s \,\mathrm{d}S_s + c^H_2 \quad \text{with } c^H_2 := -\frac{1}{\gamma}\log c^H_1,$$

and hence $\big(N^H, -\frac{1}{\gamma}\zeta^H, c^H_2\big)$ is an $FER^\star(H)$. Note that $\int \zeta^H \,\mathrm{d}S$ is a $P\big(N^H\big)$-martingale because the H-MEM Q^E_H equals the probability measure $P\big(N^H\big)$ associated with $\big(N^H, -\frac{1}{\gamma}\zeta^H, c^H_2\big)$ by construction; compare (3.16).

To establish the equivalences of Theorem 3.5, it remains to show that the existence of $FER(H)$ implies $\mathbb{P}^{e,f}_H \neq \emptyset$, because every $FER^\star(H)$ is obviously an $FER(H)$. So let $\big(N^H, \eta^H, k^H_0\big)$ be an $FER(H)$ and recall that its associated measure $P\big(N^H\big)$ is defined by $\frac{\mathrm{d}P(N^H)}{\mathrm{d}P} := \mathcal{E}\big(N^H\big)_T$. We prove that $P\big(N^H\big) \in \mathbb{P}^{e,f}_H$. By condition (i) on $FER(H)$, $P\big(N^H\big)$ is a probability measure equivalent to P and S is a $P\big(N^H\big)$-sigma-martingale. To show that $P\big(N^H\big)$ has finite relative entropy with respect to P_H, we write

$$\begin{aligned}\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H} &= \frac{\mathrm{d}P(N^H)}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}P_H} = \mathcal{E}\big(N^H\big)_T \exp(-\gamma H)E_P\big[\exp(\gamma H)\big] \\ &= \exp\big(-\gamma k_0^H\big)E_P\big[\exp(\gamma H)\big]\exp\bigg(-\gamma\int_0^T \eta_s^H \,\mathrm{d}S_s\bigg), \qquad (3.18)\end{aligned}$$

where the last equality is due to the decomposition (3.7) in $FER(H)$. This yields by (ii) of $FER(H)$ that

$$\begin{aligned}I\Big(P\big(N^H\big)\Big|P_H\Big) &= E_{P(N^H)}\bigg[\log\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H}\bigg] \\ &= -\gamma k_0^H + \log E_P\big[\exp(\gamma H)\big] - \gamma E_{P(N^H)}\bigg[\int_0^T \eta_s^H\,\mathrm{d}S_s\bigg] \\ &< \infty.\end{aligned}$$

Finally, the last assertion follows directly from the first part of Lemma 3.4. □

While the *existence* of $FER(H)$ and of $FER^\star(H)$ is equivalent by Theorem 3.5, the two representations are obviously different since $FER^\star(H)$ imposes more stringent conditions. The next result serves to clarify this difference.

Proposition 3.6. *Assume $\mathbb{P}_H^{e,f} \neq \emptyset$ and let $\big(N^H,\eta^H,k_0^H\big)$ be an $FER(H)$ with associated measure $P\big(N^H\big)$. Then the following are equivalent:*

(a) $\big(N^H,\eta^H,k_0^H\big)$ is an $FER^\star(H)$, i.e., $\big(N^H,\eta^H,k_0^H\big)$ satisfies (3.8);

(b) $P\big(N^H\big)$ equals the H-MEM Q_H^E, and $\int \eta^H\,\mathrm{d}S$ is a $P\big(N^H\big)$-martingale;

(c) $\int \eta^H\,\mathrm{d}S$ is a Q_H^E-martingale and $E_{P(N^H)}\big[\int_0^T \eta_s^H\,\mathrm{d}S_s\big] = 0$;

(d) $\int \eta^H\,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$.

Moreover, the class of $FER^\star(H)$ consists of a singleton.

Proof. Clearly, (d) implies (a), and also (c) since Q_H^E exists by Proposition 3 of Biagini and Frittelli [8], using $\mathbb{P}_H^{e,f} \neq \emptyset$ and the standing assumption $\mathcal{W}_H \neq \emptyset$. We prove "(a) $\Longrightarrow$ (b)", "(c) $\Longrightarrow$ (b)" and finally "(b) $\Longrightarrow$ (d)". The first implication goes as in the proof of Theorem 2.3 of Frittelli [25], because we have by (3.18) that

$$\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H} = c_3^H \exp\bigg(-\gamma\int_0^T \eta_s^H\,\mathrm{d}S_s\bigg) \quad \text{with } c_3^H := \exp\big(-\gamma k_0^H\big)E_P\big[\exp(\gamma H)\big]. \qquad (3.19)$$

The implication "(c) $\Longrightarrow$ (b)" follows from the first part of the proof of Proposition 3.2 of Grandits and Rheinländer [27], which does not use the assumption that S is locally bounded. To show "(b) $\Longrightarrow$ (d)", note that (b), (3.17) and (3.19) yield

$$c_3^H \exp\biggl(-\gamma \int_0^T \eta_s^H \,\mathrm{d}S_s\biggr) = c^H \exp\biggl(\int_0^T \zeta_s^H \,\mathrm{d}S_s\biggr) \quad P\text{-a.s.}, \tag{3.20}$$

where ζ^H in $L(S)$ is such that $\int \zeta^H \,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$. Taking logarithms and $P(N^H)$-expectations in (3.20), we get $c_3^H = c^H$ since $P(N^H) \in \mathbb{P}_H^{e,f}$ by the proof of Theorem 3.5. Thus $\int_0^T \eta_s^H \,\mathrm{d}S_s = -\frac{1}{\gamma}\int_0^T \zeta_s^H \,\mathrm{d}S_s$ P-a.s. and hence $\int \eta^H \,\mathrm{d}S = -\frac{1}{\gamma}\int \zeta^H \,\mathrm{d}S$ since both $\int \eta^H \,\mathrm{d}S$ and $\int \zeta^H \,\mathrm{d}S$ are $P(N^H)$-martingales. Therefore, $\int \eta^H \,\mathrm{d}S = -\frac{1}{\gamma}\int \zeta^H \,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^f$.

Theorem 3.5 implies the existence of $FER^\star(H)$ because $\mathbb{P}_H^{e,f} \neq \emptyset$. To show uniqueness, let (N^H, η^H, k_0^H) and $(\widetilde{N}^H, \widetilde{\eta}^H, \widetilde{k}_0^H)$ be two $FER^\star(H)$. Since the minimal H-entropy measure is unique by Proposition 3 of Biagini and Frittelli [8], we have from "(a) $\Longrightarrow$ (b)" that

$$\mathcal{E}(N^H)_T = \frac{\mathrm{d}Q_H^E}{\mathrm{d}P} = \mathcal{E}(\widetilde{N}^H)_T.$$

So $\mathcal{E}(\widetilde{N}^H)$ is a version of $\mathcal{E}(N^H)$ since both are P-martingales, and taking stochastic logarithms implies that $\widetilde{N}^H$ is a version of N^H. Similarly, (3.19) and (c) yield

$$-\gamma k_0^H + \log\Bigl(E_P\bigl[\exp(\gamma H)\bigr]\Bigr) = E_{Q_H^E}\biggl[\log\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H}\biggr] = -\gamma \widetilde{k}_0^H + \log\Bigl(E_P\bigl[\exp(\gamma H)\bigr]\Bigr),$$

thus $\widetilde{k}_0^H = k_0^H$, and therefore again from (3.19) that

$$\int_0^T \eta_s^H \,\mathrm{d}S_s = -\frac{1}{\gamma}\log\biggl(\frac{1}{c_3^H}\frac{\mathrm{d}Q_H^E}{\mathrm{d}P_H}\biggr) = \int_0^T \widetilde{\eta}_s^H \,\mathrm{d}S_s.$$

But both $\int \eta^H \,\mathrm{d}S$ and $\int \widetilde{\eta}^H \,\mathrm{d}S$ are Q_H^E-martingales due to (d), and so $\int \widetilde{\eta}^H \,\mathrm{d}S$ is a version of $\int \eta^H \,\mathrm{d}S$. $\square$

Remark 3.7. Exploiting Proposition 3.4 of Grandits and Rheinländer [27], applied to P_H instead of P, gives a sufficient condition for $FER^\star(H)$ by using our Proposition 3.6. Indeed, assume that S is locally bounded and $\mathbb{P}_H^{e,f} \neq \emptyset$. If for an $FER(H)$ (N^H, η^H, k_0^H), $\int \eta^H \,\mathrm{d}S$ is a $BMO\bigl(P(N^H)\bigr)$-martingale and $E_{P_H}\Bigl[\Bigl|\frac{\mathrm{d}P(N^H)}{\mathrm{d}P_H}\Bigr|^{-\varepsilon}\Bigr] < \infty$ for some $\varepsilon > 0$, then (N^H, η^H, k_0^H) is the $FER^\star(H)$.

Another sufficient criterion is obtained from Proposition 3.2 of Rheinländer [50] in view of our Proposition 3.6. Namely, if S is a locally bounded semimartingale and for an $FER(H)$ $\big(N^H, \eta^H, k_0^H\big)$ there exists $\varepsilon > 0$ such that $E_{P_H}\Big[\exp\Big(\varepsilon\big[\int \eta^H \,\mathrm{d}S\big]_T\Big)\Big] < \infty$, then $\big(N^H, \eta^H, k_0^H\big)$ is the $FER^\star(H)$. $\Diamond$

While there is always at most one $FER^\star(H)$ by Proposition 3.6, the next example shows that there may be several $FER(H)$. This also illustrates that the uniqueness for $FER^\star(H)$ is closely related to integrability properties.

Example 3.8. Take two independent P-Brownian motions W and $W^\perp$, denote by $\mathbb{F}$ their P-augmented filtration and choose $d = 1$, $S = W$ and $H \equiv 0$. The MEM Q_0^E then equals P since S is a P-martingale, and $(0, 0, 0)$ is the unique $FER^\star(0)$.

To construct another $FER(0)$, set $N^0 := W^\perp$. Then $\mathcal{E}\big(N^0\big) = \mathcal{E}\big(W^\perp\big)$ is clearly a positive P-martingale strongly P-orthogonal to $S = W$ so that condition (i) in $FER(0)$ holds. Define $P\big(N^0\big)$ by $\frac{\mathrm{d}P(N^0)}{\mathrm{d}P} := \mathcal{E}\big(N^0\big)_T = \mathcal{E}\big(W^\perp\big)_T$ as usual. By Girsanov's theorem, W and $\widetilde{W}_t^\perp := W_t^\perp - t$, $0 \le t \le T$, are then $P\big(N^0\big)$-Brownian motions and we can explicitly compute

$$\begin{aligned} E_P\Big[\log \mathcal{E}\big(N^0\big)_T\Big] &= E_P\big[W_T^\perp - T/2\big] = -T/2, \\ I\Big(P\big(N^0\big)\Big|P\Big) &= E_{P(N^0)}\Big[\log \mathcal{E}\big(N^0\big)_T\Big] = E_{P(N^0)}\Big[\widetilde{W}_T^\perp + T/2\Big] = T/2. \end{aligned} \tag{3.21}$$

This shows that $P\big(N^0\big) \in \mathbb{P}_0^{e,f}$. Since $S = W$ is a P-Brownian motion, Proposition 1 of Emery et al. [19] now yields for every $c \in \mathbb{R}$ a process $\eta^0(c)$ in $L(S)$ such that

$$-\frac{1}{\gamma} \log \mathcal{E}\big(W^\perp\big)_T - c = \int_0^T \eta_s^0(c)\,\mathrm{d}S_s \quad P\text{-a.s.} \tag{3.22}$$

Because $I\big(P\big(N^0\big)\big|P\big) < \infty$, using the inequality $x|\log x| \le x \log x + 2\mathrm{e}^{-1}$ shows that $\int_0^T \eta_s^0(c)\,\mathrm{d}S_s$ is in $L^1\big(P\big(N^0\big)\big)$ so that (ii) of $FER(0)$ is also satisfied. Hence $\big(N^0, \eta^0(c), c\big)$ is an $FER(0)$, but does not coincide with $(0, 0, 0)$ which is the $FER^\star(0)$. To check that property (3.8) indeed fails, we can easily see from (3.21) and (3.22) that $\int \eta^0(c)\,\mathrm{d}S$ cannot be a $P\big(N^0\big)$-martingale if $c \neq -\frac{1}{2\gamma}T$. If $c = -\frac{1}{2\gamma}T$, we can simply compute, for $P \in \mathbb{P}_0^f$, that

$$E_P\left[\int_0^T \eta_s^0(c)\,\mathrm{d}S_s\right] = -\frac{1}{\gamma} E_P\Big[\log \mathcal{E}\big(N^0\big)_T\Big] + \frac{1}{2\gamma}T = \frac{1}{\gamma}T > 0.$$

We have just constructed an $FER(0)$ different from $FER^\star(0)$. Yet another $FER(0)$ can be obtained by choosing for $k \in \mathbb{R}\backslash\{0\}$ a process $\beta^0(k)$ in

$L(S)$ such that

$$\int_0^{T/2} \beta^0_s(k)\,\mathrm{d}S_s = k \quad \text{and} \quad \int_{T/2}^{T} \beta^0_s(k)\,\mathrm{d}S_s = -k \quad P\text{-a.s.},$$

which is possible by Proposition 1 of Emery et al. [19]. Clearly, we have $\int_0^T \beta^0_s(k)\,\mathrm{d}S_s = 0$ P-a.s. and $\big(0, \beta^0(k), 0\big)$ is an $FER(0)$ (with associated measure P), which even satisfies $E_Q\big[\int_0^T \beta^0_s(k)\,\mathrm{d}S_s\big] = 0$ for all $Q \in \mathbb{P}^f_0$; but $\int \beta^0(k)\,\mathrm{d}S$ is not a P-martingale. This ends the example. $\diamondsuit$

Example 3.8 shows that we should focus on $FER^\star(H)$ if we want to obtain good results. If S is continuous and we impose additional assumptions, the next result gives BMO-properties for the components of $FER^\star(H)$. This will be used later when we give a BSDE description for the exponential utility indifference value process. We first recall some definitions.

Let Q be a probability measure on $(\Omega, \mathcal{F})$ equivalent to P and $p > 1$. An adapted positive RCLL stochastic process Z is said to satisfy the *reverse Hölder inequality* $R_p(Q)$ if there exists a positive constant C such that

$$\operatorname*{ess\,sup}_{\substack{\tau \text{ stopping}\\ \text{time}}} E_Q\left[\left(\frac{Z_T}{Z_\tau}\right)^p \middle| \mathcal{F}_\tau\right] = \operatorname*{ess\,sup}_{\substack{\tau \text{ stopping}\\ \text{time}}} E_Q\big[(Z_{\tau,T})^p \big| \mathcal{F}_\tau\big] \le C.$$

Recall that $Z_{\tau,T} = Z_T/Z_\tau$ for a positive process Z. We say that Z satisfies the *reverse Hölder inequality* $R_{L\log L}(Q)$ if there exists a positive constant C such that

$$\operatorname*{ess\,sup}_{\substack{\tau \text{ stopping}\\ \text{time}}} E_Q[Z_{\tau,T} \log^+ Z_{\tau,T} | \mathcal{F}_\tau] \le C.$$

Z satisfies *condition (J)* if there exists a positive constant C such that

$$\frac{1}{C} Z_- \le Z \le C Z_-.$$

Theorem 3.9. *Assume that S is continuous, H is bounded and there exists $Q \in \mathbb{P}^{e,f}_0$ whose P-density process satisfies $R_{L\log L}(P)$. Let $\big(N^H, \eta^H, k^H_0\big)$ be an $FER(H)$. Then the following are equivalent:*

(a) $\big(N^H, \eta^H, k^H_0\big)$ *is the* $FER^\star(H)$*;*

(b) N^H *is a $BMO(P)$-martingale,* $\mathcal{E}\big(N^H\big)$ *satisfies condition (J),* $\int \eta^H\,\mathrm{d}S$ *is a* $P\big(N^H\big)$*-martingale;*

(c) N^H *is a $BMO(P)$-martingale,* $\mathcal{E}\big(N^H\big)$ *satisfies condition (J),* $\int \eta^H\,\mathrm{d}S$ *is a* $BMO\big(P\big(N^H\big)\big)$*-martingale;*

(d) $\int \eta^H \,\mathrm{d}M$ *is a* $BMO(P)$*-martingale, where* M *is the* P*-local martingale part of* S*;*

(e) there exists $\varepsilon > 0$ *such that* $E_P\Big[\exp\Big(\varepsilon\big[\int \eta^H \,\mathrm{d}S\big]_T\Big)\Big] < \infty$.

The hypotheses of Theorem 3.9 are for instance fulfilled if H is bounded, S is continuous and satisfies (SC), and $\int \lambda \,\mathrm{d}M$ is a $BMO(P)$-martingale. To see this, note that $\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)$ then satisfies the reverse Hölder inequality $R_p(P)$ for some $p > 1$ by Theorem 3.4 of Kazamaki [40]. The fact that there exists $k < \infty$ such that $x \log x \leq k + x^p$ for all $x > 0$ now implies that $\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)$ also satisfies $R_{L\log L}(P)$. Hence the minimal local martingale measure $\hat{P}$ given by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)_T$ is in $\mathbb{P}_0^{e,f}$ and its P-density process satisfies $R_{L\log L}(P)$.

Proof of Theorem 3.9. By Lemma 3.4, $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f} \neq \emptyset$ so that there exists an $FER(H)$ $\big(N^H, \eta^H, k_0^H\big)$ by Theorem 3.5. Before we show that (a)–(e) are equivalent, we need some preparation. Let $\tilde{Q}$ be a probability measure equivalent to P. Denoting by Z the P-density process of $\tilde{Q}$ and by Y the P_H-density process of $\tilde{Q}$, we prove that

$$Z \text{ satisfies } R_{L\log L}(P) \text{ if and only if } Y \text{ satisfies } R_{L\log L}(P_H), \tag{3.23}$$

$$Z \text{ satisfies condition (J) if and only if } Y \text{ satisfies condition (J)}. \tag{3.24}$$

To that end, observe first that because H is bounded, there exists a positive constant k with $\frac{1}{k} \leq \frac{\mathrm{d}P_H}{\mathrm{d}P} \leq k$, which yields

$$\frac{1}{k} Z \leq Y \leq kZ. \tag{3.25}$$

For any stopping time τ, (3.25) implies

$$E_{P_H}[Y_{\tau,T} \log^+ Y_{\tau,T} | \mathcal{F}_\tau] \leq E_P\Big[Z_{\tau,T} \log^+\big(Z_{\tau,T} k^2\big)\Big|\mathcal{F}_\tau\Big],$$

and so the inequality $\log^+(ab) \leq \log^+ a + \log b$ for $a > 0$ and $b \geq 1$ yields

$$E_P\Big[Z_{\tau,T} \log^+\big(Z_{\tau,T} k^2\big)\Big|\mathcal{F}_\tau\Big] \leq E_P[Z_{\tau,T} \log^+ Z_{\tau,T} | \mathcal{F}_\tau] + 2\log k,$$

which is bounded independently of τ if Z satisfies $R_{L\log L}(P)$. If Z satisfies condition (J) with constant C, then (3.25) gives

$$Y \leq kZ \leq kCZ_- \leq k^2 C Y_- \quad \text{and} \quad Y \geq \frac{1}{k} Z \geq \frac{1}{kC} Z_- \geq \frac{1}{k^2 C} Y_-.$$

So the "only if" part of both (3.23) and (3.24) is clear, and the "if" part is proved symmetrically.

By assumption, there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$, and so the P_H-density process of Q satisfies $R_{L\log L}(P_H)$ by (3.23). Because $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ is nonempty, the unique minimal H-entropy measure Q_H^E exists, and its P_H-density process also satisfies $R_{L\log L}(P_H)$ by Lemma 3.1 of Delbaen et al. [16], used for P_H instead of P. Since S is continuous, the P_H-density process of Q_H^E also satisfies condition (J) by Lemma 4.6 of Grandits and Rheinländer [27]. It follows from (3.23), (3.24) and Lemma 2.2 of Grandits and Rheinländer [27] that

$$\begin{aligned}&\text{the } P\text{-density process } Z^{Q_H^E,P} \text{ of } Q_H^E \text{ satisfies } R_{L\log L}(P), \text{ condition (J)},\\ &\quad\text{and the stochastic logarithm of } Z^{Q_H^E,P} \text{ is a } BMO(P)\text{-martingale.}\end{aligned} \tag{3.26}$$

"(a) $\Longrightarrow$ (b)". Since (N^H, η^H, k_0^H) is the $FER^\star(H)$, Proposition 3.6 implies that the P-density process $Z^{Q_H^E,P}$ of Q_H^E is given by $\mathcal{E}(N^H)$ and that $\int \eta^H \,\mathrm{d}S$ is a $P(N^H)$-martingale. We deduce (b) from (3.26).

"(b) $\Longrightarrow$ (c)". We have to show that $\int \eta^H \,\mathrm{d}S$ is in $BMO\big(P(N^H)\big)$. By conditioning (3.7) under $P(N^H)$ on $\mathcal{F}_\tau$ for a stopping time τ, we obtain by (b)

$$\int_0^\tau \eta_s^H \,\mathrm{d}S_s = -\frac{1}{\gamma} E_{P(N^H)}\Big[\log \mathcal{E}(N^H)_T \Big| \mathcal{F}_\tau\Big] + E_{P(N^H)}[H|\mathcal{F}_\tau] - k_0^H,$$

and hence

$$\begin{aligned}\int_\tau^T \eta_s^H \,\mathrm{d}S_s = &-\frac{1}{\gamma}\log \mathcal{E}(N^H)_T + \frac{1}{\gamma} E_{P(N^H)}\Big[\log \mathcal{E}(N^H)_T \Big| \mathcal{F}_\tau\Big]\\ &+ H - E_{P(N^H)}[H|\mathcal{F}_\tau].\end{aligned}$$

By Proposition 6 of Doléans-Dade and Meyer [17], there is a $BMO\big(P(N^H)\big)$-martingale $\hat{N}^H$ with $\mathcal{E}(N^H)^{-1} = \mathcal{E}(\hat{N}^H)$. This uses that $Z^{Q_H^E,P} = \mathcal{E}(N^H)$ satisfies condition (J) and N^H is a $BMO(P)$-martingale by (3.26). Since H is bounded, we get

$$\begin{aligned}&E_{P(N^H)}\left[\left|\int_\tau^T \eta_s^H \,\mathrm{d}S_s\right| \middle| \mathcal{F}_\tau\right]\\ &\le 2\|H\|_{L^\infty(P)} + \frac{1}{\gamma} E_{P(N^H)}\left[\left|\log \mathcal{E}(N^H)_T - E_{P(N^H)}\Big[\log \mathcal{E}(N^H)_T \Big| \mathcal{F}_\tau\Big]\right| \middle| \mathcal{F}_\tau\right]\\ &= 2\|H\|_{L^\infty(P)} + \frac{1}{\gamma} E_{P(N^H)}\left[\left|\log \mathcal{E}(\hat{N}^H)_T - E_{P(N^H)}\Big[\log \mathcal{E}(\hat{N}^H)_T \Big| \mathcal{F}_\tau\Big]\right| \middle| \mathcal{F}_\tau\right],\end{aligned} \tag{3.27}$$

and now we proceed like on page 1031 in Grandits and Rheinländer [27] to show that (3.27) is bounded uniformly in τ. This proves the assertion since S is continuous.

"(c) $\Longrightarrow$ (d)". Due to (3.26), Proposition 7 of Doléans-Dade and Meyer [17] implies that $\int \eta^H \,\mathrm{d}S + \left[\int \eta^H \,\mathrm{d}S, N^H\right]$ is a $BMO(P)$-martingale. By Proposition 3.2, S satisfies (SC) and $N^H = \widetilde{N}^H - \int \lambda \,\mathrm{d}M$ for a local P-martingale $\widetilde{N}^H$ null at 0 and strongly P-orthogonal to each component of M. Since S is continuous and satisfies (SC),

$$\left[\int \eta^H \,\mathrm{d}S, N^H\right] = \left[\int \eta^H \,\mathrm{d}M, N^H\right] = -\left[\int \eta^H \,\mathrm{d}M, \int \lambda \,\mathrm{d}M\right]$$
$$= -\sum_{i,j=1}^{d} \int \left(\eta^H\right)^i \lambda^j \,\mathrm{d}\langle M^i, M^j\rangle.$$

Hence $\int \eta^H \,\mathrm{d}S + \left[\int \eta^H \,\mathrm{d}S, N^H\right] = \int \eta^H \,\mathrm{d}M$ is a $BMO(P)$-martingale.

"(d) $\Longrightarrow$ (e)". We set

$$\varepsilon := \frac{1}{2\|\int \eta^H \,\mathrm{d}M\|^2_{BMO_2(P)}} \quad \text{and} \quad L := \sqrt{\epsilon}\int \eta^H \,\mathrm{d}M.$$

Clearly, L is like $\int \eta^H \,\mathrm{d}M$ a continuous $BMO(P)$-martingale and we have that $\|L\|_{BMO_2(P)} = 1/\sqrt{2} < 1$. Since S is continuous, the John-Nirenberg inequality (see Theorem 2.2 of Kazamaki [40]) yields

$$E_P\left[\exp\left(\varepsilon\left[\int \eta^H \,\mathrm{d}S\right]_T\right)\right] = E_P\left[\exp\left([L]_T\right)\right] \le \frac{1}{1 - \|L\|^2_{BMO_2(P)}} < \infty.$$

"(e) $\Longrightarrow$ (a)". This is based on the same idea as the proof of Proposition 3.2 of Rheinländer [50]. Lemma 3.5 of Delbaen et al. [16] yields

$$E_Q\left[\varepsilon\left[\int \eta^H \,\mathrm{d}S\right]_T\right] \le I(Q|P_H) + \frac{1}{\mathrm{e}} E_{P_H}\left[\exp\left(\varepsilon\left[\int \eta^H \,\mathrm{d}S\right]_T\right)\right] < \infty$$

for any $Q \in \mathbb{P}^f_H$ because H is bounded and (e) holds. So $\left[\int \eta^H \,\mathrm{d}S\right]_T$ is Q-integrable and thus the local Q-martingale $\int \eta^H \,\mathrm{d}S$ is a square-integrable Q-martingale for any $Q \in \mathbb{P}^f_H$. This concludes the proof in view of Proposition 3.6. □

3.4 Relating $FER^\star(H)$ and $FER^\star(0)$ to the indifference value

In this section, we establish the connection between $FER^\star(H)$, $FER^\star(0)$ and the indifference value process h. We then derive and study an interpolation formula for h. Throughout this section, we assume that

$$\mathbb{P}_H^{e,f} \neq \emptyset \quad \text{and} \quad \mathbb{P}_0^{e,f} \neq \emptyset,$$

and we denote by $\big(N^H, \eta^H, k_0^H\big)$ and $\big(N^0, \eta^0, k_0^0\big)$ the unique $FER^\star(H)$ and $FER^\star(0)$ with associated measures $P\big(N^H\big) = Q_H^E$ and $P\big(N^0\big) = Q_0^E$, respectively.

Our first result expresses the maximal expected utility and the indifference value in terms of the given $FER^\star(H)$ and $FER^\star(0)$. For a locally bounded S, this is very similar to Becherer [4]; see in particular there Propositions 2.2 and 3.5 and the discussion on page 12 at the end of Section 3. Indeed, the main differences are that the representation in [4] is given in terms of certainty equivalents instead of maximal conditional expected utilities and S is locally bounded; but the results are the same.

Theorem 3.10. *V^H, V^0 and h are well defined and, for any $t \in [0,T]$ and any $\mathcal{F}_t$-measurable random variable x_t, we have*

$$V_t^H(x_t) = -\exp\big(-\gamma x_t + \gamma k_t^H\big) \tag{3.28}$$

and

$$h_t(x_t) = h_t = k_t^H - k_t^0, \tag{3.29}$$

where k_t^H (and k_t^0, with the obvious adaptations) are defined in (3.9).

Proof. Let us first write (3.2) as

$$V_t^H(x_t) = -\exp(-\gamma x_t) \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_t^H} \varphi_t^H(\vartheta) \tag{3.30}$$

with the abbreviation

$$\varphi_t^H(\vartheta) := E_P\bigg[\exp\bigg(-\gamma \int_t^T \vartheta_s \,\mathrm{d}S_s + \gamma H\bigg)\bigg|\mathcal{F}_t\bigg].$$

Because $\big(N^H, \eta^H, k_0^H\big)$ is the $FER^\star(H)$, $\varphi_t^H(\vartheta)$ can be written by (3.10) as

$$\begin{aligned}\varphi_t^H(\vartheta) &= \exp\big(\gamma k_t^H\big) E_P\bigg[\mathcal{E}\big(N^H\big)_{t,T} \exp\bigg(\gamma \int_t^T \big(\eta_s^H - \vartheta_s\big)\,\mathrm{d}S_s\bigg)\bigg|\mathcal{F}_t\bigg] \\ &= \exp\big(\gamma k_t^H\big) E_{P(N^H)}\bigg[\exp\bigg(\gamma \int_t^T \big(\eta_s^H - \vartheta_s\big)\,\mathrm{d}S_s\bigg)\bigg|\mathcal{F}_t\bigg],\end{aligned} \tag{3.31}$$

using Bayes' formula. Since $P\big(N^H\big) = Q_H^E \in \mathbb{P}_H^{e,f}$ and $\int \vartheta \,\mathrm{d}S$ is a Q-supermartingale and $\int \eta^H \,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$, we have

$$E_{P(N^H)}\bigg[\int_t^T \big(\eta_s^H - \vartheta_s\big)\,\mathrm{d}S_s\bigg|\mathcal{F}_t\bigg] \geq 0$$

which implies $\varphi_t^H(\vartheta) \geq \exp\bigl(\gamma k_t^H\bigr)$ by Jensen's inequality and (3.31). On the other hand, the choice

$$\vartheta_s^\star := \eta_s^H, \quad s \in \,]t,T], \tag{3.32}$$

gives $\varphi_t^H(\vartheta^\star) = \exp\bigl(\gamma k_t^H\bigr)$ by (3.31). Because $\int \vartheta^\star \, \mathrm{d}S = \int \eta^H \, \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$, $\vartheta^\star$ is in $\mathcal{A}_t^H$, and (3.28) now follows from (3.30).

By the same reasoning as for (3.28), we obtain

$$V_t^0(x_t) = -\exp\bigl(-\gamma x_t + \gamma k_t^0\bigr).$$

Solving the implicit equation (3.3) for $h_t(x_t)$ then directly leads to (3.29). $\square$

The proof of Theorem 3.10, especially (3.32), gives an interpretation for the $FER^\star(H)$. An investor who must pay out the claim H at time T uses, under exponential utility preferences, the decomposition (3.7). The portion of H that he hedges by trading in S is $\int_0^T \eta_s^H \, \mathrm{d}S_s$, whereas $\frac{1}{\gamma} \log \mathcal{E}\bigl(N^H\bigr)_T$ remains unhedged. Moreover, the proof of Theorem 3.10 shows that for $t \in [0,T]$ and an $\mathcal{F}_t$-measurable x_t, the value of $V_t^H(x_t)$ is not affected if we restrict the set $\mathcal{A}_t^H$ to those $\vartheta \in \mathcal{A}_t^H$ such that $\int \vartheta \, \mathrm{d}S$ is not only a Q-supermartingale, but a Q-martingale for every $Q \in \mathbb{P}_H^{e,f}$.

Proposition 3.11. *Assume that H satisfies (3.12). Then for any $Q \in \mathbb{P}_0^f$ and $t \in [0,T]$,*

$$h_t = E_Q[H|\mathcal{F}_t] - \frac{1}{\gamma} E_Q\left[\log \frac{\mathcal{E}\bigl(N^H\bigr)_{t,T}}{\mathcal{E}\bigl(N^0\bigr)_{t,T}} \middle| \mathcal{F}_t \right]. \tag{3.33}$$

In particular,

$$h_0 = E_Q[H] + \frac{1}{\gamma}\Bigl(I\bigl(Q\big|Q_H^E\bigr) - I\bigl(Q\big|Q_0^E\bigr)\Bigr). \tag{3.34}$$

The decomposition (3.34) of the indifference value h_0 can be described as follows. The first term, $E_Q[H]$, is the expected payoff under a measure $Q \in \mathbb{P}_0^f$. This is linear in the number of claims. The second term is a nonlinear correction term or safety loading. It can be interpreted as the difference of the distances from Q_H^E and Q_0^E to Q $\bigl($although $I(\cdot|\cdot)$ is not a metric$\bigr)$. This correction term is not based on all of H, but only on the processes N^H and N^0 from the $FER^\star(H)$ and $FER^\star(0)$, i.e., on the unhedged parts of H and 0, respectively. A similar decomposition also appears for indifference pricing under quadratic preferences; see Schweizer [53].

If H satisfies (3.12), then the indifference value process h is a Q_0^E-supermartingale. In fact, Jensen's inequality and (3.33) with $Q = Q_0^E$ yield

$h_t \geq E_{Q_0^E}[H|\mathcal{F}_t]$ P-a.s. for $t \in [0,T]$ and so $h_t^- \in L^1\big(Q_0^E\big)$ since H is Q_0^E-integrable due to (3.12); compare (3.15). Moreover, $Z := \mathcal{E}\big(N^H\big)/\mathcal{E}\big(N^0\big)$ is a Q_0^E-martingale as it is the Q_0^E-density process of Q_H^E. Thus $\log Z$ has the Q_0^E-supermartingale property by Jensen's inequality, and so has h since $h_t = E_{Q_0^E}[H|\mathcal{F}_t] - \frac{1}{\gamma}E_{Q_0^E}[\log Z_T|\mathcal{F}_t] + \frac{1}{\gamma}\log Z_t$ for $t \in [0,T]$ by (3.33). Now $E_{Q_0^E}[h_t] \leq h_0 < \infty$ shows that h_t is Q_0^E-integrable for every $t \in [0,T]$.

Proof of Proposition 3.11. Since $Q \in \mathbb{P}_0^f \subseteq \mathbb{P}_H^f$ by Lemma 3.4, $\int \eta^H \,\mathrm{d}S$ is a Q-martingale by Proposition 3.6. Moreover, H is Q-integrable due to (3.12); compare (3.15). From (3.10), we thus obtain for $t \in [0,T]$ that

$$k_t^H = E_Q\left[H - \frac{1}{\gamma}\log \mathcal{E}\big(N^H\big)_{t,T}\middle|\mathcal{F}_t\right]. \tag{3.35}$$

Plugging (3.35) and the analogous expression for k_t^0 into (3.29) leads to (3.33).

To prove (3.34), we first show that $I\big(Q\big|Q_0^E\big)$ is finite. We can write

$$I\big(Q\big|Q_0^E\big) = E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}P} + \log\frac{\mathrm{d}P}{\mathrm{d}Q_0^E}\right] = I(Q|P) - E_Q\left[\log \mathcal{E}\big(N^0\big)_T\right] < \infty \tag{3.36}$$

because $Q \in \mathbb{P}_0^f$ and $-E_Q\big[\log \mathcal{E}\big(N^0\big)_T\big] = \gamma k_0^0$ by (3.35) for $H = 0$ and $t = 0$. Moreover, $Q \ll P \approx Q_H^E$ gives $\frac{\mathrm{d}Q}{\mathrm{d}P} > 0$ Q-a.s. and thus from

$$\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} = \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}Q_H^E} = \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{1}{\mathcal{E}(N^H)_T} \quad Q\text{-a.s.}$$

that

$$-\log \mathcal{E}\big(N^H\big)_T = \log\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} - \log\frac{\mathrm{d}Q}{\mathrm{d}P} \quad Q\text{-a.s.},$$

and analogously for 0 instead of H. Hence

$$E_Q\left[-\log\frac{\mathcal{E}\big(N^H\big)_T}{\mathcal{E}\big(N^0\big)_T}\right] = E_Q\left[\log\frac{\mathrm{d}Q}{\mathrm{d}Q_H^E} - \log\frac{\mathrm{d}Q}{\mathrm{d}Q_0^E}\right] = I\big(Q\big|Q_H^E\big) - I\big(Q\big|Q_0^E\big),$$

where we have used (3.36) for the last equality. Now (3.33) implies (3.34). $\square$

We next come to the announced interpolation formula for the indifference value.

Theorem 3.12. *Let $Q \in \mathbb{P}_H^{e,f}$ and φ in $L(S)$ be such that $\int \varphi \,\mathrm{d}S$ is a Q- and Q_H^E-martingale. Fix $t \in [0,T]$, denote by Z the P-density process of Q, set*

$$\Psi_t^H := \frac{\exp\big(\gamma H + \int_t^T \varphi_s \,\mathrm{d}S_s\big)}{Z_{t,T}} \tag{3.37}$$

and assume that Ψ_t^H and $\log \Psi_t^H$ are Q-integrable. Then there exists an $\mathcal{F}_t$-measurable random variable $\delta_t^H : \Omega \to [1, \infty]$ such that for almost all $\omega \in \Omega$,

$$k_t^H(\omega) = \frac{1}{\gamma} \log\Big(E_Q\Big[\big|\Psi_t^H\big|^{1/\delta} \Big| \mathcal{F}_t \Big](\omega) \Big)^\delta \Big|_{\delta = \delta_t^H(\omega)}, \tag{3.38}$$

where

$$\begin{aligned} \log\Big(E_Q\Big[\big|\Psi_t^H\big|^{1/\delta} \Big| \mathcal{F}_t \Big](\omega) \Big)^\delta \Big|_{\delta=\infty} &:= \lim_{\delta \to \infty} \log\Big(E_Q\Big[\big|\Psi_t^H\big|^{1/\delta} \Big| \mathcal{F}_t \Big](\omega) \Big)^\delta \\ &= E_Q\big[\log \Psi_t^H \big| \mathcal{F}_t \big](\omega) \end{aligned} \tag{3.39}$$

for almost all $\omega \in \Omega$.

In view of $h_t = k_t^H - k_t^0$ by Theorem 3.10, (3.38) gives us a quasi-explicit formula for the exponential utility indifference value if H is bounded and if we can find a measure $Q \in \mathbb{P}_0^{e,f}$ such that the corresponding Ψ_t^0 given in (3.37) and $\log \Psi_t^0$ are Q-integrable for some predictable φ such that $\int \varphi \, \mathrm{d}S$ is a Q-, Q_0^E- and Q_H^E-martingale. For $t = 0$, one possible choice is the minimal 0-entropy measure Q_0^E which is by (3.19) and Proposition 3.6 of the form $\frac{\mathrm{d}Q_0^E}{\mathrm{d}P} = c_3^0 \exp\big(\int_0^T \zeta_s^0 \, \mathrm{d}S_s \big)$ for a constant c_3^0 and a predictable process ζ^0 such that $\int \zeta^0 \, \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^f$. One disadvantage of this choice is that Q_0^E is in general unknown; a second is that we still need to find some φ, and we know almost nothing about the potential candidate ζ^0. In Corollary 3.13, we give conditions under which the explicitly known minimal local martingale measure $\hat{P}$ satisfies the assumptions of Theorem 3.12.

Proof of Theorem 3.12. From (3.10) and (3.37), we obtain via $\frac{\mathrm{d}Q_H^E}{\mathrm{d}P} = \mathcal{E}\big(N^H\big)_T$ and Bayes' formula that

$$\begin{aligned} \exp\big(-\gamma k_t^H\big) E_Q\big[\Psi_t^H \big| \mathcal{F}_t\big] &= E_Q\left[\frac{\mathcal{E}\big(N^H\big)_{t,T}}{Z_{t,T}} \exp\bigg(\int_t^T \big(\varphi_s + \gamma \eta_s^H\big) \, \mathrm{d}S_s \bigg) \bigg| \mathcal{F}_t \right] \\ &= E_{Q_H^E}\left[\exp\bigg(\int_t^T \big(\varphi_s + \gamma \eta_s^H\big) \, \mathrm{d}S_s \bigg) \bigg| \mathcal{F}_t \right] \\ &\geq \exp\bigg(E_{Q_H^E}\bigg[\int_t^T \big(\varphi_s + \gamma \eta_s^H\big) \, \mathrm{d}S_s \bigg| \mathcal{F}_t \bigg] \bigg) \\ &= 1 \end{aligned} \tag{3.40}$$

by Jensen's inequality and because $\int \varphi \, \mathrm{d}S$ and $\int \eta^H \, \mathrm{d}S$ are Q_H^E-martingales. Hence

$$k_t^H \leq \frac{1}{\gamma} \log E_Q\big[\Psi_t^H \big| \mathcal{F}_t\big]. \tag{3.41}$$

On the other hand, (3.35), (3.37) and Jensen's inequality yield

$$\begin{aligned}\gamma k_t^H &= E_Q\Big[\gamma H - \log \mathcal{E}\big(N^H\big)_{t,T}\Big|\mathcal{F}_t\Big]\\ &= E_Q\left[\log \Psi_t^H - \log \frac{\mathcal{E}\big(N^H\big)_{t,T}}{Z_{t,T}}\middle|\mathcal{F}_t\right]\\ &\geq E_Q\big[\log \Psi_t^H\big|\mathcal{F}_t\big]. \end{aligned}\tag{3.42}$$

Consider the stochastic process $f(\cdot,\cdot) : [1,\infty)\times\Omega \to \mathbb{R}$ defined by

$$f(\delta,\omega) := \log\Big(E_Q\Big[\big|\Psi_t^H\big|^{\frac{1}{\delta}}\Big|\mathcal{F}_t\Big](\omega)\Big)^{\delta}, \quad (\delta,\omega)\in[1,\infty)\times\Omega.$$

Because $|\Psi_t^H|^{1/\delta} \leq 1 + \Psi_t^H \in L^1(Q)$ for all $\delta \in [1,\infty)$, Lebesgue's dominated convergence theorem and Jensen's inequality for conditional expectations allow us to choose a version of f which is continuous and nonincreasing in δ for all fixed $\omega \in \Omega$, so that by monotonicity, the limit $f(\infty,\omega) := \lim_{\delta\to\infty} f(\delta,\omega)$ exists for all $\omega\in\Omega$. We next show that

$$f(\infty,\omega) = E_Q\big[\log \Psi_t^H\big|\mathcal{F}_t\big](\omega) \quad \text{for almost all } \omega\in\Omega. \tag{3.43}$$

To ease the notation, we define $g(\cdot,\cdot) : [1,\infty)\times\Omega\to\mathbb{R}$ by

$$g(\delta,\omega) := \Big(\exp\big(f(\delta,\omega)\big)\Big)^{\frac{1}{\delta}} = E_Q\Big[\big|\Psi_t^H\big|^{\frac{1}{\delta}}\Big|\mathcal{F}_t\Big](\omega), \quad (\delta,\omega)\in[1,\infty)\times\Omega$$

so that $f(\delta,\omega) = \delta\log g(\delta,\omega)$. Again since $|\Psi_t^H|^{1/\delta} \leq 1+\Psi_t^H \in L^1(Q)$ for all $\delta\in[1,\infty)$, dominated convergence gives

$$\lim_{n\to\infty} g(n,\omega) = 1 \quad \text{for almost all } \omega\in\Omega. \tag{3.44}$$

For $x > 1/2$ we have $x-1 \geq \log x \geq x-1-|x-1|^2$, from which we obtain by (3.44) that for almost all $\omega\in\Omega$, there exists $n_0(\omega)\in\mathbb{N}$ such that

$$n\big(g(n,\omega)-1\big) \geq f(n,\omega) \geq n\big(g(n,\omega)-1\big) - n\big|g(n,\omega)-1\big|^2, \quad n\geq n_0(\omega). \tag{3.45}$$

In view of (3.44) and (3.45), we get (3.43) if we show that

$$\lim_{n\to\infty} n\big(g(n,\omega)-1\big) = E_Q\big[\log\Psi_t^H\big|\mathcal{F}_t\big](\omega) \quad \text{for almost all } \omega\in\Omega. \tag{3.46}$$

But (3.46) follows from Lebesgue's convergence theorem and

$$\lim_{n\to\infty} n\Big(\big|\Psi_t^H\big|^{\frac{1}{n}} - 1\Big) = \lim_{n\to\infty} n\left(\exp\left(\frac{1}{n}\log\Psi_t^H\right) - 1\right) = \log\Psi_t^H \quad P\text{-a.s.}$$

if we show that $n\Big|\big|\Psi_t^H\big|^{1/n}-1\Big|$, $n \in \mathbb{N}$, is dominated by a Q-integrable random variable. Due to $\mathrm{e}^x - 1 \geq x$ for $x \in \mathbb{R}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}x\Big(a^{\frac{1}{x}}-1\Big) = a^{\frac{1}{x}}\Big(1-\frac{1}{x}\log a\Big)-1 \leq a^{\frac{1}{x}}\exp\Big(-\frac{1}{x}\log a\Big)-1 = 0$$

for $a > 0$ and $x > 0$, it follows for $a = \Psi_t^H$ that

$$\log \Psi_t^H \leq n\bigg(\exp\Big(\frac{1}{n}\log\Psi_t^H\Big)-1\bigg) \leq \Psi_t^H - 1, \quad n \in \mathbb{N}.$$

This gives $n\Big|\big|\Psi_t^H\big|^{1/n}-1\Big| \leq \big|\log\Psi_t^H\big|+\Psi_t^H \in L^1(Q)$, $n \in \mathbb{N}$, and proves (3.43).

Combining (3.41), (3.42) and (3.43) yields $f(\infty,\omega) \leq \gamma k_t^H(\omega) \leq f(1,\omega)$ for almost all $\omega \in \Omega$. By the intermediate value theorem, the set

$$\Delta(\omega) := \big\{\delta \in [1,\infty] \;\big|\; f(\delta,\omega) = \gamma k_t^H(\omega)\big\}$$

is thus nonempty for almost all $\omega \in \Omega$. Define $\delta_t^H : \Omega \to [1,\infty]$ by

$$\delta_t^H(\omega) := \sup \Delta(\omega), \quad \omega \in \Omega, \tag{3.47}$$

setting $\delta_t^H := 1$ on the P-null set $\{\omega \in \Omega \,|\, \Delta(\omega) = \emptyset\}$. By continuity of f in δ, $\Delta(\omega)$ is closed in $\mathbb{R} \cup \{+\infty\}$ for all $\omega \in \Omega$, and we get for almost all $\omega \in \Omega$,

$$f\big(\delta_t^H(\omega),\omega\big) = \gamma k_t^H(\omega). \tag{3.48}$$

It remains to prove that the mapping $\omega \mapsto \delta_t^H(\omega)$ is $\mathcal{F}_t$-measurable. Because f is nonincreasing and due to (3.47) and (3.48), we have for any $a \in [1,\infty]$ that

$$\begin{aligned}\big\{\omega \in \Omega \,\big|\, \delta_t^H(\omega) < a\big\} &= \big\{\omega \in \Omega \,\big|\, f\big(\delta_t^H(\omega),\omega\big) > f(a,\omega)\big\}\\ &= \big\{\omega \in \Omega \,\big|\, \gamma k_t^H(\omega) > f(a,\omega)\big\}\\ &= \bigcup_{q \in \mathbb{Q}}\Big(\big\{\omega \in \Omega \,\big|\, \gamma k_t^H(\omega) > q\big\} \cap \big\{\omega \in \Omega \,\big|\, q > f(a,\omega)\big\}\Big)\end{aligned}$$

up to a P-null set. The last set is in $\mathcal{F}_t$ because k_t^H and $f(a,\cdot)$ for fixed $a \in [1,\infty]$ are $\mathcal{F}_t$-measurable random variables. Since $\mathcal{F}_t$ is complete, the set $\big\{\omega \in \Omega \,\big|\, \delta_t^H(\omega) < a\big\}$ is in $\mathcal{F}_t$ for every $a \in \mathbb{R} \cup \{+\infty\}$, and so δ_t^H is $\mathcal{F}_t$-measurable. □

The next result provides a simplified version of Theorem 3.12 based on the use of the minimal local martingale measure $\hat{P}$.

Corollary 3.13. *Fix $t \in [0,T]$ and assume that H is bounded and S satisfies (SC). Suppose further that $\hat{P}$ given by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)_T$ is in $\mathbb{P}_0^{e,f}$, that $\int \lambda \,\mathrm{d}S$ is a $\hat{P}$-, Q_0^E- and Q_H^E-martingale, and that the random variable*

$$\exp\Big(-\Big\langle \int \lambda \,\mathrm{d}M \Big\rangle + \frac{1}{2}\Big[\int \lambda \,\mathrm{d}M\Big]^c\Big)_{t,T} \prod_{t<s\le T} \frac{\mathrm{e}^{-\lambda_s' \Delta M_s}}{1-\lambda_s' \Delta M_s}$$

and its logarithm are $\hat{P}$-integrable. Then there exist $\mathcal{F}_t$-measurable random variables δ_t^0, $\delta_t^H : \Omega \to [1,\infty]$ such that for almost all $\omega \in \Omega$,

$$\begin{aligned} h_t(\omega) = {} & \frac{1}{\gamma}\log\Big(E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1/\delta}\Big|\mathcal{F}_t\Big](\omega)\Big)^{\delta}\Big|_{\delta=\delta_t^H(\omega)} \\ & - \frac{1}{\gamma}\log\Big(E_{\hat{P}}\Big[\big|\Psi_t^0\big|^{1/\delta'}\Big|\mathcal{F}_t\Big](\omega)\Big)^{\delta'}\Big|_{\delta'=\delta_t^0(\omega)}, \end{aligned}$$

where we use the convention (3.39) and the definition

$$\Psi_t^H := \frac{\exp\big(\gamma H - \int_t^T \lambda_s \,\mathrm{d}S_s\big)}{\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)_{t,T}} = \frac{\mathrm{e}^{\gamma H}\exp(-\int \lambda \,\mathrm{d}S)_{t,T}}{\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)_{t,T}}. \tag{3.49}$$

Proof. We only need to check that Ψ_t^0, Ψ_t^H given by (3.49) and $\log \Psi_t^0$, $\log \Psi_t^H$ are $\hat{P}$-integrable as the result then follows from Theorems 3.10 and 3.12 with the choice $Q := \hat{P}$ and $\varphi := -\lambda$. Using the formula for the stochastic exponential and (SC), we get

$$\Psi_t^0 = \exp\Big(-\Big\langle \int \lambda \,\mathrm{d}M \Big\rangle + \frac{1}{2}\Big[\int \lambda \,\mathrm{d}M\Big]^c\Big)_{t,T} \prod_{t<s\le T} \frac{\mathrm{e}^{-\lambda_s' \Delta M_s}}{1-\lambda_s' \Delta M_s},$$

and thus $\Psi_t^0, \log \Psi_t^0 \in L^1\big(\hat{P}\big)$ by assumption. The same is true for Ψ_t^H because H is bounded by assumption. □

To the best of our knowledge, results like Theorem 3.12 and Corollary 3.13 have not been available in the literature so far. A closed-form expression for the exponential utility indifference value has been known only in specific cases when the asset prices are modeled by continuous semimartingales; see for example Theorems 2.9 and 2.10 for explicit expressions of the indifference value in two Brownian settings. There the adapted process δ^H, called the *distortion power*, is closely related to the instantaneous correlation between the driving Brownian motions. The model of Chapter 2 consists of a risk-free bank account and a stock $S = S^1$ driven by a Brownian motion W. The claim H depends on another Brownian motion Y which has a time-dependent and fairly general instantaneous stochastic correlation ρ with W, with $|\rho|$ uniformly bounded away from 1. Theorem 2.9 proves that the indifference value is of the form of Corollary 3.13 above, with δ_t^H and δ_t^0 taking values between

$$\underline{\delta}_t := \inf_{s \in [t,T]} \frac{1}{\|1-|\rho_s|^2\|_{L^\infty(P)}} \quad \text{and} \quad \overline{\delta}_t := \sup_{s \in [t,T]} \left\| \frac{1}{1-|\rho_s|^2} \right\|_{L^\infty(P)}.$$

$\big($The random variables $\delta_t^{\hat{H}}$, $\delta_t^{\hat{0}}$ in Theorem 2.9 correspond to δ_t^H, δ_t^0 in Corollary 3.13.$\big)$ For small $|\rho|$ (uniformly in s, in the L^∞-norm), the claim H is almost unhedgeable and $1/\delta^H$ is nearly 1, whereas for $|\rho|$ close to 1, the claim H is well hedgeable and $1/\delta^H$ is nearly 0. So in that Brownian model, $1/\delta^H$ is closely related to some kind of distance of H from being attainable or hedgeable. In the subsequent discussion, we extend this idea to a more general setting, while we come back to the Brownian model in Section 3.6. Proposition 3.28 in Appendix A makes more precise the range of δ_t^H in a continuous filtration, where all local P-martingales are continuous.

Consider the setting of Corollary 3.13 where S is (in addition) continuous and satisfies (SC), and H is bounded. Then the P-martingale part M of S is continuous and the mean-variance tradeoff process $K = \langle \int \lambda \,\mathrm{d}M \rangle = \langle \int \lambda \,\mathrm{d}S \rangle$ is P-a.s. finite by (SC). The quantity Ψ_t^H from (3.49) then reduces to

$$\Psi_t^H = \exp\Big(\gamma H - \frac{1}{2}(K_T - K_t)\Big),$$

and the assumptions of Corollary 3.13 are satisfied if K_T is bounded, because $\int \lambda \,\mathrm{d}M$ is then a $BMO(P)$-martingale. If we now even suppose that K_T is deterministic, the indifference value at time 0 simplifies to

$$h_0 = \frac{1}{\gamma} \log\Big(E_{\hat{P}}\Big[\exp\big(\gamma H/\delta\big)\Big]\Big)^{\delta}\bigg|_{\delta=\delta_0^H} \tag{3.50}$$

by Corollary 3.13. If $\delta_0^H < \infty$, we can write

$$h_0 = -\tilde{U}_H^{-1}\Big(E_{\hat{P}}\big[\tilde{U}_H(-H)\big]\Big), \quad \text{where} \quad \tilde{U}_H(x) := -\exp\big(-\gamma x/\delta_0^H\big), \ x \in \mathbb{R},$$

which means that $-h_0$ is a certainty equivalent of $-H$. Note, however, that this is done under $\hat{P}$, not P, and with respect to the utility function $\tilde{U}_H$, not U, where $\tilde{U}_H$ depends itself on the claim H. If $\delta_0^H = 1$, then $\tilde{U}_H$ and U coincide and H is valued by the U-certainty equivalent under $\hat{P}$. Moreover, (3.38) shows that we then must have equality in (3.40) for $t = 0$, which implies that $\int_0^T \big(\gamma\eta_s^H - \lambda_s\big)\,\mathrm{d}S_s$ is deterministic, hence $\int \big(\gamma\eta^H - \lambda\big)\,\mathrm{d}S = 0$.

In other words, the equivalent formulation (3.11) of $FER(H)$ in Proposition 3.2 simplifies in this case to

$$H = \frac{1}{\gamma} \log \mathcal{E}\big(\widetilde{N}^H\big)_T + \frac{1}{2\gamma} K_T + k_0^H,$$

which means that H consists only of a constant plus an unhedged term. This may be interpreted as saying that H has maximal distance to attainability. On the opposite extreme, the case $\delta_0^H = \infty$ leads by (3.50) and (3.39) (and still under the same assumptions) to $h_0 = E_{\hat{P}}[H]$. Hence for $\delta_0^H = \infty$, we get a familiar no-arbitrage value for H. In this case, (3.38) and (3.39) show that we must have equality in (3.42) for $t = 0$; hence $\mathcal{E}\big(N^H\big) = \mathcal{E}\big(-\int \lambda \, \mathrm{d}M\big)$ and thus (3.11) simplifies to

$$H = \int_0^T \widetilde{\eta}_s^H \, \mathrm{d}S + \frac{1}{2\gamma} K_T + k_0^H,$$

showing that H is attainable. Summing up, we can interpret $1/\delta^H$ as the distance of H from being attainable; for $1/\delta^H = 0$ (convention: $1/\infty = 0$), the distance is minimal, whereas for $1/\delta^H = 1$, it is maximal. The following remark shows how this idea can be made mathematically more precise.

Remark 3.14. Assume that S is continuous and satisfies (SC) and that $K_T = \big\langle \int \lambda \, \mathrm{d}M \big\rangle_T$ is bounded, but not necessarily deterministic. By Theorem 3.12 and Corollary 3.13, we can attribute to any $H \in L^\infty(P)$ a number $\delta(H) := \delta_0^H$ in $[1, \infty]$ uniquely defined via (3.47) with $Q = \hat{P}$ and $\varphi = -\lambda$. Defining for $G, H \in L^\infty(P)$

$$G \sim H \quad :\Longleftrightarrow \quad \delta\Big(G + \frac{1}{2\gamma} K_T\Big) = \delta\Big(H + \frac{1}{2\gamma} K_T\Big)$$

gives an equivalence relation on $L^\infty(P)$. We denote by $D := L^\infty(P)/\!\sim$ the set of its equivalence classes and associate to each equivalence class a representative. We further define the mapping $d : D \times D \to [0, 1]$ for $G, H \in D$ by

$$d(G, H) := \left| \frac{1}{\delta\big(G + \frac{1}{2\gamma} K_T\big)} - \frac{1}{\delta\big(H + \frac{1}{2\gamma} K_T\big)} \right|.$$

Clearly, d is a metric on D. A claim $G \in L^\infty(P)$ is called $\big(\hat{P}\text{-}\big)$*attainable* if it can be written as $G = E_{\hat{P}}[G] + \int_0^T \beta_s \, \mathrm{d}S_s$ for a predictable process β such that $\int \beta \, \mathrm{d}S$ is a $\hat{P}$-martingale, which is then even a $BMO\big(\hat{P}\big)$-martingale. If G is attainable, the $FER^\star$ of $G + \frac{1}{2\gamma} K_T$ equals $\big(-\int \lambda \, \mathrm{d}M, \beta + \frac{1}{\gamma}\lambda, E_{\hat{P}}[G]\big)$, and so the term $\log \frac{\mathcal{E}(N^H)_T}{\mathcal{E}(-\int \lambda \,\mathrm{d}M)_T}$ vanishes identically. This implies $\delta\big(G + \frac{1}{2\gamma}K_T\big) = \infty$ by the proof of Theorem 3.12, hence $G \sim 0$. Therefore,

$$d(0, H) = \frac{1}{\delta\big(H + \frac{1}{2\gamma}K_T\big)}$$

is a distance of $H \in L^\infty(P)$ from attainability.

The maximal value of $d(0, \cdot)$ depends on the diversity of the filtration $\mathbb{F}$. If S has the predictable representation property in $\mathbb{F}$ in the sense that any $H \in L^\infty(P)$ is attainable (as above), then $\sim$ has only one equivalence class and $d \equiv 0$. On the other hand, suppose that there exists a nondeterministic local $\hat{P}$-martingale N null at 0 and strongly $\hat{P}$-orthogonal to each component of S such that $\mathcal{E}(N)$ is a $\hat{P}$-martingale bounded away from zero and infinity. The maximal distance to attainability is then attained by $\frac{1}{\gamma}\log \mathcal{E}(N)_T$ since $d\big(0, \frac{1}{\gamma}\log \mathcal{E}(N)_T\big) = 1$. $\diamondsuit$

3.5 A BSDE characterisation of the indifference value process

In this section, we prove that the indifference value process h is (the first component of) the unique solution, in a suitable class of processes, of a backward stochastic differential equation (BSDE). This result is similar to Becherer [5] and Mania and Schweizer [44], but obtained here in a general (not even locally bounded) semimartingale model.

We assume throughout this section that

$$\mathbb{P}_0^{e,f} \neq \emptyset$$

and denote by Q_0^E the minimal 0-entropy measure. Let us consider the BSDE

$$\Gamma_t = \Gamma_0 + \frac{1}{\gamma}\log \mathcal{E}(L)_t + \int_0^t \psi_s \,\mathrm{d}S_s, \quad t \in [0, T] \tag{3.51}$$

with the boundary condition

$$\Gamma_T = H. \tag{3.52}$$

We introduce three different notions of solutions to (3.51), (3.52).

Definition 3.15. We say that the triple (Γ, ψ, L) is a *solution* of (3.51), (3.52) if

Si) Γ is a real-valued semimartingale;

Sii) ψ is in $L(S)$;

Siii) L is a local Q_0^E-martingale null at 0 such that $\mathcal{E}(L)$ is a positive Q_0^E-martingale and S is a $Q(L)$-sigma-martingale, where $Q(L)$ is defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathcal{E}(L)_T$.

We call (Γ, ψ, L) a *special solution* of (3.51), (3.52) if furthermore

Siv) $\int \psi \, \mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}_0^{e,f}$;

Sv) $E_P\Big[\mathcal{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P} \log\Big(\mathcal{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P}\Big)\Big] < \infty$, i.e., the probability measure $Q(L)$ defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathcal{E}(L)_T$ has finite relative entropy with respect to P.

If S is locally bounded, we say that (Γ, ψ, L) is an *orthogonal solution* of (3.51), (3.52) if it satisfies (3.51), (3.52), Si), Sii) and

Siii′) L is a local Q_0^E-martingale null at 0 and strongly Q_0^E-orthogonal to every component of S and such that $\mathcal{E}(L)$ is positive.

Under the assumption that S is locally bounded,

$$\begin{gathered}\text{a triple } (\Gamma, \psi, L) \text{ is a solution of (3.51), (3.52) if and only if} \\ \text{it is an orthogonal solution and } \mathcal{E}(L) \text{ is a } Q_0^E\text{-martingale.}\end{gathered} \tag{3.53}$$

To see this, note first that a locally bounded S is a $Q(L)$-sigma-martingale if and only if $\mathcal{E}(L)S$ is a local Q_0^E-martingale, under the assumption that $Q(L)$ is a probability measure. If (Γ, ψ, L) is a solution, then Siii) holds and all of $\mathcal{E}(L)S$, $\mathcal{E}(L)$ and S are local Q_0^E-martingales. Hence $\mathcal{E}(L)$ is strongly Q_0^E-orthogonal to every component of S, and therefore so is L. Conversely, if Siii′) holds, then $\mathcal{E}(L)$ is like L strongly Q_0^E-orthogonal to every component of the local Q_0^E-martingale S. Hence $\mathcal{E}(L)S$ is a local Q_0^E-martingale and thus S is a $Q(L)$-sigma-martingale if $\mathcal{E}(L)$ is a Q_0^E-martingale.

Our main result in this section is then

Theorem 3.16. *Assume that H satisfies (3.13). Then the indifference value process h is the first component of the unique special solution of the BSDE (3.51), (3.52).*

Theorem 3.16 looks at first glance like Theorem 13 of Mania and Schweizer [44]. The important difference, however, is that we do not suppose that the filtration $\mathbb{F}$ is continuous, i.e., that all local P-martingales are continuous. If $\mathbb{F}$ is continuous, then $\frac{1}{\gamma} \log \mathcal{E}(L) = L/\gamma - \frac{\gamma}{2}\langle L/\gamma \rangle$ and Theorem 3.16 corresponds to Theorem 13 of Mania and Schweizer [44]. (Since H is allowed to be unbounded in Theorem 3.16, there are some differences in the integrability properties.) However, recovering the latter result in precise form and almost full strength from Theorem 3.16 requires some additional work which we discuss at the end of this section. The derivation in [44] uses the martingale optimality principle, the existence of an optimal strategy for the indifference value process, and a comparison theorem for BSDEs. Our proof is completely different; it is based on our results for the $FER^\star(H)$ and its relation to the indifference value.

Theorem 4.4 of Becherer [5] is another similar result. Instead of a continuous filtration, the framework in [5] has a continuous price process driven by Brownian motions, and a filtration generated by these and a random measure allowing the modeling of nonpredictable events. Again, to regain from Theorem 3.16 the same statement as in Theorem 4.4 of Becherer [5], some additional work is necessary, which is carried out in Appendix B.

In Corollary 3.6 of the earlier paper [4], Becherer gives a characterisation of $\frac{\mathrm{d}Q^E_H}{\mathrm{d}Q^E_0}$ in a locally bounded semimartingale model. Theorem 3.16 can be viewed as a dynamic extension of that result to a general semimartingale model.

Proof of Theorem 3.16. By Lemma 3.4, (3.13) implies that $\mathbb{P}^{e,f}_H = \mathbb{P}^{e,f}_0 \neq \emptyset$, and so Theorem 3.10 and (3.9) yield

$$h_t = k^H_t - k^0_t = h_0 + \frac{1}{\gamma}\log\frac{\mathcal{E}\big(N^H\big)_t}{\mathcal{E}\big(N^0\big)_t} + \int_0^t \big(\eta^H_s - \eta^0_s\big)\,\mathrm{d}S_s, \quad 0 \le t \le T,$$

where $\big(N^H, \eta^H, k^H_0\big)$ and $\big(N^0, \eta^0, k^0_0\big)$ are the $FER^\star(H)$ and $FER^\star(0)$; see Proposition 3.6 for their properties. Then $\psi := \eta^H - \eta^0$ is in $L(S)$ and $\int \psi\,\mathrm{d}S$ is a Q-martingale for every $Q \in \mathbb{P}^{e,f}_0 = \mathbb{P}^{e,f}_H$. By Bayes' formula, $\mathcal{E}\big(N^H\big)/\mathcal{E}\big(N^0\big)$ is the Q^E_0-density process of Q^E_H, and so it is a positive Q^E_0-martingale and its stochastic logarithm L, defined by $\mathcal{E}(L) = \mathcal{E}\big(N^H\big)/\mathcal{E}\big(N^0\big)$, is a local Q^E_0-martingale null at 0. Moreover, $\frac{\mathrm{d}Q(L)}{\mathrm{d}P} = \mathcal{E}(L)_T\frac{\mathrm{d}Q^E_0}{\mathrm{d}P} = \frac{\mathrm{d}Q^E_H}{\mathrm{d}P}$ shows $Q(L) = Q^E_H$. Hence S is a $Q(L)$-sigma-martingale and Sv) is satisfied because Q^E_H has finite relative entropy with respect to P. Since $h_T = H$ by definition, we see that h is the first component of a special solution of the BSDE (3.51), (3.52).

To prove uniqueness, let (Γ, ψ, L) be any special solution of (3.51), (3.52). Denote by $\big(N^0, \eta^0, k^0_0\big)$ the unique $FER^\star(0)$, and define

$$N := N^0 + L + \big[N^0, L\big], \quad \eta := \eta^0 + \psi \;\text{ and }\; k_0 := k^0_0 + \Gamma_0. \tag{3.54}$$

We claim that

$$(N, \eta, k_0) \text{ is the unique } FER^\star(H). \tag{3.55}$$

For the proof, we first note that $\mathcal{E}\big(N^0\big)\mathcal{E}(L) = \mathcal{E}\big(N^0 + L + \big[N^0, L\big]\big) = \mathcal{E}(N)$ by Yor's formula. Using (3.51), (3.52) and (3.7) for $H = 0$ thus yields

$$
\begin{aligned}
H &= \frac{1}{\gamma}\log\Big(\mathcal{E}\big(N^0\big)_T\mathcal{E}(L)_T\Big) + \int_0^T \big(\eta^0_s + \psi_s\big)\,\mathrm{d}S_s + k^0_0 + \Gamma_0 \\
&= \frac{1}{\gamma}\log\mathcal{E}(N)_T + \int_0^T \eta_s\,\mathrm{d}S_s + k_0.
\end{aligned}
$$

Therefore (N, η, k_0) satisfies (3.7) for H, and it is enough to show that the assumptions on N and η for $FER^\star(H)$ are fulfilled. By Bayes' formula, $\mathcal{E}(N) = \mathcal{E}\big(N^0\big)\mathcal{E}(L)$ is a positive P-martingale, because $\mathcal{E}\big(L\big)$ is a positive Q^E_0-martingale by Siii) and $\mathcal{E}\big(N^0\big)$ is the P-density process of Q^E_0. Writing next

$$
\frac{\mathrm{d}P(N)}{\mathrm{d}Q^E_0} = \frac{\mathrm{d}P(N)}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}Q^E_0} = \mathcal{E}(N)_T/\mathcal{E}\big(N^0\big)_T = \mathcal{E}(L)_T,
$$

we see that $P(N) = Q(L)$ which implies that

$$
I\big(P(N)\big|P\big) = E_P\bigg[\mathcal{E}(L)_T\frac{\mathrm{d}Q^E_0}{\mathrm{d}P}\log\bigg(\mathcal{E}(L)_T\frac{\mathrm{d}Q^E_0}{\mathrm{d}P}\bigg)\bigg] < \infty
$$

by Sv) and that S is a $P(N)$-sigma-martingale by Siii). Because $\big(N^0, \eta^0, k^0_0\big)$ is the $FER^\star(0)$, $\int\eta\,\mathrm{d}S = \int\eta^0\,\mathrm{d}S + \int\psi\,\mathrm{d}S$ is by Proposition 3.6 and Siv) a Q-martingale for every $Q \in \mathbb{P}^{e,f}_0 = \mathbb{P}^{e,f}_H$, hence also for $P(N)$ and Q^E_H, and so (N, η, k_0) is an $FER(H)$ satisfying (c) from Proposition 3.6. This implies (3.55). Uniqueness of the $FER^\star(H)$ and (3.54) now imply that Γ_0, ψ are unique; so is L due to $\mathcal{E}(L) = \mathcal{E}(N)/\mathcal{E}\big(N^0\big)$, and finally also Γ by (3.51). This ends the proof. □

The above argument shows in particular a close link between the $FER^\star(H)$ and the BSDE (3.51), (3.52). Provided we have the $FER^\star(0)$, we can construct $FER^\star(H)$ from the special solution of (3.51), (3.52), and vice versa. This is familiar from exponential utility indifference valuation; indeed, knowing $FER^\star(0)$ corresponds to knowing the minimal 0-entropy measure Q^E_0.

Remark 3.17. If S is locally bounded and H is bounded, there is another way to prove uniqueness of the first component of a special solution of the BSDE (3.51), (3.52), which we briefly sketch here. If (Γ, ψ, L) is a special solution of (3.51), (3.52), the idea is to show that Γ equals the indifference value process h, which then yields the desired uniqueness result. Let $t \in [0, T]$ and replace in the definition of $\mathcal{A}^H_t$ the condition that $\int\vartheta\,\mathrm{d}S$ is a Q-supermartingale for every $Q \in \mathbb{P}^{e,f}_H$ by assuming that it is a Q-martingale for every $Q \in \mathbb{P}^{e,f}_H$. We do the analogous change for $\mathcal{A}^0_t$ and note that this does not affect the values of V_t^H and V_t^0, as mentioned after the proof of Theorem 3.10. We now apply Proposition 3 of Mania and Schweizer [44] to obtain

$$h_t = \frac{1}{\gamma} \log \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_t^H} E_{Q_0^E}\left[\exp\left(\gamma H - \gamma \int_t^T \vartheta_s \,\mathrm{d}S_s\right)\middle|\mathcal{F}_t\right]. \tag{3.56}$$

Using (3.51), (3.52) gives

$$\gamma H = \gamma \Gamma_0 + \log \mathcal{E}(L)_T + \gamma \int_0^T \psi_s \,\mathrm{d}S_s = \gamma \Gamma_t + \log \frac{\mathcal{E}(L)_T}{\mathcal{E}(L)_t} + \gamma \int_t^T \psi_s \,\mathrm{d}S_s,$$

which we plug into (3.56) to obtain

$$h_t = \Gamma_t + \frac{1}{\gamma} \log \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_t^H} E_{Q(L)}\left[\exp\left(\gamma \int_t^T (\psi_s - \vartheta_s) \,\mathrm{d}S_s\right)\middle|\mathcal{F}_t\right] =: \Gamma_t + \frac{1}{\gamma} \log \Lambda,$$

where the probability measure $Q(L)$ is defined by $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathcal{E}(L)_T$. To show that $\Lambda = 1$, we first note that $Q(L) \in \mathbb{P}_0^{e,f}$ by Sv), $\mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ by Lemma 3.4, and $\int \psi \,\mathrm{d}S$ as well as $\int \vartheta \,\mathrm{d}S$ are Q-martingales for every $Q \in \mathbb{P}_H^{e,f} = \mathbb{P}_0^{e,f}$ by Siv) and because $\vartheta \in \mathcal{A}_t^H$. Jensen's inequality then yields $\Lambda \geq 1$, and we obtain $\Lambda \leq 1$ by the choice $\vartheta^\star := \psi \in \mathcal{A}_t^H$. Note that also for this uniqueness proof, we have used the assumption that (Γ, ψ, L) is a *special* solution of the BSDE (3.51), (3.52), i.e., that it also satisfies Siv), Sv). ◊

We have seen in Section 3.3 that the difference between $FER(H)$ and the (unique) $FER^\star(H)$ is an issue of integrability. The same thing happens here: The next example shows that the BSDE (3.51), (3.52) may have many solutions if we omit the requirement Siv) (which corresponds to (d) in Proposition 3.6).

Example 3.18. As in Example 3.8, take independent P-Brownian motions W and $W^\perp$, their P-augmented filtration $\mathbb{F}$ and $d = 1$, $S = W$, $H \equiv 0$. Then $Q_0^E = P$ and $(0, 0, 0)$ is the unique special solution of (3.51), (3.52).

As in Example 3.8, take $N^0 = W^\perp$ and use Proposition 1 of Emery et al. [19] to find for any $c \in \mathbb{R}$ a process $\psi(c)$ in $L(S)$ such that

$$-\frac{1}{\gamma} \log \mathcal{E}\left(N^0\right)_T - c = \int_0^T \psi_s(c) \,\mathrm{d}S_s \quad P\text{-a.s.}$$

If we then set $\Gamma_t(c) := c + \frac{1}{\gamma} \log \mathcal{E}\left(N^0\right)_t + \int_0^t \psi_s(c) \,\mathrm{d}S_s$ for $t \in [0, T]$, we easily see as in Example 3.8 that $\left(\Gamma(c), \psi(c), N^0\right)$ is a solution to (3.51), (3.52) and satisfies Sv), but not Siv). So we clearly have multiple solutions. ◊

Theorem 3.16 allows us to obtain a result similar to Proposition 3.11.

Corollary 3.19. *Assume that H satisfies (3.13). Then we have for any probability measure $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$ and $t \in [0,T]$ that*

$$h_t = E_Q[H|\mathcal{F}_t] - \frac{1}{\gamma} E_Q\big[\log \mathcal{E}(L)_{t,T}\big|\mathcal{F}_t\big], \tag{3.57}$$

where L is the third component of the unique special solution of the BSDE (3.51), (3.52). In particular,

$$h_0 = E_{Q_0^E}[H] + \frac{1}{\gamma} I\big(Q_0^E\big|Q(L)\big), \tag{3.58}$$

where $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} := \mathcal{E}(L)_T$.

Proof. Theorem 3.16 implies (3.57) by taking conditional Q-expectations between t and T in (3.51), using (3.52) and Siv). (3.58) follows for $Q = Q_0^E$. $\square$

Remark 3.20. Corollary 3.19 raises the question if one can find a probability measure $Q \in \mathbb{P}_0^{e,f}$ such that the indifference value is the Q-conditional expectation of H. From (3.57) we see that $\log \mathcal{E}(L)$ must then be a Q-martingale, and if we write the Q_0^E-density process of Q as $\mathcal{E}(R)$ for some local Q_0^E-martingale R, Bayes' formula tells us that we want $\mathcal{E}(R)\log\mathcal{E}(L)$ to be a Q_0^E-martingale. Itô's formula gives

$$\begin{aligned}
\mathrm{d}\big(\mathcal{E}(R)\log\mathcal{E}(L)\big)_t = {} & \log\mathcal{E}(L)_{t-}\,\mathrm{d}\mathcal{E}(R)_t + \frac{\mathcal{E}(R)_{t-}}{\mathcal{E}(L)_{t-}}\,\mathrm{d}\mathcal{E}(L)_t \\
& + \mathcal{E}(R)_{t-}\,\mathrm{d}\left[L^c, R^c - \frac{1}{2}L^c\right]_t \\
& + \mathcal{E}(R)_{t-}\big((\Delta R_t + 1)\log(1+\Delta L_t) - \Delta L_t\big),
\end{aligned}$$

where L^c and R^c denote the continuous local Q_0^E-martingale parts of L and R. For $\mathcal{E}(R)\log\mathcal{E}(L)$ to be a local Q_0^E-martingale, we must have that $R^c = \frac{1}{2}L^c$ on $\{L^c \neq 0\}$ and $\Delta R_t = \frac{\Delta L_t - \log(1+\Delta L_t)}{\log(1+\Delta L_t)}$ on $\{\Delta L_t \neq 0\}$. Therefore, we define $R = R^c + R^d$ by

$$R_t^c := \frac{1}{2}L_t^c \quad\text{and}\quad R_t^d := \sum_{0<s\leq t} \frac{\Delta L_s - \log(1+\Delta L_s)}{\log(1+\Delta L_s)} \mathbb{1}_{\Delta L_s \neq 0} - A_t, \tag{3.59}$$

where A is the dual predictable projection under Q_0^E of the sum in (3.59). Note that R^d is well defined, since $\Delta L_s > -1$, $\Delta L_s \neq 0$ implies that

$$\left|\frac{\Delta L_s - \log(1+\Delta L_s)}{\log(1+\Delta L_s)}\right| \leq |\Delta L_s|;$$

in fact, $\log(1+x) \geq \frac{x}{1+x}$ for $x > -1$ implies that $\left|\frac{x-\log(1+x)}{\log(1+x)}\right| \leq |x|$ for $x > -1$, $x \neq 0$. By this construction, $\mathcal{E}(R)$ and $\mathcal{E}(R)\log\mathcal{E}(L)$ are local Q_0^E-martingales, but it is not clear whether they are true Q_0^E-martingales. If they are and if Q defined by $\frac{\mathrm{d}Q}{\mathrm{d}Q_0^E} := \mathcal{E}(R)_T$ is in $\mathbb{P}_0^{e,f}$, then we obtain indeed $h_t = E_Q[H|\mathcal{F}_t]$ for all $t \in [0,T]$. In general, this representation is not linear in H since the probability measure Q may (via L) depend on H. Mania and Schweizer [44] showed in their Proposition 11 that a representation of this type exists if the filtration is continuous and H is bounded, in which case $R = \frac{1}{2}L$. $\diamondsuit$

Becherer [5] and Mania and Schweizer [44] show BMO-estimates for all components of the solution to the BSDE for the indifference value process h. It seems doubtful if one can obtain such results in our general framework here, but under a mild additional assumption, we can still characterise Siv) via BMO-properties without being more specific about the filtration $\mathbb{F}$; see Theorem 3.21 below.

The indifference hedging strategy β is defined as the difference of the strategies which attain $V_0^H(h_0)$ and $V_0^0(0)$, i.e., as that extra trading we do in the optimisation which can be attributed to the presence of a claim. If H satisfies (3.13), we have $\beta = \eta^H - \eta^0 = \psi$ by (3.32) and the proof of Theorem 3.16, where ψ is the second component of the unique special solution of the BSDE (3.51), (3.52). Hence it is of particular interest to know when $\int \psi \,\mathrm{d}S$ is a $BMO\big(Q_0^E\big)$-martingale.

Theorem 3.21. *Assume that S is continuous, H is bounded and there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. Let (Γ, ψ, L) be a solution of the BSDE (3.51), (3.52) which satisfies Sv). Then the following are equivalent:*

(a) (Γ, ψ, L) is the special solution of (3.51), (3.52), i.e., it also satisfies Siv);

(b) L is a $BMO\big(Q_0^E\big)$-martingale, $\mathcal{E}(L)$ satisfies condition (J), and $\int \psi \,\mathrm{d}S$ is a Q_0^E-martingale;

(c) $\int \psi \,\mathrm{d}S$ is a $BMO\big(Q_0^E\big)$-martingale;

(d) $\int \psi \,\mathrm{d}M$ is a $BMO(P)$-martingale, where M is the P-local martingale part of S;

(e) there exists $\varepsilon > 0$ such that $E_P\Big[\exp\Big(\varepsilon\big[\int \psi \,\mathrm{d}S\big]_T\Big)\Big] < \infty$.

Proof. "(a) $\Longrightarrow$ (b)". Denote by $\left(N^H, \eta^H, k_0^H\right)$ and $\left(N^0, \eta^0, k_0^0\right)$ the unique $FER^\star(H)$ and $FER^\star(0)$. Theorem 3.9 implies that N^H, N^0 are $BMO(P)$-martingales and $\mathcal{E}\left(N^H\right)$, $\mathcal{E}\left(N^0\right)$ satisfy condition (J), say with constants C^H and C^0. By the proof of Theorem 3.16, we have $\mathcal{E}(L) = \mathcal{E}\left(N^H\right)/\mathcal{E}\left(N^0\right)$ and thus $\mathcal{E}(L)$ satisfies condition (J) with constant $C^H C^0$. Since $1/\mathcal{E}\left(N^0\right)$ is the Q_0^E-density process of P, $\mathcal{E}\left(N^0\right)^{-1} = \mathcal{E}\left(\hat{N}^0\right)$ for a local Q_0^E-martingale $\hat{N}^0$, and so $\mathcal{E}(L) = \mathcal{E}\left(N^H + \hat{N}^0 + \left[N^H, \hat{N}^0\right]\right)$ by Yor's formula. Due to the properties of N^0 and N^H, both $\hat{N}^0$ and $N^H + \left[N^H, \hat{N}^0\right]$ are $BMO\left(Q_0^E\right)$-martingales by Propositions 6 and 7 of Doléans-Dade and Meyer [17], and hence so is $L = \hat{N}^0 + N^H + \left[N^H, \hat{N}^0\right]$. Finally, $\int \psi \,\mathrm{d}S$ is a Q_0^E-martingale by Siv).

"(b) $\Longrightarrow$ (c)", "(c) $\Longrightarrow$ (d)" and "(d) $\Longrightarrow$ (e)". These go along the same lines as the proofs of the corresponding implications in Theorem 3.9. Instead of (3.7) we take (3.51), (3.52), and we replace $P\left(N^H\right)$ by Q_0^E.

"(e) $\Longrightarrow$ (a)". Like for the corresponding implication in Theorem 3.9, we obtain that $\int \psi \,\mathrm{d}S$ is a square-integrable Q-martingale for any $Q \in \mathbb{P}_0^{e,f} = \mathbb{P}_H^{e,f}$, which implies Siv). □

Remark 3.22. Example 3.18 also shows that even if the assumptions of Theorem 3.21 are satisfied, none of the equivalent statements (a)–(e) need hold. This is another way of saying that there exist solutions of (3.51), (3.52) which are not special solutions. ◊

Corollary 3.23. *Suppose the assumptions of Theorem 3.21 hold. Let* (Γ, ψ, L) *be an orthogonal solution of the BSDE (3.51), (3.52). Then* (Γ, ψ, L) *is the special solution of (3.51), (3.52) if and only if both* L *and* $\int \psi \,\mathrm{d}S$ *are* $BMO\left(Q_0^E\right)$*-martingales and* $\mathcal{E}(L)$ *is a* Q_0^E*-martingale which satisfies condition (J).*

Proof. The "only if" part follows immediately from Theorem 3.21. For the "if" part, note first that (Γ, ψ, L) is a solution of (3.51), (3.52) by (3.53). So we need only show that (Γ, ψ, L) satisfies Sv) in view of Theorem 3.21. We first prove that $\int \psi \,\mathrm{d}S$ is a $BMO\big(Q(L)\big)$-martingale, where $\frac{\mathrm{d}Q(L)}{\mathrm{d}Q_0^E} = \mathcal{E}(L)_T$. Because $1/\mathcal{E}(L)$ is the $Q(L)$-density process of Q_0^E, it can be written as $\mathcal{E}(L)^{-1} = \mathcal{E}\left(\hat{L}\right)$ for a local $Q(L)$-martingale $\hat{L}$ satisfying $L + \hat{L} + \left[L, \hat{L}\right] = 0$ by Yor's formula. The continuity of S and the strong Q_0^E-orthogonality of L to S entail

$$\left[\int \psi \,\mathrm{d}S, \hat{L}\right] = -\left[\int \psi \,\mathrm{d}S, L\right] = 0.$$

This yields by Proposition 7 of Doléans-Dade and Meyer [17] that $\int \psi \,\mathrm{d}S$ is a $BMO\big(Q(L)\big)$-martingale. For the second component η^0 of the $FER^\star(0)$, we similarly have that $\int \eta^0 \, \mathrm{d}S$ is a $BMO\big(Q(L)\big)$-martingale since $\int \eta^0 \, \mathrm{d}S$ is a $BMO\big(Q_0^E\big)$-martingale by Theorem 3.9. Because (Γ, ψ, L) is a solution of (3.51), (3.52), we can write

$$\log \mathcal{E}(L)_T = -\gamma \int_0^T \psi_s \, \mathrm{d}S_s + \gamma H - \gamma \Gamma_0,$$

and similarly, we have for the $FER^\star(0)$ $\big(N^0, \eta^0, k_0^0\big)$ that

$$\log \frac{\mathrm{d}Q_0^E}{\mathrm{d}P} = \log \mathcal{E}\big(N^0\big)_T = -\gamma \int_0^T \eta_s^0 \, \mathrm{d}S_s - \gamma k_0^0.$$

Because $\int \big(\eta^0 + \psi\big) \, \mathrm{d}S$ is a $BMO\big(Q(L)\big)$-martingale, we thus obtain

$$\begin{aligned} E_{Q(L)}\left[\log\left(\mathcal{E}(L)_T \frac{\mathrm{d}Q_0^E}{\mathrm{d}P}\right)\right] &= -\gamma\Gamma_0 - \gamma k_0^0 + \gamma E_{Q(L)}\left[H - \int_0^T \big(\eta_s^0 + \psi_s\big) \, \mathrm{d}S_s\right] \\ &= -\gamma\Gamma_0 - \gamma k_0^0 + \gamma E_{Q(L)}[H] < \infty \end{aligned}$$

since H is bounded. Hence (Γ, ψ, L) satisfies Sv) and we are done. □

Corollary 3.23 allows us to recover Theorem 13 of Mania and Schweizer [44] from our Theorem 3.16. However, this still requires some work which is done in the next two results. A similar approach presented in Appendix B can be used to recover Theorem 4.4 of Becherer [5] from our Theorem 3.16. Although the following lemma is a special case of Proposition 7 of Mania and Schweizer [44], we give the proof here as well, both for completeness and because it is quite simple in this case.

Lemma 3.24. *Assume that the filtration $\mathbb{F}$ is continuous, H is bounded and let (Γ, ψ, L) be an orthogonal solution of the BSDE (3.51), (3.52) with bounded first component Γ. Then L and $\int \psi \, \mathrm{d}S$ are $BMO\big(Q_0^E\big)$-martingales.*

Proof. If L and $\int \psi \, \mathrm{d}S$ are true Q_0^E-martingales, (3.51) yields by continuity of L that

$$E_{Q_0^E}\big[\langle L\rangle_T - \langle L\rangle_\tau \big| \mathcal{F}_\tau\big] = 2\gamma E_{Q_0^E}[\Gamma_\tau - \Gamma_T | \mathcal{F}_\tau] \quad \text{for any stopping time } \tau. \tag{3.60}$$

Because Γ is bounded, the right-hand side of (3.60) is bounded independently of τ, and thus L is a $BMO\big(Q_0^E\big)$-martingale. Therefore, $\big(E_{Q_0^E}\big[\langle L\rangle_T \big| \mathcal{F}_s\big]\big)_{0\le s\le T}$ is also a continuous $BMO\big(Q_0^E\big)$-martingale, because

$$E_{Q_0^E}\Big[\Big|\langle L\rangle_T - E_{Q_0^E}\big[\langle L\rangle_T \big| \mathcal{F}_\tau\big]\Big| \Big| \mathcal{F}_\tau\Big] \le 2E_{Q_0^E}\big[\langle L\rangle_T - \langle L\rangle_\tau \big| \mathcal{F}_\tau\big] \le 2\|L\|^2_{BMO_2(Q_0^E)}$$

for any stopping time τ. Taking conditional Q_0^E-expectations in (3.51) with $t = T$ gives

$$\int_0^s \psi_y \,\mathrm{d}S_y = E_{Q_0^E}[\Gamma_T - \Gamma_0 | \mathcal{F}_s] - \frac{1}{\gamma} L_s + \frac{1}{2\gamma} E_{Q_0^E}\big[\langle L\rangle_T \big| \mathcal{F}_s\big], \quad 0 \le s \le T,$$

and so $\int \psi \,\mathrm{d}S$ is a $BMO\big(Q_0^E\big)$-martingale as well. Note that we obtain bounds for the $BMO_2\big(Q_0^E\big)$-norms of L and $\int \psi \,\mathrm{d}S$ that depend on Γ (and γ) alone.

For general L and $\int \psi \,\mathrm{d}S$, we stop at τ_n and apply the above argument with T replaced by τ_n. Letting $n \to \infty$ then completes the proof. □

A closer look at the proof of Lemma 3.24 shows that we did not use the property that L is strongly Q_0^E-orthogonal to S. However, this is of course necessary if we want to prove a uniqueness result. By combining Lemma 3.24 and Corollary 3.23, we obtain the following sufficient conditions for the uniqueness of an orthogonal solution of (3.51), (3.52) with bounded first component.

Proposition 3.25. *Assume that $\mathbb{F}$ is continuous, H is bounded, and there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. Then the indifference value process h is the first component of the unique orthogonal solution of (3.51), (3.52) with bounded first component. Moreover, L and $\int \psi \,\mathrm{d}S$ are $BMO\big(Q_0^E\big)$-martingales.*

Proof. By Theorem 3.16 and (3.53), h is the first component of an orthogonal solution of (3.51), (3.52). Using $V_t^H(h_t) = \exp(-\gamma h_t) V_t^H(0)$ and the definition (3.3) of h easily implies that the indifference value process h is bounded by $\|H\|_{L^\infty(P)}$. If (Γ, ψ, L) is any orthogonal solution of the BSDE (3.51), (3.52) with bounded Γ, then L and $\int \psi \,\mathrm{d}S$ are $BMO\big(Q_0^E\big)$-martingales by Lemma 3.24. By Corollary 3.23, (Γ, ψ, L) is then a special solution, which is unique by Theorem 3.16. □

Proposition 3.25 is almost identical to Theorem 13 in Mania and Schweizer [44]; the only difference is that we have here the additional assumption that there exists $Q \in \mathbb{P}_0^{e,f}$ whose P-density process satisfies $R_{L\log L}(P)$. The explanation for this is that we actually prove more than we really need for Proposition 3.25. Mania and Schweizer [44] use a comparison result for BSDEs (their Theorem 8) to deduce directly that one has uniqueness of orthogonal solutions to the BSDE within the class of those with bounded first component. In contrast, the proof of Proposition 3.25 actually shows that under the $R_{L\log L}$-condition, any solution with bounded first component is even a special solution — and then one appeals to Theorem 3.16 which asserts uniqueness within that class.

3.6 Application to a Brownian setting

In this section, we consider as a special case a model with one risky asset driven by a Brownian motion and a claim coming from a second, correlated Brownian motion. All processes are indexed by $0 \le s \le T$. Let W and Y be two Brownian motions with constant instantaneous correlation ρ satisfying $|\rho| < 1$. Choose as $\mathbb{F}$ the P-augmentation of the filtration generated by the pair (W, Y), and denote by $\mathbb{Y} = (\mathcal{Y}_s)_{0 \le s \le T}$ the P-augmentation of the filtration generated by Y alone.

As usual, the risk-free *bank account* has zero interest rate. The single *tradable stock* has a price process given by

$$\mathrm{d}S_s = \mu_s S_s \,\mathrm{d}s + \sigma_s S_s \,\mathrm{d}W_s, \quad 0 \le s \le T,\ S_0 > 0, \tag{3.61}$$

where drift μ and volatility σ are $\mathbb{F}$-predictable processes. We assume for simplicity that μ is bounded and σ is bounded away from zero and infinity. We further assume that

the instantaneous Sharpe ratio $\frac{\mu}{\sigma}$ of the tradable stock is $\mathbb{Y}$-predictable.

In the notation of Section 3.2, $S = S_0 + M + \int \lambda \,\mathrm{d}\langle M\rangle$, where $M := \int \sigma S \,\mathrm{d}W$ is a local $(\mathbb{F}, P)$-martingale and $\lambda := \frac{\mu}{\sigma}\frac{1}{\sigma S}$ is $\mathbb{F}$-predictable. Since μ is bounded and σ is bounded away from zero, the Sharpe ratio $\frac{\mu}{\sigma}$ is also bounded, and thus $\int \lambda \,\mathrm{d}M = \int \frac{\mu}{\sigma} \,\mathrm{d}W$ is a $BMO(\mathbb{F}, P)$-martingale and $\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)$ is an $(\mathbb{F}, P)$-martingale. We suppose that the contingent claim H is a bounded $\mathcal{Y}_T$-measurable random variable. Together with the structure of S in (3.61), this assumption on H formalises the idea that the payoff H is driven by Y, whereas hedging can only be done in S which is imperfectly correlated with the factor Y.

In the literature, there are three main approaches to obtain explicit formulas for the resulting optimisation problem (3.2). In a Markovian setting, Henderson [31], Henderson and Hobson [33, 34], and Musiela and Zariphopoulou [47], among others, first derive the Hamilton-Jacobi-Bellman nonlinear partial differential equation (PDE) for the value function of the underlying stochastic control problem. This PDE is then linearised by a power transformation with a constant exponent, called the *distortion power*, which corresponds to δ_0^H from Theorem 3.12 and Corollary 3.13. This method works only if one has a Markovian model. Using general techniques, Tehranchi [56] first proves a Hölder-type inequality, which he then applies to the portfolio optimisation problem. The distortion power there arises as an exponent in the Hölder-type inequality. A third approach based on martingale arguments has allowed us in Chapter 2 to consider a more general framework with a fairly general stochastic correlation ρ. Theorems 2.9 and 2.10 prove that the explicit form of the indifference value from Musiela and Zariphopoulou [47] or Tehranchi [56] is preserved, except that the distortion power, which is shown to exist but not explicitly determined, may be random and depend on H like in our general semimartingale model; compare Theorem 3.12 and Corollary 3.13.

We give here another proof based on the results of the previous sections. While there are no new results, the arguments in comparison to Chapter 2 are easier and shorter, give new insights, and show the advantage of $FER^\star(H)$ compared to the BSDE formulation (3.51), (3.52) in Section 3.5. Indeed, $FER^\star(H)$ is a representation under the original probability measure P, whereas in the BSDE formulation (3.51), (3.52), one must first determine the minimal 0-entropy measure.

Proposition 3.26. *For $t \in [0,T]$ and any $\mathcal{F}_t$-measurable random variable x_t,*

$$V_t^H(x_t) = -\exp(-\gamma x_t) E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1-|\rho|^2}\Big|\mathcal{Y}_t\Big]^{\frac{1}{1-|\rho|^2}},$$

where $\Psi_t^H = \exp\Big(\gamma H - \frac{1}{2}\int_t^T \big|\frac{\mu_s}{\sigma_s}\big|^2 \,\mathrm{d}s\Big)$ and the minimal martingale measure $\hat{P}$ is given by

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} = \mathcal{E}\Big(-\int \frac{\mu}{\sigma}\,\mathrm{d}W\Big)_T. \tag{3.62}$$

The exponential utility indifference value h_t of H at time t equals

$$h_t = \frac{1}{\gamma(1-|\rho|^2)} \log \frac{E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1-|\rho|^2}\Big|\mathcal{Y}_t\Big]}{E_{\hat{P}}\Big[\big|\Psi_t^0\big|^{1-|\rho|^2}\Big|\mathcal{Y}_t\Big]}.$$

In Corollary 3.13, we have shown that

$$\begin{aligned} h_t(\omega) = {} & \frac{1}{\gamma}\log\Big(E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1/\delta}\Big|\mathcal{F}_t\Big](\omega)\Big)^{\delta}\Big|_{\delta=\delta_t^H(\omega)} \\ & - \frac{1}{\gamma}\log\Big(E_{\hat{P}}\Big[\big|\Psi_t^0\big|^{1/\delta'}\Big|\mathcal{F}_t\Big](\omega)\Big)^{\delta'}\Big|_{\delta'=\delta_t^0(\omega)}, \end{aligned}$$

and have related $1/\delta^H$ to a kind of distance of H from attainability. Here we have $1/\delta^H = 1-|\rho|^2$, which confirms our interpretation: The closer $1/\delta^H$ is to one, the greater is the distance of H from being attainable, because a smaller correlation ρ between W and Y makes hedging more difficult.

Proof of Proposition 3.26. The idea is to explicitly derive the $FER^\star(H)$ and $FER^\star(0)$, from which the result follows by Theorem 3.10. In view of Proposition 3.2 and (3.10), we thus look for suitable real-valued processes $\widetilde{N}^H$ and $\widetilde{\eta}^H$ and an $\mathcal{F}_t$-measurable random variable k_t^H such that

$$H = \frac{1}{\gamma} \log \mathcal{E}\big(\widetilde{N}^H\big)_{t,T} + \int_t^T \widetilde{\eta}_s^H \sigma_s S_s \,\mathrm{d}\hat{W}_s + \frac{1}{2\gamma} \int_t^T \left| \frac{\mu_s}{\sigma_s} \right|^2 \mathrm{d}s + k_t^H, \tag{3.63}$$

where $\hat{W} := W + \int \frac{\mu}{\sigma} \,\mathrm{d}s$ is by Girsanov's theorem a Brownian motion under the minimal martingale measure $\hat{P}$ given by (3.62). Using Itô's representation theorem as in Lemma 1.6.7 of Karatzas and Shreve [39] for $\big|\Psi_t^H\big|^{1-|\rho|^2}$ under $\mathbb{Y}$ and $\hat{P}$ restricted to $\mathcal{Y}_T$, we can find a $\mathbb{Y}$-predictable process ζ with $E_{\hat{P}}\big[\int_0^T |\zeta_s|^2 \,\mathrm{d}s\big] < \infty$ such that

$$\big|\Psi_t^H\big|^{1-|\rho|^2} = E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1-|\rho|^2} \Big| \mathcal{Y}_t\Big] \mathcal{E}\bigg(\int \zeta \,\mathrm{d}\hat{Y}\bigg)_{t,T}, \tag{3.64}$$

where the $\big(\mathbb{Y}, \hat{P}\big)$-Brownian motion $\hat{Y}$ is defined by

$$\hat{Y}_s := Y_s + \int_0^s \rho \frac{\mu_y}{\sigma_y} \,\mathrm{d}y \quad \text{for } s \in [0, T].$$

For this, we have used that Ψ_t^H is $\mathcal{Y}_T$-measurable since $\frac{\mu}{\sigma}$ is $\mathbb{Y}$-predictable and H is $\mathcal{Y}_T$-measurable by assumption. We can write $\hat{Y} = \rho \hat{W} + \sqrt{1 - |\rho|^2} \hat{W}^\perp$ for an $\big(\mathbb{F}, \hat{P}\big)$-Brownian motion $\hat{W}^\perp$ independent of $\hat{W}$. Taking the logarithm in (3.64) results in

$$H = \frac{1}{\gamma} \int_t^T \frac{\zeta_s}{1 - |\rho|^2} \,\mathrm{d}\hat{Y}_s - \frac{1}{2\gamma} \int_t^T \frac{|\zeta_s|^2}{1 - |\rho|^2} \,\mathrm{d}s + \frac{1}{2\gamma} \int_t^T \left| \frac{\mu_s}{\sigma_s} \right|^2 \mathrm{d}s + k_t^H,$$

where

$$k_t^H := \frac{1}{\gamma(1 - |\rho|^2)} \log E_{\hat{P}}\Big[\big|\Psi_t^H\big|^{1-|\rho|^2} \Big| \mathcal{Y}_t\Big].$$

But this is (3.63) with

$$\widetilde{N}^H := \int \frac{\zeta}{\sqrt{1 - |\rho|^2}} \,\mathrm{d}\hat{W}^\perp \quad \text{and} \quad \widetilde{\eta}^H := \frac{\rho\, \zeta}{\gamma(1 - |\rho|^2)} \frac{1}{\sigma S}.$$

Clearly, $\widetilde{N}^H$ is a local $\hat{P}$-martingale strongly $\hat{P}$-orthogonal to S, hence also a local P-martingale strongly P-orthogonal to M. Moreover, Ψ_t^H is bounded away from zero and infinity, which implies by (3.64) that $\mathcal{E}\big(\int \zeta \,\mathrm{d}\hat{Y}\big)$ is uniformly bounded away from zero and infinity. By Theorem 3.4 of Kazamaki [40], $\int \zeta \,\mathrm{d}\hat{Y}$ is then a $BMO\big(\mathbb{F},\hat{P}\big)$-martingale and thus so is $\widetilde{N}^H$ because

$$\big\langle \widetilde{N}^H \big\rangle = \frac{1}{1-|\rho|^2}\int |\zeta|^2 \,\mathrm{d}s = \frac{1}{1-|\rho|^2}\left\langle \int \zeta \,\mathrm{d}\hat{Y} \right\rangle.$$

This implies that $\mathcal{E}\big(\widetilde{N}^H\big)$ is an $\big(\mathbb{F},\hat{P}\big)$-martingale so that $\mathcal{E}\big(\widetilde{N}^H\big)\mathcal{E}\big(-\int \lambda \,\mathrm{d}M\big)$ is an $(\mathbb{F},P)$-martingale, and then that also

$$\int \big(\gamma\widetilde{\eta}^H + \lambda\big)\,\mathrm{d}S = \int \gamma\widetilde{\eta}^H \sigma S \,\mathrm{d}\hat{W} + \int \frac{\mu}{\sigma}\,\mathrm{d}\hat{W} = \frac{1}{1-|\rho|^2}\int \zeta \,\mathrm{d}\hat{Y} - \widetilde{N}^H + \int \frac{\mu}{\sigma}\,\mathrm{d}\hat{W}$$

is a $BMO\big(\mathbb{F},\hat{P}\big)$-martingale. So if we set $\frac{\mathrm{d}P(N^H)}{\mathrm{d}\hat{P}} = \mathcal{E}\big(\widetilde{N}^H\big)_T$, then the process $\int \big(\widetilde{\eta}^H + \frac{1}{\gamma}\lambda\big)\,\mathrm{d}S$ is also a $BMO\big(\mathbb{F},P\big(N^H\big)\big)$-martingale by Theorem 3.6 of Kazamaki [40]. By Proposition 3.2, $\big(\widetilde{N}^H - \int \frac{\mu}{\sigma}\,\mathrm{d}W, \widetilde{\eta}^H + \frac{\mu}{\gamma\sigma}\frac{1}{\sigma S}, k_t^H\big)$ is thus an $FER(H)$ on $[t,T]$, and because the P-density process of $\hat{P}$ satisfies $R_{L\log L}(P)$ since $\frac{\mu}{\sigma}$ is bounded, this $FER(H)$ is even the unique $FER^\star(H)$ on $[t,T]$ by Theorem 3.9. The unique $FER^\star(0)$ $\big(N^0,\eta^0,k_t^0\big)$ on $[t,T]$ is constructed analogously, with Ψ_t^H replaced by Ψ_t^0. This concludes the proof in view of Theorem 3.10. □

Remark 3.27. Proposition 3.26 can be extended to the more general framework of case (I) in Section 2.4.1 where the correlation ρ is no longer constant, but $\mathbb{Y}$-predictable with absolute value uniformly bounded away from one. The explicit form of the indifference value is then essentially preserved; see Theorem 2.9 for the precise formulation. This can also be proved with our methods here, but we only sketch the main steps for $t=0$ since the full details are a bit technical. First, one calls a triple $\big(N^H,\eta^H,k_0^H\big)$ an *upper* (or *lower*) $FER^\star(H)$ if it has the properties of an $FER^\star(H)$, except that the equality sign in (3.7) is replaced by "$\geq$" (or "$\leq$"). One then shows that for an upper (lower) $FER^\star(H)$, (3.28) is satisfied with "$\leq$" ("$\geq$") instead of equality. In a third step, one defines constants

$$\overline{\delta} := \sup_{s\,\in\,[0,T]} \left\| \frac{1}{1-|\rho_s|^2} \right\|_{L^\infty(P)} \quad \text{and} \quad \underline{\delta} := \inf_{s\,\in\,[0,T]} \frac{1}{\|1-|\rho_s|^2\|_{L^\infty(P)}}$$

and finds, in the spirit of (3.64), $\mathbb{Y}$-predictable processes $\overline{\zeta}$ and $\underline{\zeta}$ such that

$$\big|\Psi_0^H\big|^{1/\overline{\delta}} = E_{\hat{P}}\Big[\big|\Psi_0^H\big|^{1/\overline{\delta}}\Big]\mathcal{E}\bigg(\int \overline{\zeta}\,\mathrm{d}\hat{Y}\bigg)_T \quad \text{and} \quad E_{\hat{P}}\bigg[\int_0^T \big|\overline{\zeta}_s\big|^2\,\mathrm{d}s\bigg] < \infty,$$

with an analogous construction for $\underline{\zeta}$. For this one uses that $\hat{Y}$ is $\mathbb{Y}$-adapted because ρ is $\mathbb{Y}$-predictable. Similarly to the proof of Proposition 3.26, one shows that $\left(\overline{N}^H, \overline{\eta}^H, \overline{k}_0^H\right)$ is an upper $FER^\star(H)$, where

$$\overline{N}^H = \int \overline{\delta}\,\overline{\zeta}\sqrt{1-|\rho|^2}\,\mathrm{d}\hat{W}^\perp - \int \frac{\mu}{\sigma}\,\mathrm{d}W, \quad \overline{\eta}^H = \frac{\overline{\delta}\rho\,\overline{\zeta}}{\gamma}\frac{1}{\sigma S} + \frac{\mu}{\gamma\sigma}\frac{1}{\sigma S}$$

and $\overline{k}_0^H = \frac{\overline{\delta}}{\gamma}\log E_{\hat{P}}\left[\left|\Psi_0^H\right|^{1/\overline{\delta}}\right]$. A completely analogous result holds for $\underline{\delta}$. Therefore, one obtains

$$-\exp\left(-\gamma x_0 + \gamma \underline{k}_0^H\right) \le V_0^H(x_0) \le -\exp\left(-\gamma x_0 + \gamma \overline{k}_0^H\right)$$

by the above versions of (3.28). Because $\delta \mapsto \delta \log E_{\hat{P}}\left[\left|\Psi_0^H\right|^{1/\delta}\right]$ is continuous on $\left[\underline{\delta}, \overline{\delta}\right]$, interpolation then yields the existence of $\delta_0^H \in \left[\underline{\delta}, \overline{\delta}\right]$ such that

$$V_0^H(x_0) = -\exp(-\gamma x_0) E_{\hat{P}}\left[\left|\Psi_0^H\right|^{1/\delta_0^H}\right]^{\delta_0^H}$$

Solving the implicit equation (3.3) with respect to h_0 finally gives an explicit expression for h_0. $\diamondsuit$

3.7 Appendix A: The distortion power δ_t^H in a continuous filtration

In this appendix, we come back to the distortion power δ_t^H used in Theorem 3.12 and Corollary 3.13 in Section 3.4. If the filtration $\mathbb{F}$ is continuous, that is, all local P-martingales are continuous, then we can make more precise the range of δ_t^H. In the next proposition and its proof, we use the abbreviation $\langle N\rangle_{t,T} := \langle N\rangle_T - \langle N\rangle_t$ for a continuous P-semimartingale N, differing from the notation $Z_{t,s} := Z_s/Z_t$, $0 \le t \le s \le T$.

Proposition 3.28. *Assume that $\mathbb{F}$ is continuous, S satisfies (SC) and that $\left\langle \int \lambda\,\mathrm{d}M\right\rangle_T$ and H are bounded. Let $\left(N^H, \eta^H, k_0^H\right)$ be the $FER^\star(H)$, set $\widetilde{N}^H := N^H + \int \lambda\,\mathrm{d}M$ and $\widetilde{\eta}^H := \eta^H - \frac{1}{\gamma}\lambda$ and fix $t \in [0,T]$. Then δ_t^H defined via (3.47) with the choice $Q := \hat{P}$ and $\varphi := -\lambda$ is valued almost surely in $\left[\underline{\delta}_t^H, \overline{\delta}_t^H\right]$, where*

$$\overline{\delta}_t^H := \left\| \frac{\langle \widetilde{N}^H\rangle_{t,T} + \langle \int \gamma\widetilde{\eta}^H\,\mathrm{d}S\rangle_{t,T}}{\langle \widetilde{N}^H\rangle_{t,T}} \right\|_{L^\infty}, \quad \frac{1}{\underline{\delta}_t^H} := \left\| \frac{\langle \widetilde{N}^H\rangle_{t,T}}{\langle \widetilde{N}^H\rangle_{t,T} + \langle \int \gamma\widetilde{\eta}^H\,\mathrm{d}S\rangle_{t,T}} \right\|_{L^\infty}; \tag{3.65}$$

if $P\big[\langle \widetilde{N}^H\rangle_{t,T} + \langle \int \gamma\widetilde{\eta}^H \,\mathrm{d}S\rangle_{t,T} = 0\big] > 0$, *we set* $\underline{\delta}_t^H = 1$ *and* $\overline{\delta}_t^H = \infty$ *by convention.*

Proof. If $\delta_t^H = \infty$ with positive P-probability, then

$$E_{\hat{P}}\Big[\log \mathcal{E}\big(\widetilde{N}^H\big)_{t,T}\Big|\mathcal{F}_t\Big] = E_{\hat{P}}\left[\log \frac{\mathcal{E}\big(N^H\big)_{t,T}}{\mathcal{E}(-\int\lambda\,\mathrm{d}M)_{t,T}}\middle|\mathcal{F}_t\right] = 0 \quad \text{on a set } A \in \mathcal{F}_t \tag{3.66}$$

with $P[A] > 0$ by (3.42) and (3.43) with $Q = \hat{P}$, and because $\widetilde{N}^H$ is strongly P-orthogonal to $\int\lambda\,\mathrm{d}M$ by Proposition 3.2. Since $\big\langle \int\lambda\,\mathrm{d}M\big\rangle_T$ is bounded, the P-density process of $\hat{P}$ satisfies $R_{L\log L}(P)$, and Theorem 3.9 yields that N^H is a $BMO(P)$-martingale. Therefore, $\widetilde{N}^H = N^H + \int\lambda\,\mathrm{d}M$ is both a $BMO(P)$- and a $BMO\big(\hat{P}\big)$-martingale by Theorem 3.6 of Kazamaki [40]. Jensen's inequality and Theorem 2.3 of Kazamaki [40] yield

$$\begin{aligned} E_{\hat{P}}\Big[\mathbb{1}_A \log \mathcal{E}\big(\widetilde{N}^H\big)_{t,T}\Big] &= E_{\hat{P}}\Big[\log\Big(\mathcal{E}\big(\widetilde{N}^H\big)_{t,T}\mathbb{1}_A + \mathbb{1}_{A^c}\Big)\Big] \\ &\le \log\Big(E_{\hat{P}}\Big[\mathbb{1}_A E_{\hat{P}}\big[\mathcal{E}\big(\widetilde{N}^H\big)_{t,T}\big|\mathcal{F}_t\big]\Big] + \hat{P}[A^c]\Big) = 0. \end{aligned} \tag{3.67}$$

Now (3.66) shows that we have equality in (3.67). Hence $\mathcal{E}\big(\widetilde{N}^H\big)_{t,T}\mathbb{1}_A + \mathbb{1}_{A^c}$ is deterministic, so $\mathcal{E}\big(\widetilde{N}^H\big)_{t,T} = 1$ on A and thus $\big\langle\widetilde{N}^H\big\rangle_{t,T} = 0$ on A, which implies $\overline{\delta}_t^H = \infty$. Consequently, we may assume without loss of generality that $\delta_t^H < \infty$ a.s.

In view of the proof of Theorem 3.12 and using (3.39), we have to show

$$\left(E_{\hat{P}}\left[\big|\Psi_t^H\big|^{\frac{1}{\overline{\delta}_t^H}}\middle|\mathcal{F}_t\right]\right)^{\overline{\delta}_t^H} \le \exp\big(\gamma k_t^H\big) \le \left(E_{\hat{P}}\left[\big|\Psi_t^H\big|^{\frac{1}{\underline{\delta}_t^H}}\middle|\mathcal{F}_t\right]\right)^{\underline{\delta}_t^H}, \tag{3.68}$$

where $\Psi_t^H = \exp\big(\gamma H - \frac{1}{2}\langle\int\lambda\,\mathrm{d}M\rangle_{t,T}\big)$ is given in (3.49). To establish the first half of (3.68), we can assume $\overline{\delta}_t^H < \infty$ (otherwise there is nothing to prove). As in Proposition 3.2, we deduce from (3.10) and the definitions of $\widetilde{N}^H$ and $\widetilde{\eta}^H$ that

$$\begin{aligned} \log\Psi_t^H &= \gamma k_t^H + \log\mathcal{E}\big(\widetilde{N}^H\big)_{t,T} + \int_t^T \gamma\widetilde{\eta}_s^H\,\mathrm{d}S_s \\ &= \gamma k_t^H + \widetilde{N}_T^H - \widetilde{N}_t^H - \frac{1}{2}\big\langle\widetilde{N}^H\big\rangle_{t,T} + \int_t^T \gamma\widetilde{\eta}_s^H\,\mathrm{d}S_s. \end{aligned}$$

Therefore, we have by (3.65) that

$$\begin{aligned} &\log\Psi_t^H \\ &\le \gamma k_t^H + \widetilde{N}_T^H - \widetilde{N}_t^H - \frac{1}{2\,\overline{\delta}_t^H}\left(\big\langle\widetilde{N}^H\big\rangle_{t,T} + \left\langle\int\gamma\widetilde{\eta}^H\,\mathrm{d}S\right\rangle_{t,T}\right) + \int_t^T \gamma\widetilde{\eta}_s^H\,\mathrm{d}S_s. \end{aligned} \tag{3.69}$$

By Proposition 3.2, $\widetilde{N}^H$ is strongly P-orthogonal to each component of M, which implies $\langle \widetilde{N}^H, \int \gamma\widetilde{\eta}^H \,\mathrm{d}S\rangle = 0$ by (SC) and the continuity of S. So (3.69) leads to

$$\big|\Psi_t^H\big|^{\frac{1}{\overline{\delta}_t^H}} \leq \exp\left(\frac{\gamma k_t^H}{\overline{\delta}_t^H}\right)\mathcal{E}\left(\frac{\widetilde{N}^H + \int \gamma\widetilde{\eta}^H \,\mathrm{d}S}{\overline{\delta}_t^H}\right)_{t,T}. \tag{3.70}$$

We have already seen that $\widetilde{N}^H$ is a $BMO(\hat{P})$-martingale. Similarly, $\int \eta^H \,\mathrm{d}M$ is a $BMO(P)$-martingale by Theorem 3.9, hence $\int \eta^H \,\mathrm{d}S$ is a $BMO(\hat{P})$-martingale by Kazamaki's Theorem 3.6. Moreover, $\int \lambda \,\mathrm{d}S$ is a $BMO(\hat{P})$-martingale since $\langle \int \lambda \,\mathrm{d}M\rangle_T$ is bounded by assumption. Thus $\widetilde{N}^H + \int \gamma\widetilde{\eta}^H \,\mathrm{d}S$ is a $BMO(\hat{P})$-martingale, and we obtain from (3.70) that

$$E_{\hat{P}}\left[\big|\Psi_t^H\big|^{\frac{1}{\overline{\delta}_t^H}}\,\middle|\,\mathcal{F}_t\right] \leq \mathrm{e}^{\gamma k_t^H/\overline{\delta}_t^H} E_{\hat{P}}\left[\mathcal{E}\left(\frac{\widetilde{N}^H + \int \gamma\widetilde{\eta}^H \,\mathrm{d}S}{\overline{\delta}_t^H}\right)_{t,T}\,\middle|\,\mathcal{F}_t\right] = \mathrm{e}^{\gamma k_t^H/\overline{\delta}_t^H}$$

by Theorem 2.3 of Kazamaki [40]. The second inequality in (3.68) is proved analogously, using that $\underline{\delta}_t^H = \infty$ implies $\langle \widetilde{N}^H\rangle_{t,T} = 0$ and $\mathcal{E}(\widetilde{N}^H)_{t,T} = 1$ a.s., hence $\delta_t^H = \infty$ a.s. by (3.42) and (3.43). □

3.8 Appendix B: Specialisation to the model of Becherer [5]

In this appendix, we recover Theorem 4.4 of Becherer [5] from our Theorem 3.16. This works in two steps, like in the approach used in Section 3.5 to regain Theorem 13 of Mania and Schweizer [44]. In the first step, we show for the specific models that if (Γ, ψ, L) is an orthogonal solution of the BSDE (3.51), (3.52) with bounded Γ, then L and $\int \psi \,\mathrm{d}S$ are $BMO(Q_0^E)$-martingales and $\mathcal{E}(L)$ is a Q_0^E-martingale which satisfies condition (J). This is stated, for the setting of [44], in Lemma 3.24 in Section 3.5 and, for [5], in Lemma 3.29 below, which additionally imposes that L and $\int \psi \,\mathrm{d}S$ are square-integrable Q_0^E-martingales. In the second step, we use Theorem 3.16 and Corollary 3.23 to show uniqueness of an orthogonal solution (Γ, ψ, L) with bounded Γ (and square-integrable Q_0^E-martingales L and $\int \psi \,\mathrm{d}S$ for the setting of [5]). This procedure yields Proposition 3.25, corresponding to Theorem 13 in [44], and Proposition 3.30 below, corresponding to Theorem 4.4 in [5].

The framework of Becherer [5] is a specialisation of our model presented in Section 3.2. For convenience of the reader, we recall the model of [5]. It consists of a d-dimensional Brownian motion W and an integer-valued random measure μ on $\big([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{B}(E)\big)$ with compensator ν^P under P, where $E := \mathbb{R}^\ell \backslash \{0\}$ is equipped with its Borel σ-field $\mathcal{B}(E)$. Set $\widetilde{\mu}^P := \mu - \nu^P$ and denote by $\mathcal{P}$ the predictable σ-field on $\Omega \times [0,T]$. In [5], ν^P is supposed to be equivalent to a product measure $\lambda \otimes \mathrm{Leb}$ with a density ζ^P such that

$$\nu^P(\omega, \mathrm{d}t, \mathrm{d}e) = \zeta^P(\omega, t, e)\lambda(\mathrm{d}e)\,\mathrm{d}t, \quad \omega \in \Omega,\ t \in [0,T],\ e \in E, \tag{3.71}$$

where λ is a bounded measure on $\big(E, \mathcal{B}(E)\big)$, and the density ζ^P is a bounded, nonnegative and $\big(\mathcal{P} \otimes \mathcal{B}(E)\big)$-measurable function such that for a constant c^P,

$$0 \le \zeta^P \le c^P < \infty, \quad (P \otimes \lambda \otimes \mathrm{Leb})\text{-almost everywhere.} \tag{3.72}$$

It is further assumed that, with respect to $\mathbb{F}$ and P, W and $\widetilde{\mu}^P$ have the weak property of representation. This means that every square-integrable P-martingale L has a representation $L = L_0 + \int Z\,\mathrm{d}W + A * \widetilde{\mu}^P$, where Z is $\mathcal{P}$- and A is $\big(\mathcal{P} \otimes \mathcal{B}(E)\big)$-measurable such that $E_P\big[\int_0^T |Z_t|^2\,\mathrm{d}t\big] < \infty$ and $E_P\big[|A|^2 * \nu^P_T\big] < \infty$, and $A * \widetilde{\mu}^P$ is the integral process of A with respect to $\widetilde{\mu}^P$.

The price process S is given by

$$\mathrm{d}S^i_t = S^i_t \sum_{j=1}^d \sigma^{ij}_t \big(\varphi^j_t\,\mathrm{d}t + \mathrm{d}W^j_t\big), \quad t \in [0,T], \quad S^i_0 > 0, \quad i = 1, \dots, d,$$

where φ is an $\mathbb{R}^d$-valued $\mathcal{P}$-measurable process which is bounded $(P \otimes \mathrm{Leb})$-a.e. and σ is an $\mathbb{R}^{d \times d}$-valued $\mathcal{P}$-measurable process such that σ is invertible $(P \otimes \mathrm{Leb})$-a.e. and integrable with respect to $\hat{W} := W + \int \varphi\,\mathrm{d}t$. The minimal local martingale measure $\hat{P}$ given by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} = \mathcal{E}\big(-\int \varphi\,\mathrm{d}W\big)_T$ satisfies $R_{L\log L}(P)$ since φ is bounded.

Lemma 3.29. *Consider the above specialisation of the model and assume that H is bounded. Let (Γ, ψ, L) be an orthogonal solution of the BSDE (3.51), (3.52) with bounded Γ and square-integrable Q^E_0-martingales L and $\int \psi\,\mathrm{d}S$. Then L and $\int \psi\,\mathrm{d}S$ are $BMO\big(Q^E_0\big)$-martingales and $\mathcal{E}(L)$ is a Q^E_0-martingale which satisfies condition (J).*

Proof. We first study the form of Q^E_0. By Proposition 3.6, Q^E_0 is given by $\frac{\mathrm{d}Q^E_0}{\mathrm{d}P} = \mathcal{E}\big(N^0\big)_T$, where N^0 is the first component of the unique $FER^\star(0)$ $\big(N^0, \eta^0, k^0_0\big)$. Let us write the equivalent form (3.11) of $FER^\star(0)$ from Proposition 3.2 for $t \in [0,T]$ as

$$\widetilde{k}^0_t = \frac{1}{\gamma} \log \mathcal{E}\big(\widetilde{N}^0\big)_t + \int_0^t \widetilde{\eta}^0_s\,\mathrm{d}S_s + k^0_0, \quad \widetilde{k}^0_T = -\frac{1}{2\gamma}\left\langle \int \varphi\,\mathrm{d}W \right\rangle_T, \tag{3.73}$$

where $\widetilde{N}^0 = N^0 + \int \varphi \,\mathrm{d}W$ and $\widetilde{\eta}^0 = \eta^0 - \frac{1}{\gamma}\mathrm{diag}(S)^{-1}(\sigma')^{-1}\varphi$. The process $\widetilde{k}^0$ is bounded; indeed, (3.73) and Jensen's inequality yield for $t \in [0,T]$ that

$$\mathrm{e}^{\gamma \widetilde{k}^0_t} \le E_{Q_0^E}\left[\mathcal{E}\big(\widetilde{N}^0\big)_t \exp\!\left(\gamma k^0_0 + \gamma \int_0^T \widetilde{\eta}^0_s \,\mathrm{d}S_s\right)\middle|\mathcal{F}_t\right] = E_{\hat{P}}\left[\mathrm{e}^{\gamma \widetilde{k}^0_T}\middle|\mathcal{F}_t\right] \le 1.$$

Because $-\widetilde{k}^0_T = \frac{1}{2\gamma}\int_0^T |\varphi_s|^2 \,\mathrm{d}s$ is bounded, say by c, we similarly obtain for $t \in [0,T]$ that

$$\mathrm{e}^{-\gamma \widetilde{k}^0_t} \le E_{\hat{P}}\left[\frac{1}{\mathcal{E}\big(\widetilde{N}^0\big)_t} \exp\!\left(-\gamma k^0_0 - \gamma \int_0^T \widetilde{\eta}^0_s \,\mathrm{d}S_s\right)\middle|\mathcal{F}_t\right] = E_{Q_0^E}\left[\mathrm{e}^{-\gamma \widetilde{k}^0_T}\middle|\mathcal{F}_t\right] \le \mathrm{e}^{\gamma c},$$

using that $\int \widetilde{\eta}^0 \,\mathrm{d}S$ is a $\big(BMO(\hat{P})\text{-}\big)$martingale by Theorem 3.9 above and Proposition 7 of Doléans-Dade and Meyer [17]; hence $\widetilde{k}^0$ is bounded. Moreover, Theorem 3.9 yields that N^0 is a $BMO(P)$-martingale. Because of the weak property of representation and since Q_0^E is a local martingale measure, we can write $N^0 = -\int \varphi \,\mathrm{d}W + A * \widetilde{\mu}^P$ for a $\big(\mathcal{P}\otimes\mathcal{B}(E)\big)$-measurable A such that $E_{Q_0^E}\big[|A|^2 * \nu^P_T\big] < \infty$. Using the formula for the stochastic exponential, we can derive from (3.73) that

$$\begin{aligned}\widetilde{k}^0_t = \frac{1}{\gamma}\int_0^t\int_E \log\big(A_s(e)+1\big)\widetilde{\mu}^P(\mathrm{d}s,\mathrm{d}e) + \int_0^t \Big(\eta^0_s - \frac{1}{\gamma}\mathrm{diag}(S_s)^{-1}(\sigma'_s)^{-1}\varphi_s\Big)\,\mathrm{d}S_s \\ - \frac{1}{\gamma}\int_0^t\int_E \Big(A_s(e) - \log\big(A_s(e)+1\big)\Big)\nu^P(\mathrm{d}s,\mathrm{d}e) + k^0_0,\ \ t \in [0,T]. \quad (3.74)\end{aligned}$$

The jumps of $\widetilde{k}^0$ are given by

$$\Delta\widetilde{k}^0_t = \frac{1}{\gamma}\int_E \log\big(A_t(e)+1\big)\mu\big(\{t\},\mathrm{d}e\big), \quad t \in [0,T],$$

which is bounded uniformly in t since so is $\widetilde{k}^0$. Because μ is an integer-valued random measure, $\log(A+1)$ is bounded $(\mu\otimes P)$-a.e.; hence A is bounded away from -1 and ∞ $(\mu\otimes P)$-a.e. and thus also $(\nu^P\otimes P)$-a.e. By (4.24) in Becherer [5], the Q_0^E-compensator $\nu^{Q_0^E}$ of μ is given by

$$\nu^{Q_0^E}(\mathrm{d}t,\mathrm{d}e) = (A_t(e)+1)\nu^P(\mathrm{d}t,\mathrm{d}e), \quad t \in [0,T],\ e \in E, \tag{3.75}$$

and $\nu^{Q_0^E}\big([0,T],E\big)$ is bounded due to (3.71), (3.72) and since λ is bounded by assumption and A is bounded $(\nu^P\otimes P)$-a.e.

We now study an orthogonal solution (Γ,ψ,L) of the BSDE (3.51), (3.52) with bounded Γ and square-integrable Q_0^E-martingales L and $\int \psi \,\mathrm{d}S$. By Theorem 13.22 of He et al. [30], we have $L = \int Z \,\mathrm{d}\hat{W} + D * \widetilde{\mu}^{Q_0^E}$ for a $\mathcal{P}$-measurable Z and a $\big(\mathcal{P}\otimes\mathcal{B}(E)\big)$-measurable D with $E_{Q_0^E}\Big[\int_0^T |Z_t|^2 \,\mathrm{d}t\Big] < \infty$

and $E_{Q_0^E}\Big[|D|^2 * \nu_T^{Q_0^E}\Big] < \infty$, because W and $\widetilde{\mu}^P$ have the weak property of representation with respect to $\mathbb{F}$ and P. Since L is strongly Q_0^E-orthogonal to every component of S and σ is invertible, we have $Z \equiv 0$. Similarly to (3.74), we can write (3.51) as

$$\begin{aligned}\Gamma_t = \Gamma_0 &+ \frac{1}{\gamma}\int_0^t \int_E \log\big(D_s(e)+1\big)\widetilde{\mu}^{Q_0^E}(\mathrm{d}s,\mathrm{d}e) + \int_0^t \psi_s\,\mathrm{d}S_s \\ &- \frac{1}{\gamma}\int_0^t \int_E \Big(D_s(e) - \log\big(D_s(e)+1\big)\Big)\nu^{Q_0^E}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,T]. \end{aligned} \tag{3.76}$$

Because $\nu^{Q_0^E}$ is equivalent to $\lambda\otimes$Leb by (3.71) and (3.75), the jumps of Γ are given by

$$\Delta\Gamma_t = \frac{1}{\gamma}\int_E \log\big(D_t(e)+1\big)\mu\big(\{t\},\mathrm{d}e\big), \quad t\in[0,T],$$

which is bounded uniformly in t since so is Γ. Because μ is an integer-valued random measure, $\log(D+1)$ is bounded $(\mu\otimes P)$-a.e., and thus D is bounded away from -1 and ∞ $(\mu\otimes P)$-a.e. and also $\big(\nu^{Q_0^E}\otimes P\big)$-a.e. by (3.75). Therefore, the process $\int\int_E \Big(D_s(e)-\log\big(D_s(e)+1\big)\Big)\nu^{Q_0^E}(\mathrm{d}s,\mathrm{d}e)$ is bounded since $\nu^{Q_0^E}\big([0,T]\times E\big)$ is bounded. This implies by (3.76) that

$$\frac{1}{\gamma}\int_0^t \int_E \log\big(D_s(e)+1\big)\widetilde{\mu}^{Q_0^E}(\mathrm{d}s,\mathrm{d}e) + \int_0^t \psi_s\,\mathrm{d}S_s \ \text{ is bounded unif. in } t\in[0,T]$$

and thus both $\int \psi\,\mathrm{d}S$ and $\log(D+1)*\widetilde{\mu}^{Q_0^E}$ are $BMO\big(Q_0^E\big)$-martingales by the same argument as in Lemma 3.4 of Becherer [5]. Since μ is an integer-valued random measure and D is bounded away from -1 and ∞, $L = D*\widetilde{\mu}^{Q_0^E}$ has jumps bounded away from -1 and ∞, and $\langle L\rangle_T = \int_0^T\int_E |D_s(e)|^2\,\nu^{Q_0^E}(\mathrm{d}s,\mathrm{d}e)$ is bounded since so is $\nu^{Q_0^E}\big([0,T]\times E\big)$. Therefore, L is a $BMO\big(Q_0^E\big)$-martingale and $\mathcal{E}(L)$ is a Q_0^E-martingale which satisfies condition (J) by Shimbo (cited in Protter [49] on p. 142 in the second remark after Theorem III.45) and Proposition 6 of Doléans-Dade and Meyer [17]. □

We can now recover Theorem 4.4 of Becherer [5], which we formulate as a proposition. It can here be proved like Proposition 3.25, applying Lemma 3.29 instead of Lemma 3.24 and using that the minimal local martingale measure $\hat{P}$ satisfies $R_{L\log L}(P)$.

Proposition 3.30. *Consider the above specialisation of the model and assume that H is bounded. Then the indifference value process h is the first component of the unique orthogonal solution (Γ,ψ,L) of (3.51), (3.52) with bounded Γ and square-integrable Q_0^E-martingales L and $\int\psi\,\mathrm{d}S$. Moreover, L and $\int\psi\,\mathrm{d}S$ are $BMO\big(Q_0^E\big)$-martingales.*

Chapter 4

Convexity bounds for BSDE solutions

Proving new results for Brownian BSDEs with a particular quadratic generator, we derive in this chapter bounds for the indifference value in a multidimensional Brownian model.

4.1 Introduction

Backward stochastic differential equations (BSDEs) play an important role in mathematical finance; see El Karoui et al. [26] for an early overview. Existence and uniqueness results are well known both for Lipschitz and for quadratic drivers; see Kobylanski [42]. In this chapter, we study a particular class of quadratic BSDEs of the form

$$\Gamma_s = H - \int_s^T \big(f(\Lambda_r, Z_r + \alpha_r) + \chi_r\big)\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}B_r, \quad 0 \le s \le T, \tag{4.1}$$

where $f(\Lambda_r, Z_r + \alpha_r) := \frac{1}{2}(Z_r + \alpha_r)'\Lambda_r^{-1}(Z_r + \alpha_r)$ and the processes χ, α, Λ take values in $\mathbb{R}$, $\mathbb{R}^n$ and the set $\mathcal{S}^n$ of symmetric strictly positive definite matrices, respectively. Since there is no general formula for the solution Γ of (4.1), we want to find bounds on Γ that can be computed more explicitly. To that end, we first show that $f(A, z)$ is jointly convex, deduce that Γ is jointly concave in $(H, \Lambda, \alpha, \chi)$, and then prove convexity bounds via three different routes, as follows.

In general, a BSDE is based on a probability space, a filtration and a probability measure. By changing in (4.1) each of these ingredients in a suitable way, we obtain other BSDEs whose solutions are upper bounds for Γ due to concavity. Finding bounds for these changed BSDEs or solving them is easier than for the original (4.1), because they are driven by a lower-dimensional Brownian motion or, in some sense, their matrix-valued process Λ is more regular.

We start by changing the probability measure. Our first main result, Theorem 4.5, characterises Γ as the essential infimum and supremum of certain conditional expectations. In particular, it gives upper bounds for Γ, which depend on the maximal eigenvalue of Λ. This shows that Λ is the crucial factor in finding good bounds, or even an explicit formula for Γ. The latter is easy if $\Lambda = cI$ for some constant c, and we prove in Corollary 4.6 that the converse holds as well. As a consequence, we then focus on improving the form of Λ by projecting and/or symmetrising the BSDE (4.1).

For the projection, we change the filtration. The solution Γ of (4.1) relates to the filtration $\mathbb{F}^B$ generated by $B = \big(\overline{B}, \underline{B}\big)'$, and our second main result, Theorem 4.7, gives an upper bound for Γ in terms of the solution $\overline{\Gamma}$ to the BSDE ($\overline{4.1}$) obtained by projecting (4.1) onto $\mathbb{F}^{\overline{B}}$. The projected BSDE ($\overline{4.1}$) is in general easier to solve and the maximal eigenvalue of $\overline{\Lambda}$ is lower because the dimension $\overline{n}$ of $\overline{B}$ is smaller.

Finally, we change the probability space. We work on Wiener space and study how symmetrisation operations via orthogonal transformations there affect the BSDE (4.1). Our third main result, Theorem 4.11, gives an explicit upper bound for Γ in terms of the symmetrised parameters $(H, \Lambda, \alpha, \chi)^{\mathrm{Sym}}$. The proof combines Theorem 4.5 with a result showing that, due to concavity, averaging the probability space over a set of orthogonal transformations increases the solution of (4.1).

This chapter is structured as follows. We lay out preliminaries and prove the basic concavity property in Section 4.2.1. All our main results for the BSDE (4.1) have analogues in terms of solutions to partial differential equations (PDEs), which actually provided the original motivation and inspiration; see for instance Alvino et al. [1]. Section 4.2.2 discusses these connections in some more detail, and Section 4.3 contains the main results explained above. In Section 4.4, we briefly recall the concept of exponential utility indifference valuation for a contingent claim H in an incomplete financial market. It is well known that the corresponding dynamic value process V^H, or rather $\Gamma = -\frac{1}{\gamma}\log\big(-V^H\big)$, satisfies a quadratic BSDE; see for instance Hu et al. [37]. But since this BSDE is *not* of the form (4.1), we still have to do some work in Section 4.5 before we can apply our main results. We also discuss there in a concrete example why the symmetrisation techniques may, but need not lead to better bounds for Γ. Finally, the Appendix contains some proofs and auxiliary results.

4.2 A quadratic convex BSDE

This section serves as preparation for the main results. We first introduce notation and show some properties of quadratic BSDEs in Section 4.2.1, and then motivate in Section 4.2.2 the BSDE results of Section 4.3 by presenting their PDE analogues.

4.2.1 Preliminaries

We work on a finite time interval $[0,T]$ for a fixed $T>0$ and a filtered probability space $\big(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_s)_{0\le s\le T},P\big)$, where $\mathcal{F}=\mathcal{F}_T$ and $\mathbb{F}$ is the augmented filtration generated by an n-dimensional Brownian motion B. Unless specified differently, all notions depending on a filtered probability space refer in Sections 4.2 and 4.3 to $(\Omega,\mathcal{F},\mathbb{F},P)$, *and* $t\in[0,T]$ *is fixed.* For $(n\times n)$-matrices, we denote by $\mathcal{S}^n$ the set of symmetric strictly positive definite ones, by $\mathrm{GL}(n)$ and $\mathrm{O}(n)$ the invertible respectively orthogonal ones, and by I the identity. For a diagonalisable matrix A, we write $\mathrm{spec}(A)$ for the spectrum (the set of eigenvalues) and $\mathrm{tr}(A)$ for the trace of A. We shall use several times that standard operations from linear algebra can be done in a measurable way. This includes eigenvalues, eigenvectors and diagonalisation; see Corollary 4 of Azoff [3]. Finally, we denote by $\mathcal{E}(N)_s:=\exp\big(N_s-\frac{1}{2}\langle N\rangle_s\big)$, $0\le s\le T$, the stochastic exponential of a continuous semimartingale N.

Let us consider the BSDE

$$\Gamma_s=H-\int_s^T\big(f(\Lambda_r,Z_r+\alpha_r)+\chi_r\big)\,\mathrm{d}r+\int_s^T Z_r\,\mathrm{d}B_r,\quad 0\le s\le T,\tag{4.2}$$

where the function $f:\mathcal{S}^n\times\mathbb{R}^n\to\mathbb{R}$ is given by

$$f(A,z):=\frac{1}{2}z'A^{-1}z\quad\text{for }(A,z)\in\mathcal{S}^n\times\mathbb{R}^n.\tag{4.3}$$

The terminal value H is (usually) in L^∞, the process Λ is $\mathcal{S}^n$-valued and predictable with eigenvalues uniformly bounded away from zero and infinity, and α, χ are $\mathbb{R}^n$-, $\mathbb{R}$-valued uniformly bounded predictable processes. A *(generalised) solution* of (4.2) is a pair (Γ,Z) satisfying (4.2), where Γ is a real-valued *(not necessarily) bounded* continuous semimartingale and Z is an $\mathbb{R}^n$-valued predictable process with $\int_0^T|Z_s|^2\,\mathrm{d}s<\infty$ almost surely. To emphasise the dependence on H, Λ, α and χ, we write $\big(\Gamma(H,\Lambda,\alpha,\chi),Z(H,\Lambda,\alpha,\chi)\big)$ for a solution of (4.2), and we sometimes call $\Gamma(H,\Lambda,\alpha,\chi)$ alone a solution of (4.2).

Remark 4.1. For ease of exposition, we formulate and prove all our results for bounded data H, Λ, α, χ. Extensions to unbounded settings with exponential moment conditions are partly possible; this is discussed in more detail in Remark 4.12. ◊

Lemma 4.2. *There exists a unique solution* (Γ, Z) *of (4.2), and* $\int Z\,\mathrm{d}B$ *is a* BMO*-martingale.*

Proof. Existence follows from Theorem 2.3 of Kobylanski [42], and uniqueness and the BMO-property from Proposition 7 and Theorem 8 of Mania and Schweizer [44]. □

In Lemma 4.21 in the Appendix, we show that f is jointly convex. This is the basis for the following result.

Proposition 4.3. *The solution* $\Gamma(H, \Lambda, \alpha, \chi)$ *of (4.2) is jointly concave in* $(H, \Lambda, \alpha, \chi)$.

Remark 4.4. It is BSDE folklore that convexity of the generator implies (under some assumptions) that the solution is concave; see for instance Proposition 3.5 of El Karoui et al. [26], where the generator is fairly general, but must satisfy a Lipschitz condition in Z_r and in Γ_r. We need the variant in Proposition 4.3 with a specific quadratic generator for our later results. ◊

Proof of Proposition 4.3. Let $\mu \in [0,1]$, $H^i \in L^\infty$, let Λ^i be predictable $\mathcal{S}^n$-valued with eigenvalues bounded away from zero and infinity and let bounded predictable α^i be $\mathbb{R}^n$-valued and χ^i be $\mathbb{R}$-valued, $i = 1, 2$. We set $H^3 := \mu H^1 + (1-\mu)H^2$, define Λ^3, α^3, χ^3 analogously and denote by (Γ^i, Z^i), $i = 1, 2, 3$, the solutions of (4.2) corresponding to $(H^i, \Lambda^i, \alpha^i, \chi^i)$. By Lemma 4.2, each of these is unique and $\int Z^i\,\mathrm{d}B$ are BMO-martingales. Since $\mu\Gamma^1_T + (1-\mu)\Gamma^2_T = \mu H^1 + (1-\mu)H^2 = H^3$, (4.2) and Lemma 4.21 yield

$$
\begin{aligned}
&\Gamma^3_s - \big(\mu\Gamma^1_s + (1-\mu)\Gamma^2_s\big)\\
&= \int_s^T \big(\mu f(\Lambda^1_r, Z^1_r + \alpha^1_r) + (1-\mu) f(\Lambda^2_r, Z^2_r + \alpha^2_r) - f(\Lambda^3_r, Z^3_r + \alpha^3_r)\big)\,\mathrm{d}r\\
&\quad - \int_s^T \big(\mu Z^1_r + (1-\mu)Z^2_r - Z^3_r\big)\,\mathrm{d}B_r\\
&\geq \int_s^T \Big(f\big(\Lambda^3_r, \mu Z^1_r + (1-\mu)Z^2_r + \alpha^3_r\big) - f\big(\Lambda^3_r, Z^3_r + \alpha^3_r\big)\Big)\,\mathrm{d}r\\
&\quad - \int_s^T \big(\mu Z^1_r + (1-\mu)Z^2_r - Z^3_r\big)\,\mathrm{d}B_r\\
&= -\int_s^T \big(\mu Z^1_r + (1-\mu)Z^2_r - Z^3_r\big)\,(\mathrm{d}B_r - \kappa_r\,\mathrm{d}r), \qquad 0 \leq s \leq T, \qquad (4.4)
\end{aligned}
$$

with $\kappa := \frac{1}{2}(\Lambda^3)^{-1}\big(\mu Z^1+(1-\mu)Z^2+Z^3+2\alpha^3\big)$. Since the eigenvalues of Λ^3 are bounded away from zero and α^3 is bounded, $\int \kappa\,\mathrm{d}B$ is a BMO-martingale. By Theorem 3.6 of Kazamaki [40] and the $BMO(P)$-property of $\int Z^i\,\mathrm{d}B$, the process $\int\big(\mu Z^1 + (1-\mu)Z^2 - Z^3\big)\,(\mathrm{d}B - \kappa\,\mathrm{d}r)$ is thus also a $BMO\big(\tilde{P}\big)$-martingale for the probability measure $\tilde{P}$ given by $\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} := \mathcal{E}\big(\int \kappa\,\mathrm{d}B\big)_T$. Taking $\big(\tilde{P},\mathcal{F}_s\big)$-conditional expectations in (4.4) yields $\Gamma^3_s - \big(\mu\Gamma^1_s + (1-\mu)\Gamma^2_s\big) \geq 0$ for any $s \in [0,T]$, which concludes the proof since the Γ^i are continuous. $\square$

The basic and well-known case is when $\alpha \equiv 0$, $\chi \equiv 0$ and $\Lambda = cI$ for a fixed $c > 0$. The BSDE (4.2) then simplifies to

$$\Gamma_s = H - \int_s^T \frac{1}{2c}|Z_r|^2\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}B_r = \Gamma_0 - c\log\mathcal{E}\left(\int \frac{1}{c}Z\,\mathrm{d}B\right)_s, \quad 0 \leq s \leq T.$$

Due to Itô's formula, its explicit solution is

$$\Gamma_s = -c\log E[\exp(-H/c)|\mathcal{F}_s], \quad 0 \leq s \leq T \tag{4.5}$$

because $\int Z\,\mathrm{d}B$ is a BMO-martingale by Lemma 4.2, and hence $\mathcal{E}\big(\int \frac{1}{c}Z\,\mathrm{d}B\big)$ is a martingale by Theorem 2.3 of Kazamaki [40].

4.2.2 Motivation for the convexity results

Before we state and prove in Section 4.3 convexity results for the solution of the BSDE (4.2), we explain the basic ideas using PDEs. Since we only want to provide motivation, we look at the results exclusively for time 0 and ignore here all technical issues like existence of smooth solutions, interchanging expectation and differential, etc.

Assume in (4.2) that α, χ and Λ are all deterministic and $H = g(B_T)$ for a smooth function $g : \mathbb{R} \to \mathbb{R}$. In this Markovian setting, one can derive from Itô's formula that the solution (Γ, Z) of (4.2) satisfies

$$\Gamma_s = u(s,B_s), \quad Z_s = -\nabla_x u(s,B_s) \quad \text{for } s \in [0,T],$$

where $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ solves the PDE

$$\left.\begin{aligned} &\frac{\partial}{\partial s}u(s,x) + \frac{1}{2}\Delta_x u(s,x) - f\big(\Lambda(s),\alpha(s) - \nabla_x u(s,x)\big) - \chi(s) = 0, \\ &u(T,x) = g(x) \quad \text{for } s \in [0,T) \text{ and } x \in \mathbb{R}^n. \end{aligned}\right\} \tag{4.6}$$

Each of our three main results yields an upper bound for Γ. We look in the following as illustration at the PDE analogue of the symmetrisation result in Theorem 4.11. The other BSDE theorems have similar PDE analogues. For ease of notation, we take α, χ, Λ all constant.

Symmetrisation inequalities play an important role in the theory of *linear* parabolic PDEs; see e.g. Alvino et al. [1] and the references therein. They show that in some sense, the solution of a symmetrised PDE dominates the symmetrised solution of the original PDE. Theorem 4.11 below can be viewed as an analogue of these results for nonlinear parabolic PDEs. To explain the connection, let Perm $\subseteq$ O(n) be the group of permutations of length n, where we identify permutations with orthogonal matrices. We define

$$\Lambda^{\mathrm{Sym}} = \frac{1}{n!}\sum_{O\in\mathrm{Perm}} O'\Lambda O, \quad \alpha^{\mathrm{Sym}} := \frac{1}{n!}\sum_{O\in\mathrm{Perm}} O'\alpha, \quad g^{\mathrm{Sym}} := \frac{1}{n!}\sum_{O\in\mathrm{Perm}} (g\circ O).$$

Let $\tilde{u} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ solve the symmetrised PDE

$$\left.\begin{aligned} &\frac{\partial}{\partial s}\tilde{u}(s,x) + \frac{1}{2}\Delta_x\tilde{u}(s,x) - f\big(\Lambda^{\mathrm{Sym}}, \alpha^{\mathrm{Sym}} - \nabla_x\tilde{u}(s,x)\big) - \chi = 0,\\ &\tilde{u}(T,x) = g^{\mathrm{Sym}}(x) \quad \text{for } s\in[0,T) \text{ and } x\in\mathbb{R}^n. \end{aligned}\right\} \tag{4.7}$$

Then Proposition 4.10 below tells us that

$$\tilde{u}(0,0) \geq u(0,0). \tag{4.8}$$

We justify this here by a PDE comparison argument. For $O\in$ Perm, we have $\nabla_{Ox}u(s,Ox) = O\nabla_x u(s,Ox)$, $\Delta_{Ox}u(s,Ox) = \Delta_x u(s,Ox)$ and, from (4.3),

$$f\big(\Lambda, \alpha - O\nabla_x u(s,Ox)\big) = f\big(O'\Lambda O, O'\alpha - \nabla_x u(s,Ox)\big).$$

Due to (4.6), the symmetrised function $\overline{u}(s,x) := \frac{1}{n!}\sum_{O\in\mathrm{Perm}} u(s,Ox)$ solves

$$\begin{aligned} &\frac{\partial}{\partial s}\overline{u}(s,x) + \frac{1}{2}\Delta_x\overline{u}(s,x) - \frac{1}{n!}\sum_{O\in\mathrm{Perm}} f\big(O'\Lambda O, O'\alpha - \nabla_x u(s,Ox)\big) - \chi = 0,\\ &\overline{u}(T,x) = g^{\mathrm{Sym}}(x) \quad \text{for } s\in[0,T) \text{ and } x\in\mathbb{R}^n. \end{aligned} \tag{4.9}$$

By Lemma 4.21 in the Appendix, f is jointly convex, which yields

$$\frac{1}{n!}\sum_{O\in\mathrm{Perm}} f\big(O'\Lambda O, O'\alpha - \nabla_x u(s,Ox)\big) \geq f\big(\Lambda^{\mathrm{Sym}}, \alpha^{\mathrm{Sym}} - \nabla_x\overline{u}(s,x)\big).$$

Since $\overline{u}(0,0) = u(0,0)$, we obtain (4.8) by comparing (4.7) and (4.9). Now fix $c>0$. One can check that the solution $\hat{u}$ of

$$\left.\begin{aligned} &\frac{\partial}{\partial s}\hat{u}(s,x) + \frac{1}{2}\Delta_x\hat{u}(s,x) - \frac{1}{2c}\big|\alpha^{\mathrm{Sym}} - \nabla_x\hat{u}(s,x)\big|^2 - \chi = 0,\\ &\hat{u}(T,x) = g^{\mathrm{Sym}}(x) \quad \text{for } s\in[0,T) \text{ and } x\in\mathbb{R}^n \end{aligned}\right\} \tag{4.10}$$

satisfies

$$\hat{u}(0,0) = -c \log E\left[\exp\left(-g^{\mathrm{Sym}}(B_T) + \int_0^T \alpha^{\mathrm{Sym}}\,\mathrm{d}B_s\right)^{\frac{1}{c}}\right] - \int_0^T \chi\,\mathrm{d}s. \quad (4.11)$$

To compare (4.7) with (4.10), we assume that $\Lambda = \mathrm{diag}(\Lambda^{11}, \dots, \Lambda^{nn})$ is of diagonal form and set $c := \sup_{s\in[0,T]} \frac{1}{n}\,\mathrm{tr}(\Lambda_s)$ (if Λ is time-dependent). Then $\Lambda^{\mathrm{Sym}} = \frac{1}{n}\,\mathrm{tr}(\Lambda)I$ since Λ is diagonal, and hence

$$f\left(\Lambda^{\mathrm{Sym}}, \alpha^{\mathrm{Sym}} - x\right) \geq \frac{1}{2c}\left|\alpha^{\mathrm{Sym}} - x\right|^2 \quad \text{for } x \in \mathbb{R}^n.$$

We thus expect by comparing (4.7) and (4.10) that $\tilde{u}(0,0) \leq \hat{u}(0,0)$, which gives via (4.8) and (4.11) an explicit upper bound for the solution of the original PDE (4.6). Theorem 4.11 makes this statement precise and provides a proof in a general BSDE setting.

4.3 Convexity results for quadratic BSDEs

This section contains our three main results. We study how the solution of the BSDE (4.2) is affected if we change the probability measure, shrink the filtration, or symmetrise the probability space.

4.3.1 Changing the probability measure

For any predictable κ such that $\int \kappa\,\mathrm{d}B$ is a BMO-martingale, we define

$$\frac{\mathrm{d}P^\kappa}{\mathrm{d}P} := \mathcal{E}\left(-\int \kappa\,\mathrm{d}B\right)_T, \quad B^\kappa := B + \int \kappa\,\mathrm{d}s \quad (4.12)$$

and note that B^κ is a Brownian motion under the probability measure P^κ. Recalling that $t \in [0,T]$ *is fixed* and spec denotes the spectrum, we define

$$\delta_t^{\max} := \sup_{s\in[t,T]} \|\max\mathrm{spec}(\Lambda_s)\|_{L^\infty}, \quad \delta_t^{\min} := \inf_{s\in[t,T]} \frac{1}{\|\max\mathrm{spec}(\Lambda_s^{-1})\|_{L^\infty}},$$

$$H_t^\kappa := H - \int_t^T \left(\chi_s + \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\right)\mathrm{d}s - \int_t^T (\alpha_s + \Lambda_s\kappa_s)\,\mathrm{d}B_s. \quad (4.13)$$

For $\delta > 0$, let $\mathcal{K}^\delta$ be the set of all predictable $\mathbb{R}^n$-valued processes κ such that $\int \kappa\,\mathrm{d}B$ is in BMO and there exist $p > 1$ and a constant C such that

$$\begin{aligned} &E_{P^\kappa}\left[\exp\left(\int_t^T \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\,\mathrm{d}s + \int_t^T \Lambda_s\kappa_s\,\mathrm{d}B_s\right)^{p/\delta}\middle|\mathcal{F}_\tau\right] \\ &\leq C E_{P^\kappa}\left[\exp\left(\int_t^T \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\,\mathrm{d}s + \int_t^T \Lambda_s\kappa_s\,\mathrm{d}B_s\right)^{1/\delta}\middle|\mathcal{F}_\tau\right]^p < \infty, \end{aligned} \quad (4.14)$$

for any stopping time τ with values in $[t,T]$. The latter condition says that the martingale

$$E_{P^\kappa}\left[\exp\left(\int_t^T \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\,\mathrm{d}s + \int_t^T \Lambda_s\kappa_s\,\mathrm{d}B_s\right)^{1/\delta}\middle|\mathcal{F}_s\right],\quad t\le s\le T,$$

satisfies the *reverse Hölder inequality* $R_p(P^\kappa)$. Each $\mathcal{K}^\delta$ contains all bounded predictable processes κ, and (4.14) is equivalent to

$$E_{P^\kappa}\big[\exp(-H_t^\kappa)^{p/\delta}\big|\mathcal{F}_\tau\big] \le C E_{P^\kappa}\big[\exp(-H_t^\kappa)^{1/\delta}\big|\mathcal{F}_\tau\big]^p < \infty, \tag{4.15}$$

since H, χ and α are bounded. We set $\mathcal{K} := \mathcal{K}^{\delta_t^{\max}} \cap \mathcal{K}^{\delta_t^{\min}}$.

Theorem 4.5. *The solution* Γ *of the BSDE (4.2) satisfies*

$$\Gamma_t = -\operatorname*{ess\,sup}_{\kappa\in\mathcal{K}} \log E_{P^\kappa}[\exp(-H_t^\kappa/\delta_t^{\max})|\mathcal{F}_t]^{\delta_t^{\max}} \tag{4.16}$$

$$= -\operatorname*{ess\,inf}_{\kappa\in\mathcal{K}} \log E_{P^\kappa}[\exp(-H_t^\kappa/\delta_t^{\min})|\mathcal{F}_t]^{\delta_t^{\min}}, \tag{4.17}$$

and for every $\kappa\in\mathcal{K}$, *there exists an* $\mathcal{F}_t$*-measurable random variable* $\delta_t^{\kappa,H}$ *with values in* $[\delta_t^{\min},\delta_t^{\max}]$ *such that*

$$\Gamma_t = -\log E_{P^\kappa}[\exp(-H_t^\kappa/\delta)|\mathcal{F}_t]^\delta\big|_{\delta=\delta_t^{\kappa,H}}. \tag{4.18}$$

Theorem 4.5 illustrates the importance of the process Λ in the BSDE (4.2). Indeed, Λ determines via (4.13) the eigenvalue bounds $\delta_t^{\min,\max}$ and hence the range of $\delta_t^{\kappa,H}$ in (4.18). If $\Lambda = cI$ for a constant c, we have $\delta_t^{\min} = \delta_t^{\max} = c = \delta_t^{\kappa,H}$, and (4.18) is an explicit formula for Γ_t as distorted conditional expectation under P^κ. Corollary 4.6 below gives a converse: If for any H, the solution Γ_t of the BSDE (4.2) is the distorted conditional expectation under some P^κ, then $\Lambda = cI$ for a constant c. Theorem 4.5 also generalises Theorem 2.9, as we explain in Section 4.5.3. Moreover, we can recover the bound in Proposition 2.1 of Kobylanski [42] applied to the BSDE (4.2); indeed, for $\kappa = -\Lambda^{-1}\alpha \in \mathcal{K}$, (4.16) yields

$$\begin{aligned}
\Gamma_t &\le -\log E_{P^\kappa}\left[\exp\left(-H + \int_t^T \Big(\chi_s + \frac{1}{2}\alpha_s'\Lambda_s^{-1}\alpha_s\Big)\,\mathrm{d}s\right)^{1/\delta_t^{\max}}\middle|\mathcal{F}_t\right]^{\delta_t^{\max}}\\
&\le -\log E_{P^\kappa}\left[\exp\left(-\|H^+\|_{L^\infty} - \int_t^T \Big\|\chi_s + \frac{1}{2}\alpha_s'\Lambda_s^{-1}\alpha_s\Big\|_{L^\infty}\mathrm{d}s\right)^{1/\delta_t^{\max}}\middle|\mathcal{F}_t\right]^{\delta_t^{\max}}\\
&\le \|H^+\|_{L^\infty} + \int_t^T \big\||\chi_s| + \alpha_s'\Lambda_s^{-1}\alpha_s\big\|_{L^\infty}\,\mathrm{d}s,
\end{aligned}$$

which one can also derive from Proposition 2.1 of Kobylanski [42].

From (4.16) we obtain upper bounds for Γ_t, which depend on the maximal eigenvalue of Λ. Our other two main results, Theorems 4.7 and 4.11, can be viewed as approaches to get better bounds by reducing $\delta_t^{\max}$ (and also changing H). In Theorem 4.7, we reduce the dimension n of the BSDE by projecting it onto the filtration of a lower-dimensional Brownian motion, and replacing Λ by its projection in principle lowers the maximal eigenvalue. Similarly, the symmetrisation in Theorem 4.11 makes the eigenvalues more similar and in particular reduces the maximal eigenvalue.

Proof of Theorem 4.5. We first show

$$\Gamma_t \le -\log E_{P^\kappa}[\exp(-H_t^\kappa/\delta_t^{\max})|\mathcal{F}_t]^{\delta_t^{\max}} \tag{4.19}$$

for any $\kappa \in \mathcal{K}$. We obtain from (4.2), (4.3) and (4.12) that

$$\begin{aligned}\Gamma_s = H &- \int_s^T \frac{1}{2}(Z_r + \alpha_r + \Lambda_r\kappa_r)'\Lambda_r^{-1}(Z_r + \alpha_r + \Lambda_r\kappa_r)\,\mathrm{d}r \\ &- \int_s^T \Big(\chi_r - \kappa_r'\alpha_r - \frac{1}{2}\kappa_r'\Lambda_r\kappa_r\Big)\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}B_r^\kappa, \quad 0 \le s \le T.\end{aligned} \tag{4.20}$$

Define

$$\begin{aligned}\Gamma_s^\kappa :=& -\log E_{P^\kappa}[\exp(-H_t^\kappa/\delta_t^{\max})|\mathcal{F}_s]^{\delta_t^{\max}} \\ &+ \int_t^s \Big(\chi_r + \frac{1}{2}\kappa_r'\Lambda_r\kappa_r\Big)\,\mathrm{d}r + \int_t^s (\alpha_r + \Lambda_r\kappa_r)\,\mathrm{d}B_r, \quad t \le s \le T.\end{aligned} \tag{4.21}$$

Using Itô's representation theorem as in Lemma 1.6.7 of Karatzas and Shreve [39] gives

$$E_{P^\kappa}[\exp(-H_t^\kappa/\delta_t^{\max})|\mathcal{F}_\cdot] = c^\kappa\,\mathcal{E}\bigg(\int Z^\kappa\,\mathrm{d}B^\kappa\bigg) \tag{4.22}$$

for a constant c^κ and a predictable $\mathbb{R}^n$-valued Z^κ such that $\mathcal{E}\big(\int Z^\kappa\,\mathrm{d}B^\kappa\big)$ is a P^κ-martingale. Since H, χ and α are bounded, (4.13) and (4.14) imply that $\mathcal{E}\big(\int Z^\kappa\,\mathrm{d}B^\kappa\big)$ satisfies the reverse Hölder inequality $R_p(P^\kappa)$ for some $p > 1$. Hence $\int Z^\kappa\,\mathrm{d}B^\kappa$ is in $BMO(P^\kappa)$ by Theorem 3.4 of Kazamaki [40], and so is $\int \bar{Z}^\kappa\,\mathrm{d}B^\kappa$ for

$$\bar{Z}^\kappa := \delta_t^{\max} Z^\kappa - \alpha - \Lambda\kappa. \tag{4.23}$$

A calculation based on (4.21) and (4.22) gives for $t \le s \le T$

$$\begin{aligned}\Gamma_s^\kappa = H &- \int_s^T \frac{1}{2\delta_t^{\max}}\big|\bar{Z}_r^\kappa + \alpha_r + \Lambda_r\kappa_r\big|^2\,\mathrm{d}r \\ &- \int_s^T \Big(\chi_r - \kappa_r'\alpha_r - \frac{1}{2}\kappa_r'\Lambda_r\kappa_r\Big)\,\mathrm{d}r + \int_s^T \bar{Z}_r^\kappa\,\mathrm{d}B_r^\kappa,\end{aligned} \tag{4.24}$$

and comparing (4.24) and (4.20) yields similarly as in Proposition 4.3 that $\Gamma_t \leq \Gamma^\kappa_t$. This is (4.19).

Now set $\hat\kappa := -\Lambda^{-1}(Z+\alpha)$ with Z from (4.20). Then $\int \hat\kappa \,\mathrm{d}B \in BMO$ since α is bounded, $\int Z \,\mathrm{d}B \in BMO$ and Λ^{-1} is bounded. Moreover, $H^{\hat\kappa}_t$ is $\mathcal{F}_t$-measurable; hence $\hat\kappa$ satisfies (4.15) and thus (4.14) for any $\delta > 0$, and so $\hat\kappa$ is in $\mathcal{K}$. Again using that $H^{\hat\kappa}_t$ is $\mathcal{F}_t$-measurable plus (4.20) and (4.13) shows that

$$\Gamma_t = H^{\hat\kappa}_t = -\log E_{P^{\hat\kappa}}[\exp(-H^{\hat\kappa}_t/\delta^{\max}_t)|\mathcal{F}_t]^{\delta^{\max}_t}.$$

Hence we have (4.16), and (4.17) is proved analogously. (4.18) now follows by the same interpolation argument as in Theorem 2.2; $\exp(-H^\kappa_t/\delta^{\min}_t)+1$ is the required P^κ-integrable majorant for $\big\{\exp(-H^\kappa_t/\delta) \,\big|\, \delta \in [\delta^{\min}_t, \delta^{\max}_t]\big\}$. □

We next study when Γ_t from (4.2) is a distorted conditional expectation under some P^κ. For $\delta > 0$ and $\kappa \in \mathcal{K}^\delta$, let $L^{\delta,\kappa}$ be the set of random variables H such that H^κ_t from (4.13) satisfies the reverse Hölder inequality (4.15) for some $p > 1$. The definition of $\mathcal{K}^\delta$ implies that $L^\infty \subseteq L^{\delta,\kappa}$, but $H \in L^{\delta,\kappa}$ need not be bounded.

Corollary 4.6. *The following are equivalent:*

(a) There exists a constant $c > 0$ such that

$$\Lambda = cI \quad \text{on }]\!]t,T]\!] \quad (P\otimes\mathrm{Leb})\text{-a.e.} \tag{4.25}$$

(b) There exists a constant $\delta \in [\delta^{\min}_t, \delta^{\max}_t]$ such that for all $\kappa \in \mathcal{K}^\delta$ and $H \in L^{\delta,\kappa}$, there exists a generalised solution (Γ, Z) on $[\![t,T]\!]$ of (4.2) such that $\int Z\,\mathrm{d}B$ is a $BMO(P)$-martingale and

$$\Gamma_t = -\log E_{P^\kappa}[\exp(-H^\kappa_t/\delta)|\mathcal{F}_t]^\delta. \tag{4.26}$$

(c) There exist a constant $\delta \in [\delta^{\min}_t, \delta^{\max}_t]$ and a process $\kappa \in \mathcal{K}^\delta$ such that for all $H \in L^{\delta,\kappa}$, there exists a generalised solution (Γ, Z) on $[\![t,T]\!]$ of (4.2) such that $\int Z\,\mathrm{d}B$ is a $BMO(P)$-martingale and (4.26) holds.

In this case, $c = \delta$.

Proof. "(b) $\Longrightarrow$ (c)" is clear. To show "(a) $\Longrightarrow$ (b)", we use a similar argument as for Theorem 4.5. Take $\kappa \in \mathcal{K}$ and define Γ^κ and $\bar Z^\kappa$ by (4.21) and (4.23) with $\delta^{\max}_t$ replaced by $\delta := c$. Then $\int \bar Z^\kappa \,\mathrm{d}B^\kappa$ is again in $BMO(P^\kappa)$ so that $\int \bar Z^\kappa \,\mathrm{d}B$ is in $BMO(P)$, and like (4.24), we get

$$\begin{aligned}\Gamma^\kappa_s = \Gamma^\kappa_t &+ \int_t^s \frac{1}{2\delta}\big|\bar Z^\kappa_r + \alpha_r + \Lambda_r\kappa_r\big|^2 \,\mathrm{d}r \\ &+ \int_t^s \Big(\chi_r - \kappa_r'\alpha_r - \frac{1}{2}\kappa_r'\Lambda_r\kappa_r\Big)\,\mathrm{d}r - \int_t^s \bar Z^\kappa_r \,\mathrm{d}B^\kappa_r, \quad t \leq s \leq T.\end{aligned}$$

Plugging in (4.25) with $\delta = c$ shows after some computation that $\big(\Gamma^\kappa, \bar{Z}^\kappa\big)$ satisfies (4.2) on $[\![t,T]\!]$. Finally, (4.26) holds for $\Gamma := \Gamma^\kappa$ by construction.

To prove "(c) $\Longrightarrow$ (a)", we define the predictable set

$$\Upsilon_1 := \big\{(\omega,s)\in]\!]t,T]\!] \,\big|\, \min\operatorname{spec}\big(\Lambda_s(\omega)\big) < \delta\big\}$$

and choose a predictable $\mathbb{R}^n$-valued process v such that $\Lambda v = \big(\min\operatorname{spec}(\Lambda)\big)v$ and $|v|=1$ on $]\!]t,T]\!]$; so $v_s(\omega)$ is an eigenvector for the smallest eigenvalue of $\Lambda_s(\omega)$. Set

$$\begin{aligned} H := &\int_t^T \Big(\chi_s + \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\Big)\,\mathrm{d}s + \int_t^T (\alpha_s + \Lambda_s\kappa_s)\,\mathrm{d}B_s \\ &- \int_t^T \mathbb{1}_{\Upsilon_1}(s)v_s\,\mathrm{d}B_s + \int_t^T \mathbb{1}_{\Upsilon_1}(s)\Big(\frac{1}{2\delta} - v_s'\kappa_s\Big)\,\mathrm{d}s \end{aligned}$$

so that the corresponding H_t^κ given by (4.13) satisfies

$$\exp(-H_t^\kappa/\delta) = \mathcal{E}\bigg(\int \frac{1}{\delta}\mathbb{1}_{\Upsilon_1} v\,\mathrm{d}B^\kappa\bigg)_T. \tag{4.27}$$

Hence H is in $L^{\delta,\kappa}$ by Theorem 3.4 of Kazamaki [40]; in fact, $\int \frac{1}{\delta}\mathbb{1}_{\Upsilon_1} v\,\mathrm{d}B^\kappa$ is a $BMO(P^\kappa)$-martingale because its integrand is bounded.

Now (4.26), (4.27) and Itô's formula, (4.13) and (4.12) give with some calculations

$$\begin{aligned} \Gamma_t &= -\log E_{P^\kappa}[\exp(-H_t^\kappa/\delta)|\mathcal{F}_t]^\delta \\ &= H - \int_t^T \Big(\chi_s - \kappa_s'\alpha_s - \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\Big)\,\mathrm{d}s - \int_t^T (\alpha_s + \Lambda_s\kappa_s)\,\mathrm{d}B_s^\kappa \\ &\quad + \int_t^T \mathbb{1}_{\Upsilon_1}(s)v_s\,\mathrm{d}B_s^\kappa - \frac{1}{2\delta}\int_t^T \mathbb{1}_{\Upsilon_1}(s)\,\mathrm{d}s \\ &\geq H - \int_t^T \Big(\chi_s - \kappa_s'\alpha_s - \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\Big)\,\mathrm{d}s - \int_t^T (\alpha_s + \Lambda_s\kappa_s)\,\mathrm{d}B_s^\kappa \\ &\quad + \int_t^T \mathbb{1}_{\Upsilon_1}(s)v_s\,\mathrm{d}B_s^\kappa - \int_t^T \frac{1}{2}\big(\mathbb{1}_{\Upsilon_1}(s)v_s\big)'\Lambda_s^{-1}\big(\mathbb{1}_{\Upsilon_1}(s)v_s\big)\,\mathrm{d}s \end{aligned} \tag{4.28}$$

by the definition of Υ_1. But we also have like in (4.20) that

$$\begin{aligned} \Gamma_t = H &- \int_t^T \Big(\chi_s - \kappa_s'\alpha_s - \frac{1}{2}\kappa_s'\Lambda_s\kappa_s\Big)\,\mathrm{d}s + \int_t^T Z_s\,\mathrm{d}B_s^\kappa \\ &- \int_t^T \frac{1}{2}(Z_s + \alpha_s + \Lambda_s\kappa_s)'\Lambda_s^{-1}(Z_s + \alpha_s + \Lambda_s\kappa_s)\,\mathrm{d}s, \end{aligned} \tag{4.29}$$

and subtracting (4.29) from (4.28), we obtain

$$0 \geq \int_t^T \big(\mathbb{1}_{\Upsilon_1}(s)v_s - \alpha_s - \Lambda_s\kappa_s - Z_s\big) \\ \times \Big(\mathrm{d}B_s^\kappa - \Lambda_s^{-1}\big(\mathbb{1}_{\Upsilon_1}(s)v_s + \alpha_s + \Lambda_s\kappa_s + Z_s\big)\,\mathrm{d}s\Big). \tag{4.30}$$

Like in the proof of Proposition 4.3, the right-hand side of (4.30) has zero expectation under some equivalent probability measure. Hence it must vanish, so we must also have equality in (4.28), and this implies $(P\otimes\mathrm{Leb})[\Upsilon_1] = 0$. Analogously, we have $(P\otimes\mathrm{Leb})[\Upsilon_2] = 0$ for

$$\Upsilon_2 := \big\{(\omega, s) \in \rrbracket t, T\rrbracket \,\big|\, \max\operatorname{spec}\big(\Lambda_s(\omega)\big) > \delta\big\}.$$

This shows (4.25) with $c := \delta$ and also gives the last assertion. □

4.3.2 Projecting the BSDE

Let us split $B = \big(\overline{B}, \underline{B}\big)'$ into $\overline{B}$ and $\underline{B}$, an $\overline{n}$- and an $\underline{n}$-dimensional $(\mathbb{F}, P)$-Brownian motion with $\overline{n} + \underline{n} = n$. What happens to the BSDE

$$\Gamma_s = H - \int_s^T \big(f(\Lambda_r, Z_r + \alpha_r) + \chi_r\big)\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}B_r, \quad 0 \leq s \leq T \tag{4.2}$$

if we project it, in a way to be specified, onto the filtration generated by $\overline{B}$? In this section, we precisely formulate and then answer this question.

Let $\overline{\mathbb{F}} = \big(\overline{\mathcal{F}}_s\big)_{0\leq s\leq T}$ be the augmented filtration generated by $\overline{B}$. For a process Z, we denote its componentwise optional $(P$-$)$projection onto $\overline{\mathbb{F}}$ by Z^o (if it exists). It is — by definition — the unique $\overline{\mathbb{F}}$-optional process satisfying $Z^o_\tau = E\big[Z_\tau\big|\overline{\mathcal{F}}_\tau\big]$ for every $\overline{\mathbb{F}}$-stopping time τ.

To compare (4.2) with a BSDE driven by $\overline{B}$, write $\alpha = (\overline{\alpha}, \underline{\alpha})'$ and denote by $\overline{\Lambda}$ the upper-left $\overline{n} \times \overline{n}$ components of Λ. A *solution*, for $s \in [0, T]$, of

$$\check{\Gamma}_s = E\big[H\big|\overline{\mathcal{F}}_T\big] - \int_s^T \Big(\frac{1}{2}\big(\check{Z}_r + \overline{\alpha}^o_r\big)'\big(\overline{\Lambda}^o_r\big)^{-1}\big(\check{Z}_r + \overline{\alpha}^o_r\big) + \chi^o_r\Big)\,\mathrm{d}r + \int_s^T \check{Z}_r\,\mathrm{d}\overline{B}_r \tag{4.31}$$

is a pair $\big(\check{\Gamma}, \check{Z}\big)$ satisfying (4.31), where $\check{\Gamma}$ is a real-valued bounded continuous $\big(\overline{\mathbb{F}}, P\big)$-semimartingale and $\check{Z}$ is an $\mathbb{R}^{\overline{n}}$-valued $\overline{\mathbb{F}}$-predictable process such that $\int_0^T \big|\check{Z}_s\big|^2\,\mathrm{d}s < \infty$ almost surely. Note that $\overline{X}^o := \big(\overline{X}\big)^o = \overline{(X^o)}$ for $X = \alpha, \Lambda$.

Theorem 4.7. *The BSDE (4.31) has a unique solution* $\big(\check{\Gamma}, \check{Z}\big)$. *It satisfies* $\Gamma^o \leq \check{\Gamma}$, *where* (Γ, Z) *is the solution of (4.2).*

Theorem 4.7 is a Jensen-type inequality for quadratic BSDEs. For a simple illustration, take $\overline{n} = \underline{n} = 1$ and $\Lambda \equiv cI$, $\alpha \equiv 0$, $\chi \equiv 0$. In this case, the solution of (4.2) has $\Gamma_0 = -c \log E[\exp(-H/c)]$ by (4.5), and analogously, $\check{\Gamma}_0 = -c \log E\big[\exp\big(-\frac{1}{c} E\big[H \big| \overline{\mathcal{F}}_T\big]\big)\big]$. So $\Gamma_0^o \le \check{\Gamma}_0$ follows here also directly from Jensen's inequality.

Proof of Theorem 4.7. As in Lemma 4.2, (4.31) has a unique solution $\big(\check{\Gamma}, \check{Z}\big)$, and $\int \check{Z} \, \mathrm{d}\overline{B} \in BMO\big(\overline{\mathbb{F}}, P\big)$. Fix $s \in [0,T]$ and condition (4.2) on $\overline{\mathcal{F}}_s$ to get

$$E\big[\Gamma_s \big| \overline{\mathcal{F}}_s\big] = \Gamma_0 + E\left[\int_0^s \big(f(\Lambda_r, Z_r + \alpha_r) + \chi_r\big) \, \mathrm{d}r \middle| \overline{\mathcal{F}}_s\right] - E\left[\int_0^s Z_r \, \mathrm{d}B_r \middle| \overline{\mathcal{F}}_s\right]. \tag{4.32}$$

Note next that χ^o exists since χ is bounded by assumption. We claim that

$$E\left[\int_0^s \chi_r \, \mathrm{d}r \middle| \overline{\mathcal{F}}_s\right] = \int_0^s \chi_r^o \, \mathrm{d}r, \tag{4.33}$$

and because $\overline{\mathbb{F}}$ is generated by $\overline{B}$, it is by Itô's representation theorem enough to show that

$$E\left[\int_0^s \chi_r \, \mathrm{d}r \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right] = E\left[\int_0^s \chi_r^o \, \mathrm{d}r \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right] \tag{4.34}$$

for any $\overline{\mathbb{F}}$-predictable β such that $\int \beta \, \mathrm{d}\overline{B}$ is bounded. By Fubini's theorem,

$$E\left[\int_0^s \chi_r \, \mathrm{d}r \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right] = \int_0^s E\left[\chi_r \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right] \mathrm{d}r, \tag{4.35}$$

and conditioning on $\overline{\mathcal{F}}_r$ for $r \in [0,s]$ yields

$$E\left[\chi_r \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right] = E\left[\chi_r \int_0^r \beta_q \, \mathrm{d}\overline{B}_q\right] = E\left[\chi_r^o \int_0^s \beta_q \, \mathrm{d}\overline{B}_q\right],$$

which implies (4.34) by using (4.35) once for χ and once for χ^o instead of χ. So we have (4.33), and using $f \ge 0$, we analogously obtain

$$E\left[\int_0^s f(\Lambda_r, Z_r + \alpha_r) \, \mathrm{d}r \middle| \overline{\mathcal{F}}_s\right] = \int_0^s \big(f(\Lambda, Z + \alpha)\big)_r^o \, \mathrm{d}r. \tag{4.36}$$

To simplify the term $E\big[\int_0^s Z_r \, \mathrm{d}B_r \big| \overline{\mathcal{F}}_s\big]$ in (4.32), we use the optional projection of Z. However, we cannot use the classical optional projection because Z is in general neither bounded nor nonnegative. We define Z^o instead by

$$Z^o := \begin{cases} (Z^+)^o - (Z^-)^o & \text{if } |Z|^o < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $Z^{\pm} := \big((Z^1)^{\pm}, \dots, (Z^n)^{\pm}\big)'$. Then Z^o is $\overline{\mathbb{F}}$-optional and $|Z|^o < \infty$ $(P\otimes\mathrm{Leb})$-a.e. since Tonelli's theorem and $\int Z\,\mathrm{d}B \in BMO(\mathbb{F}, P)$ by Lemma 4.2 give

$$\int_0^T E[|Z|^o_r]\,\mathrm{d}r = \int_0^T E[|Z_r|]\,\mathrm{d}r = E\bigg[\int_0^T |Z_r|\,\mathrm{d}r\bigg] < \infty.$$

Write $Z = \big(\overline{Z}, \underline{Z}\big)'$ and $Z^o = \big(\overline{Z}^o, \underline{Z}^o\big)'$. We then have

$$E\bigg[\int_0^s Z_r\,\mathrm{d}B_r\bigg|\overline{\mathcal{F}}_s\bigg] = \int_s^T \overline{Z}^o_r\,\mathrm{d}\overline{B}_r; \tag{4.37}$$

indeed, using $E\big[\overline{Z}_r\big|\overline{\mathcal{F}}_r\big] = \overline{Z}^o_r$ P-a.s. for Leb-a.a. $r \in [0, s]$ and the isometry property of the stochastic integral, we obtain similarly to (4.34) that

$$E\bigg[\int_0^s Z_r\,\mathrm{d}B_r \int_0^s \beta_q\,\mathrm{d}\overline{B}_q\bigg] = E\bigg[\int_0^s \beta'_r\overline{Z}_r\,\mathrm{d}r\bigg] = E\bigg[\int_0^s \overline{Z}^o_r\,\mathrm{d}\overline{B}_r \int_0^s \beta_q\,\mathrm{d}\overline{B}_q\bigg]$$

for any $\overline{\mathbb{F}}$-predictable β such that $\int \beta\,\mathrm{d}\overline{B}$ is bounded, and this implies (4.37) by Itô's representation theorem. Combining (4.32), (4.33), (4.36) and (4.37) thus yields

$$E\big[\Gamma_s\big|\overline{\mathcal{F}}_s\big] = E\big[H\big|\overline{\mathcal{F}}_T\big] - \int_s^T \Big(\big(f(\Lambda, Z+\alpha)\big)^o_r + \chi^o_r\Big)\,\mathrm{d}r + \int_s^T \overline{Z}^o_r\,\mathrm{d}\overline{B}_r. \tag{4.38}$$

Due to Lemma 4.21 in the Appendix, the function f is jointly convex. Identifying (A, z) in $\mathcal{S}^n \times \mathbb{R}^n$ with a vector in $\mathbb{R}^{\frac{n(n+1)}{2}+n}$, we view f as a function on such vectors and then apply Jensen's inequality to obtain for any $\overline{\mathbb{F}}$-stopping time τ that

$$\big(f(\Lambda, Z+\alpha)\big)^o_\tau = E\big[f(\Lambda_\tau, Z_\tau+\alpha_\tau)\big|\overline{\mathcal{F}}_\tau\big] \geq f(\Lambda^o_\tau, Z^o_\tau + \alpha^o_\tau)\mathbb{1}_{|Z|^o_\tau<\infty}.$$

Thus the optional selection theorem and $|Z|^o < \infty$ $(P\otimes\mathrm{Leb})$-a.e. yield

$$\big(f(\Lambda, Z+\alpha)\big)^o \geq f(\Lambda^o, Z^o+\alpha^o)\mathbb{1}_{|Z|^o<\infty} = f(\Lambda^o, Z^o+\alpha^o) \tag{4.39}$$

$(P\otimes\mathrm{Leb})$-a.e. A simple calculation (see Remark 4.8 below) shows that

$$f(A, z) = \frac{1}{2}z'A^{-1}z \geq \frac{1}{2}\overline{z}'\big(\overline{A}\big)^{-1}\overline{z} \tag{4.40}$$

for any $A \in \mathcal{S}^n$ and $z = (\overline{z}, \underline{z})' \in \mathbb{R}^n$, with $\overline{A}$ denoting the upper-left $\overline{n} \times \overline{n}$ components of A. In view of (4.38), we obtain from (4.39) and (4.40) that

$$\begin{aligned} E\big[\Gamma_s\big|\overline{\mathcal{F}}_s\big] \leq E\big[H\big|\overline{\mathcal{F}}_T\big] &- \int_s^T \Big(\frac{1}{2}\big(\overline{Z}^o_r + \overline{\alpha}^o_r\big)'\big(\overline{\Lambda}^o_r\big)^{-1}\big(\overline{Z}^o_r + \overline{\alpha}^o_r\big) + \chi^o_r\Big)\,\mathrm{d}r \\ &+ \int_s^T \overline{Z}^o_r\,\mathrm{d}\overline{B}_r. \end{aligned}$$

Hence (4.31) implies

$$E\big[\Gamma_s\big|\overline{\mathcal{F}}_s\big]-\check{\Gamma}_s\leq-\int_s^T\frac{1}{2}\big(\overline{Z}^o_r-\check{Z}_r\big)'\big(\overline{\Lambda}^o_r\big)^{-1}\big(\overline{Z}^o_r+\check{Z}_r+2\overline{\alpha}^o_r\big)\,\mathrm{d}r$$
$$+\int_s^T\big(\overline{Z}^o_r-\check{Z}_r\big)\,\mathrm{d}\overline{B}_r.$$

We know that $\int\check{Z}\,\mathrm{d}\overline{B}$ is in $BMO\big(\overline{\mathbb{F}},P\big)$, and so is $\int\overline{Z}^o\,\mathrm{d}\overline{B}$ because $\int Z\,\mathrm{d}B$ is in $BMO(\mathbb{F},P)$. Like in the proof of Proposition 4.3, we deduce that $E\big[\Gamma_s\big|\overline{\mathcal{F}}_s\big]\leq\check{\Gamma}_s$ for $s\in[0,T]$, and this concludes the proof because $\check{\Gamma}$, Γ and hence Γ^o are continuous. □

Remark 4.8. **1)** As the proof shows, we do not need for Theorem 4.7 that the generator of the BSDE (4.2) is purely quadratic like f in (4.3). We only need that it is jointly convex, satisfies a quadratic growth condition and dominates the generator of the projected BSDE (4.31). In particular, Theorem 4.7 also applies for a generator $\tilde{f}$ of the form $\tilde{f}(A,z)=\frac{1}{2}\overline{z}'\overline{A}^{-1}\overline{z}$ with $\overline{z}$ and $\overline{A}$ as in (4.40). This will later be used in the applications to indifference valuation.

2) In linear algebra, the *shorted operator* $\mathrm{sh}:\mathcal{S}^n\to\mathcal{S}^{\overline{n}}$ is defined by

$$\mathrm{sh}(A):=A^{11}-A^{12}(A^{22})^{-1}(A^{12})'\ \text{ for } A=\begin{pmatrix}A^{11}&A^{12}\\(A^{12})'&A^{22}\end{pmatrix}\in\mathcal{S}^n.$$

One can check that $(A^{11})^{-1}=\mathrm{sh}(A^{-1})$ and verify by completion of squares that

$$\overline{z}'\mathrm{sh}(A)\overline{z}=\min_{\underline{z}\in\mathbb{R}^{\underline{n}}}\left((\overline{z}',\underline{z}')A\begin{pmatrix}\overline{z}\\\underline{z}\end{pmatrix}\right)\qquad\text{for } \overline{z}\in\mathbb{R}^{\overline{n}} \text{ and } A\in\mathcal{S}^n.$$

The inequality (4.40) follows immediately. ◊

4.3.3 Symmetrising the BSDE

This section establishes our third main result, Theorem 4.11, giving an explicit upper bound for the solution Γ of (4.2). We first study how the BSDE (4.2) is affected by orthogonal transformations on the underlying probability space. To have some structure, we work on Wiener space, i.e., take $\Omega:=C([0,T],\mathbb{R}^n)$ with the Borel σ-field $\mathcal{F}$ and Wiener measure P so that the coordinate process B is a P-Brownian motion. Recall that $t\in[0,T]$ *is fixed.*

For an orthogonal $(n\times n)$-matrix, $u\in\mathrm{O}(n)$, we define the mapping $U_t:C([0,T],\mathbb{R}^n)\to C([0,T],\mathbb{R}^n)$ by applying u from time t on, i.e.,

$$U_t(g)(s)=\begin{cases}g(s)&\text{if }\ s\leq t,\\g(t)+u\big(g(s)-g(t)\big)&\text{if }\ s>t,\end{cases}\quad\text{for }\ g\in C([0,T],\mathbb{R}^n).$$

Then $B^u := U_t \circ B$ is an $\mathbb{R}^n$-valued $(\mathbb{F}, P)$-Brownian motion since u is orthogonal. The following result says that if one transforms by U_t the driver and the terminal value of a BSDE, the solution of the new BSDE is the U_t-transformation of the original solution. This is very intuitive and analogous to orthogonally transforming the variables in a second-order PDE; compare Section 4.2.2. The reason why this also works for BSDEs is that $B \circ U_t = U_t \circ B = B^u$, i.e., Brownian motion and the transformation U_t commute on Wiener space.

Lemma 4.9. *Let $u \in \mathrm{O}(n)$ and assume that the BSDE*

$$\Gamma_s = H - \int_s^T F_r(\Gamma_r, Z_r)\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}B_r, \quad 0 \le s \le T \tag{4.41}$$

for a general $\mathbb{F}$-predictable $F : \mathbb{R}^{n+1} \to \mathbb{R}$ has a unique solution (Γ, Z) (in the sense of Section 4.2.1). Then $(\Gamma \circ U_t, Z \circ U_t)$ is the unique solution of

$$\tilde{\Gamma}_s = H \circ U_t - \int_s^T (F \circ U_t)_r\big(\tilde{\Gamma}_r, \tilde{Z}_r\big)\,\mathrm{d}r + \int_s^T \tilde{Z}_r\,\mathrm{d}B^u_r, \quad 0 \le s \le T. \tag{4.42}$$

In particular, the solution $(\Gamma \circ U_t, Z \circ U_t)$ of (4.42) coincides on $[\![0, t]\!]$ with the solution (Γ, Z) of (4.41).

Proof. Let (Γ, Z) be the solution of (4.41) and define $\tilde{\Gamma}$ for $0 \le s \le T$ by

$$\begin{aligned}\tilde{\Gamma}_s &:= \Gamma_0 + \int_0^s (F \circ U_t)_r\,(\Gamma_r \circ U_t, Z_r \circ U_t)\,\mathrm{d}r - \int_0^s (Z_r \circ U_t)\,\mathrm{d}B^u_r \\ &= \Gamma_0 + \left(\int_0^s F_r\,(\Gamma_r, Z_r)\,\mathrm{d}r\right) \circ U_t - \int_0^s (Z_r \circ U_t)\,\mathrm{d}B^u_r.\end{aligned} \tag{4.43}$$

In Lemma 4.22 in the Appendix, we prove that, as one expects,

$$\int (Z \circ U_t)\,\mathrm{d}B^u = \left(\int Z\,\mathrm{d}B\right) \circ U_t. \tag{4.44}$$

This gives by (4.41) that $\tilde{\Gamma} = \Gamma \circ U_t$ and thus $(\Gamma \circ U_t, Z \circ U_t)$ solves (4.42). Uniqueness for (4.42) follows since U_t is bijective; indeed, if $\big(\tilde{\Gamma}, \tilde{Z}\big)$ solves (4.42), then (4.43) and (4.44) imply that $\big(\tilde{\Gamma} \circ U_t^{-1}, \tilde{Z} \circ U_t^{-1}\big)$ solves (4.41) whose unique solution is (Γ, Z). $\square$

The next proposition states that averaging in ω over a set of orthogonal transformations increases the solution of (4.2).

Proposition 4.10. *Take a finite subset $\mathcal{O}$ of $\mathrm{O}(n)$ with cardinality $|\mathcal{O}|$ and set*

$$H^{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} H\circ U_t, \qquad \Lambda^{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} u'(\Lambda\circ U_t)u,$$
$$\alpha^{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} u'(\alpha\circ U_t), \qquad \chi^{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} \chi\circ U_t.$$

Then the solutions (Γ, Z) of (4.2) and $\big(\Gamma^{\mathcal{O}}, Z^{\mathcal{O}}\big)$, for $0 \le s \le T$, of

$$\tilde{\Gamma}_s = H^{\mathcal{O}} - \int_s^T \Big(f\big(\Lambda_r^{\mathcal{O}}, \tilde{Z}_r + \alpha_r^{\mathcal{O}}\big) + \chi_r^{\mathcal{O}}\Big)\,\mathrm{d}r + \int_s^T \tilde{Z}_r\,\mathrm{d}B_r \tag{4.45}$$

satisfy $\Gamma_t \le \Gamma_t^{\mathcal{O}}$ almost surely.

Proof. By Lemma 4.2, (4.2) and (4.45) have unique solutions. For $u \in \mathcal{O}$, we denote by (Γ^u, Z^u) the solution of (4.2) corresponding to the parameters $\big(H\circ U_t, u'(\Lambda\circ U_t)u, u'(\alpha\circ U_t), \chi\circ U_t\big)$. The concavity from Proposition 4.3 gives $\Gamma^{\mathcal{O}} = \Gamma\big(H^{\mathcal{O}}, \Lambda^{\mathcal{O}}, \alpha^{\mathcal{O}}, \chi^{\mathcal{O}}\big) \ge \frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}}\Gamma^u$, and so it is enough to show $\Gamma_t^u = \Gamma_t$ for every $u \in \mathcal{O}$. Fix $u \in \mathcal{O}$. Applying Lemma 4.9 to (4.2) yields that the solution of the U_t-transformed BSDE is $\big(\tilde{\Gamma}, \tilde{Z}\big) := (\Gamma\circ U_t, Z\circ U_t)$. Setting $\hat{Z} := u'\tilde{Z}$ and using $\hat{Z}\,\mathrm{d}B = \tilde{Z}\,\mathrm{d}B^u$ and, due to (4.3),

$$f\big(\Lambda\circ U_t, \tilde{Z} + \alpha\circ U_t\big) = f\big(u'(\Lambda\circ U_t)u, \hat{Z} + u'(\alpha\circ U_t)\big),$$

we obtain that the U_t-transformed BSDE is, for $t \le s \le T$, equivalent to

$$\tilde{\Gamma}_s = H\circ U_t - \int_s^T \Big(f\big(u'(\Lambda\circ U_t)_r u, \hat{Z}_r + u'(\alpha\circ U_t)_r\big) + (\chi\circ U_t)_r\Big)\,\mathrm{d}r + \int_s^T \hat{Z}_r\,\mathrm{d}B_r.$$

But this is (4.2) with the parameters $\big(H\circ U_t, u'(\Lambda\circ U_t)u, u'(\alpha\circ U_t), \chi\circ U_t\big)$. So $\Gamma^u = \tilde{\Gamma} = \Gamma\circ U_t$ on $[\![t,T]\!]$ and thus $\Gamma_t^u = \Gamma_t$, since $\Gamma\circ U_t = \Gamma$ on $[\![0,t]\!]$. □

The idea to exploit Proposition 4.10 is now that choosing a "good" set $\mathcal{O}$ yields with (4.45) an easier BSDE than the original one in (4.2), so that an upper bound for the solution (Γ, Z) of (4.2) becomes more explicit. By Theorem 4.5, the upper bound for Γ is increasing in the maximal eigenvalue, $\max\operatorname{spec}(\Lambda)$. Assume for the moment that Λ is deterministic. If we first apply Proposition 4.10 to (4.2) and then Theorem 4.5 to (4.45), we obtain an upper bound depending on $\max\operatorname{spec}\big(\frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} u'\Lambda u\big)$. A simple calculation shows that for any matrix $A \in \mathcal{S}^n$ and finite subset $\mathcal{O}$ of $\mathrm{O}(n)$,

$$\frac{1}{n}\operatorname{tr}(A) \le \max\operatorname{spec}\bigg(\frac{1}{|\mathcal{O}|}\sum_{u\in\mathcal{O}} u'Au\bigg) \le \max\operatorname{spec}(A), \tag{4.46}$$

and so we obtain a smaller distortion power $\delta_t^{\max}$ by averaging over $\mathcal{O}$. On the other hand, however, averaging H, α and χ may worsen the bound on Γ, and an example in Section 4.5.3 shows how these two effects interact. The best lower bound for $\max\operatorname{spec}(A)$ that we can obtain by averaging over $\mathcal{O}$ is $\frac{1}{n}\operatorname{tr}(A)$ by (4.46), and if A is diagonal, this is attained for $\mathcal{O} = \mathrm{Perm}$, the symmetric group of permutations of length n. (We identify permutations with corresponding orthogonal matrices and use

$$\frac{1}{|\mathrm{Perm}|}\sum_{u\in\mathrm{Perm}} u'Au = \frac{\operatorname{tr}(A)}{n}I \quad \text{for any diagonal matrix } A.)$$

The idea to choose $\mathcal{O} = \mathrm{Perm}$ leads us to the next result.

Theorem 4.11. *Assume that* $\Lambda = (\Lambda^{ij})_{i,j=1,\dots,n}$ *is a diagonal matrix, and define*

$$H^{\mathrm{Sym}} := \frac{1}{n!}\sum_{u\in\mathrm{Perm}} H\circ U_t, \qquad d_t := \sup_{s\in[t,T]}\left\|\frac{1}{n}\sum_{j=1}^{n}\max_{u\in\mathrm{Perm}}\,(\Lambda_s^{jj}\circ U_t)\right\|_{L^\infty},$$
$$\alpha^{\mathrm{Sym}} := \frac{1}{n!}\sum_{u\in\mathrm{Perm}} u'(\alpha\circ U_t), \qquad \chi^{\mathrm{Sym}} := \frac{1}{n!}\sum_{u\in\mathrm{Perm}}\chi\circ U_t.$$

Then the solution (Γ, Z) *of (4.2) satisfies*

$$\Gamma_t \le -d_t\log E\left[\exp\left(-H^{\mathrm{Sym}} + \int_t^T \alpha_s^{\mathrm{Sym}}\,\mathrm{d}B_s + \int_t^T \chi_s^{\mathrm{Sym}}\,\mathrm{d}s\right)^{\frac{1}{d_t}}\middle|\mathcal{F}_t\right]. \quad (4.47)$$

Proof. By choosing $\mathcal{O} := \mathrm{Perm}$, we obtain from Proposition 4.10 a first upper bound $\Gamma_t \le \Gamma_t^{\mathcal{O}}$, where $\Gamma_t^{\mathcal{O}}$ solves the BSDE (4.45) with $\mathcal{O} := \mathrm{Perm}$. We now apply Theorem 4.5 to $\Gamma^{\mathcal{O}}$ with $\kappa\equiv 0$, which gives

$$\Gamma_t \le -\overline{\delta}_t\log E\left[\exp\left(-H^{\mathrm{Sym}} + \int_t^T \alpha_s^{\mathrm{Sym}}\,\mathrm{d}B_s + \int_t^T \chi_s^{\mathrm{Sym}}\,\mathrm{d}s\right)^{\frac{1}{\overline{\delta}_t}}\middle|\mathcal{F}_t\right]$$

with

$$\overline{\delta}_t := \sup_{s\in[t,T]}\left\|\max\operatorname{spec}\left(\frac{1}{n!}\sum_{u\in\mathrm{Perm}} u'(\Lambda_s\circ U_t)u\right)\right\|_{L^\infty} \le d_t$$

since Λ is diagonal. Thus (4.47) follows from Jensen's inequality. $\square$

The assumption that Λ is diagonal is less restrictive than it looks. We can always rewrite (4.2) to another BSDE of the same type with diagonal Λ by changing α and B. In fact, there exist a predictable $\mathrm{O}(n)$-valued process O and a predictable diagonal matrix D such that $\Lambda = O'DO$. If we now define an $(\mathbb{F}, P)$-Brownian motion by $\mathrm{d}B^O = O\,\mathrm{d}B$, a direct calculation shows that if (Γ, Z) solves (4.2) with parameters $(H, \Lambda, \alpha, \chi)$, then (Γ, OZ) solves (4.2) with parameters $(H, D, O\alpha, \chi)$ and with B replaced by B^O. This reduces the problem to the case of a diagonal matrix Λ, but we then have to symmetrise with respect to B^O and not B. For this, H, α and χ must be measurable for the filtration $\mathbb{F}^O$ generated by B^O, which can be smaller than $\mathbb{F}$. This limitation does not come up if Λ is deterministic, since then so is O and hence $\mathbb{F}^O = \mathbb{F}$. In Section 4.5, we relate the BSDE (4.2) to an optimisation problem where the matrix Λ is a transform of the correlation matrix of certain price processes. In applications, such matrices are often assumed to be deterministic. Similarly, things typically become less restrictive in a Markovian setting because one can often do everything in the filtration of the factor process.

Remark 4.12. One can generalise Theorems 4.7 and 4.11 to the case where H, α and χ are unbounded, but $|H|$ and $\int_0^T \big(|\alpha_s|^2 + |\chi_s|\big)\,\mathrm{d}s$ have exponential moments of all order. We sketch the procedure for such a generalisation. One first uses Corollary 6 of Briand and Hu [12] for the existence of a generalised solution (Γ, Z) of (4.2) and its uniqueness in a suitable class. Then one sets $H^j := H^+ \wedge j - H^- \wedge j$, $j \in \mathbb{N}$, defines α^j and χ^j analogously, and applies Theorems 4.7 and 4.11 when H, α and χ are replaced by H^j, α^j and χ^j. By taking limits in a suitable sense and applying Proposition 7 of Briand and Hu [12], one can deduce generalised versions of Theorems 4.7 and 4.11. We do not know whether Theorem 4.5 can also be formulated for unbounded H, α and χ, because the above generalisation procedure does not work there.

One cannot weaken in the above way the assumption that the eigenvalues of Λ are bounded away from zero, since this condition is needed to apply the results of Briand and Hu [12]. However, one can get rid of the restriction that the eigenvalues of Λ are bounded away from infinity. Theorems 4.5 and 4.11 can be formulated without this assumption similarly to Theorem 3.12. If the componentwise optional projection of Λ, whose eigenvalues are not bounded away from infinity, exists $(P\otimes\mathrm{Leb})$-a.e., one can prove Theorem 4.7 in the same way as in Section 4.3.2. $\Diamond$

4.4 Exponential utility indifference valuation

This section recalls the financial concept of indifference valuation, in preparation for applying the convexity results from Section 4.3.

We work on a finite time interval $[0, T]$ for a fixed $T > 0$, *and we fix* $t \in [0, T]$. On a complete probability space $(\Omega, \mathcal{G}, P)$, we have independent Brownian motions W and $W^\perp$ with values in $\mathbb{R}^m$ and $\mathbb{R}^n$. We denote by

$\mathbb{G} = (\mathcal{G}_s)_{0\leq s\leq T}$ the P-augmented filtration generated by $(W, W^\perp)$ and assume $\mathcal{G} = \mathcal{G}_T$. Moreover, we suppose there is an $\mathbb{R}^n$-valued $(\mathbb{G}, P)$-Brownian motion Y such that

$$\mathrm{d}Y_s = R_s\,\mathrm{d}W_s + \sqrt{I - R_s R_s'}\,\mathrm{d}W_s^\perp, \quad 0 \leq s \leq T \tag{4.48}$$

for a $\mathbb{G}$-predictable $(n \times m)$-matrix R describing correlations between W and Y. We assume that all eigenvalues of RR' are bounded away from one uniformly on $\Omega \times [0,T]$, i.e., there exists $c \in [0,1)$ with

$$\max \operatorname{spec}(RR') \leq c \;\; (P\otimes\mathrm{Leb})\text{-a.e. on } \Omega \times [0,T]. \tag{4.49}$$

For a fixed $\gamma > 0$, the $\mathcal{S}^n$-valued process

$$\Lambda = \frac{1}{\gamma}(I - RR')^{-1} \tag{4.50}$$

is well defined, $\mathbb{G}$-predictable and satisfies $\operatorname{spec}(\Lambda) \subseteq \left[\frac{1}{\gamma}, \frac{1}{\gamma(1-c)}\right]$. In the notation of Section 4.3, this implies that $\delta_t^{\min}(\Lambda) \geq 1/\gamma$.

Our financial market consists of a risk-free bank account yielding zero interest and m traded risky assets $S = (S^j)_{j=1,\dots,m}$ with dynamics

$$\mathrm{d}S_s^j = S_s^j \mu_s^j\,\mathrm{d}s + \sum_{k=1}^m S_s^j \sigma_s^{jk}\,\mathrm{d}W_s^k, \quad 0 \leq s \leq T,\; S_0^j > 0, \quad j = 1,\dots,m;$$

the drift vector $\mu = (\mu^j)_{j=1,\dots,m}$ and the volatility matrix $\sigma = (\sigma^{jk})_{j,k=1,\dots,m}$ are $\mathbb{G}$-predictable. We assume that σ is invertible, $\lambda := \sigma^{-1}\mu$ is bounded (uniformly in ω and s) and that there exists a constant C such that

$$C\beta'\beta \geq \beta'\sigma\sigma'\beta \geq \frac{1}{C}\beta'\beta \;\; \text{on } \Omega \times [0,T] \text{ for all } \beta \in \mathbb{R}^m.$$

(In other words, σ is uniformly both bounded and elliptic.) The processes

$$\hat{W} := W + \int \lambda\,\mathrm{d}s \;\; \text{and} \;\; \hat{Y} := Y + \int R\lambda\,\mathrm{d}s \tag{4.51}$$

are Brownian motions under the minimal martingale measure $\hat{P}$ given by

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\Big(-\int \lambda\,\mathrm{d}W\Big)_T. \tag{4.52}$$

Let H be a bounded $\mathcal{G}_T$-measurable random variable, interpreted as a contingent claim or payoff due at time T. To value H, we assume that our investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. He starts at time t with bounded $\mathcal{G}_t$-measurable initial capital x_t and runs a self-financing strategy $\pi = (\pi_s)_{t \le s \le T}$ so that his wealth at time $s \in [t, T]$ is

$$X_s^{x_t,\pi} = x_t + \int_t^s \sum_{j=1}^m \frac{\pi_r^j}{S_r^j}\,\mathrm{d}S_r^j = x_t + \int_t^s \pi_r'\sigma_r\,\mathrm{d}\hat{W}_r,$$

where π^j represents the amount invested in S^j, $j = 1, \dots, m$. The set $\mathcal{A}_t$ of *admissible strategies* on $[t, T]$ consists of all $\mathbb{G}$-predictable $\mathbb{R}^m$-valued processes $\pi = (\pi_s)_{t \le s \le T}$ which satisfy $\int_t^T |\pi_s|^2\,\mathrm{d}s < \infty$ a.s. and are such that

$$\exp\bigl(-\gamma X_s^{x_t,\pi}\bigr), t \le s \le T, \text{ is of class } (D) \text{ on } (\Omega, \mathcal{G}_T, \mathbb{G}, P).$$

We define V^H (and analogously V^0) by

$$\begin{aligned} V_t^H(x_t) &:= \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t} E_P\bigl[U(X_T^{x_t,\pi} + H)\big|\mathcal{G}_t\bigr] \\ &= \mathrm{e}^{-\gamma x_t} \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t} E_P\left[-\exp\left(-\gamma \int_t^T \pi_s'\sigma_s\,\mathrm{d}\hat{W}_s - \gamma H\right)\middle|\mathcal{G}_t\right] \end{aligned} \tag{4.53}$$

so that $V_t^H(x_t)$ is the maximal expected utility the investor can achieve by starting at time t with initial capital x_t, using some admissible strategy π, and receiving H at time T. For ease of notation, we write

$$V_t^H(x_t) = \mathrm{e}^{-\gamma x_t} V_t^H(0) =: \mathrm{e}^{-\gamma x_t} V_t^H.$$

Viewed over time, $V^H = \bigl(V_t^H\bigr)_{0 \le t \le T}$ is then the *dynamic value process* for the stochastic control problem associated to exponential utility maximisation. Compared to (2.6) and (3.2) in Chapters 2 and 3, we have changed the sign of H in the definition (4.53) of $V_t^H(x_t)$. This is done only for notational convenience to avoid later additional minus signs. For the same reason, we next consider the indifference buyer value and not the seller value. This is no restriction since the seller value of H equals minus the buyer value of $-H$.

The time t *indifference (buyer) value* $h_t(x_t)$ for H is implicitly defined by

$$V_t^0(x_t) = V_t^H\bigl(x_t - h_t(x_t)\bigr).$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with reduced initial capital $x_t - h_t(x_t)$ but receiving H at T. Our goal is to find bounds for $h_t(x_t)$. By (4.53),

$$h_t(x_t) = h_t = \frac{1}{\gamma} \log \frac{V_t^0}{V_t^H} \tag{4.54}$$

does not depend on x_t, but directly on V_t^H and V_t^0. We consider here V_t^0 as fixed via the financial market, and our focus lies on finding H-dependent bounds for V^H from the optimisation problem (4.53). An overview of the literature on exponential utility indifference valuation in Brownian settings can be found in Section 2.4.2.

4.5 Valuation bounds from convexity

In this section, we consider the same setup as in Section 4.4. In order to apply the convexity results from Section 4.3, we want to associate V^H to a quadratic convex BSDE of the form (4.2). We start with the following result which follows directly from Theorem 7 and Proposition 9 of Hu et al. [37].

Lemma 4.13. *The BSDE*

$$\check{\Gamma}_s = H - \int_s^T \left(\frac{\gamma}{2}\big|\check{Z}_r\big|^2 - \hat{Z}_r'\lambda_r - \frac{1}{2\gamma}|\lambda_r|^2\right) \mathrm{d}r + \int_s^T \hat{Z}_r \,\mathrm{d}W_r + \int_s^T \check{Z}_r \,\mathrm{d}W_r^\perp \quad (4.55)$$

for $s \in [0,T]$ has a unique solution $\big(\check{\Gamma}, \hat{Z}, \check{Z}\big)$ such that $\big(\hat{Z}, \check{Z}\big)$ is $(\mathbb{R}^m \times \mathbb{R}^n)$-valued and $\mathbb{G}$-predictable with $E_P\Big[\int_0^T \big(\big|\hat{Z}_s\big|^2 + \big|\check{Z}_s\big|^2\big)\,\mathrm{d}s\Big] < \infty$ and $\check{\Gamma}$ is $\mathbb{G}$-predictable and bounded. Furthermore, we have $V_\tau^H = -\exp\big(-\gamma\check{\Gamma}_\tau\big)$ for any $\mathbb{G}$-stopping time τ.

Unfortunately, we cannot (yet) apply the results from Section 4.3 to the BSDE (4.55), because its generator is quadratic in $\check{Z}$, but only linear in $\hat{Z}$. In contrast, the generator of (4.2) is quadratic in the full vector $Z = \big(\check{Z}, \hat{Z}\big)'$. The next sections present three different approaches to circumvent this problem. In Section 4.5.1, we simply add a term $\epsilon\big|\hat{Z}\big|^2$ to the generator of (4.55) and study the limit as ϵ tends to zero. Section 4.5.2 exploits the fact, pointed out in Remark 4.8, that one can apply the projection result in Theorem 4.7 to a BSDE with a more general generator. In a third approach, we impose in Section 4.5.3 measurability assumptions on the claim H and the coefficients of the asset S and then use symmetrisation arguments.

Lemma 4.13 also shows that the dynamic value process V^H has a continuous version. In the sequel, we always use this version of V^H.

4.5.1 ϵ-regularising the BSDE and changing the measure

In this approach, we add a term $\epsilon\big|\hat{Z}\big|^2$ to the generator of (4.55) to bring it to the form of (4.2). In some sense, this makes the BSDE (4.55) more regular. We first study how the solution of the changed BSDE behaves as $\epsilon \searrow 0$.

Lemma 4.14. *For each fixed $\epsilon > 0$, the BSDE*

$$\check{\Gamma}^\epsilon_s = H - \int_s^T \Big(\frac{\gamma}{2}\big|\check{Z}^\epsilon_r\big|^2 + \epsilon\big|\hat{Z}^\epsilon_r\big|^2 - (\hat{Z}^\epsilon_r)'\lambda_r - \frac{1}{2\gamma}|\lambda_r|^2\Big)\,\mathrm{d}r \\ + \int_s^T \hat{Z}^\epsilon_r\,\mathrm{d}W_r + \int_s^T \check{Z}^\epsilon_r\,\mathrm{d}W^\perp_r, \qquad 0 \le s \le T, \tag{4.56}$$

has a unique solution $\big(\check{\Gamma}^\epsilon, \hat{Z}^\epsilon, \check{Z}^\epsilon\big)$ (in the sense of Lemma 4.13). The solution $\check{\Gamma}$ to (4.55) satisfies

$$\check{\Gamma}_t = \operatorname*{ess\,sup}_{\epsilon>0} \check{\Gamma}^\epsilon_t = \lim_{\epsilon\searrow 0} \check{\Gamma}^\epsilon_t \quad a.s.$$

Lemma 4.14 is a variation of Proposition 3.1 of El Karoui et al. [26], which gives a similar conclusion for BSDEs with a Lipschitz-continuous generator.

Proof. Lemma 4.2 gives for each $\epsilon > 0$ a unique solution $\big(\check{\Gamma}^\epsilon, \hat{Z}^\epsilon, \check{Z}^\epsilon\big)$ of (4.56) with bounded $\check{\Gamma}^\epsilon$, and both $\int \hat{Z}^\epsilon\,\mathrm{d}W$ and $\int \check{Z}^\epsilon\,\mathrm{d}W^\perp$ are in $BMO(\mathbb{G}, P)$. As in the proof of Proposition 4.3, one can show that $\check{\Gamma}^\epsilon \le \check{\Gamma}$ and that $\big|\check{\Gamma}^\epsilon\big|$ is bounded by $\|H\|_{L^\infty} + \frac{1}{2\gamma}\big\|\int_0^T |\lambda_s|^2\,\mathrm{d}s\big\|_{L^\infty}$, uniformly in ϵ. Applying Itô's formula to $\exp\big(\check{\Gamma}^\epsilon\big)$ then yields like in the proof of Proposition 7 of Mania and Schweizer [44] that the $BMO(\mathbb{G}, P)$-norms of $\int \hat{Z}^\epsilon\,\mathrm{d}W$ and $\int \check{Z}^\epsilon\,\mathrm{d}W^\perp$ are bounded uniformly in ϵ. By Theorem 3.6 of Kazamaki [40], the $BMO\big(\mathbb{G}, \check{P}^\epsilon\big)$-norm of $\int \hat{Z}^\epsilon\,\mathrm{d}W$ is thus bounded uniformly in ϵ, where

$$\frac{\mathrm{d}\check{P}^\epsilon}{\mathrm{d}P} := \mathcal{E}\Big(-\int \lambda\,\mathrm{d}W + \frac{\gamma}{2}\int \big(\check{Z}^\epsilon + \check{Z}\big)\,\mathrm{d}W^\perp\Big)_T.$$

We now obtain from (4.55) and (4.56) by conditioning on $\mathcal{G}_t$ under $\check{P}^\epsilon$ that

$$0 \le \check{\Gamma}_t - \check{\Gamma}^\epsilon_t = \epsilon\, E_{\check{P}^\epsilon}\bigg[\int_t^T \big|\hat{Z}^\epsilon_s\big|^2\,\mathrm{d}s\bigg|\mathcal{G}_t\bigg] \le \epsilon\bigg\|\int \hat{Z}^\epsilon\,\mathrm{d}W\bigg\|^2_{BMO_2(\mathbb{G},\check{P}^\epsilon)},$$

and this converges almost surely to 0 for $\epsilon \searrow 0$. □

To apply the change of measure result in Theorem 4.5, we use notations analogous to Section 4.3.1, whose B corresponds to $(W, W^\perp)$. Let us set

$$\gamma H^{\kappa,\epsilon}_t := \gamma H + \frac{1}{2}\int_t^T \big(|\lambda_s|^2 + \epsilon^{-1}(|\lambda_s|^2 - |\kappa_s|^2)\big)\,\mathrm{d}s - \epsilon^{-1}\int_t^T (\kappa_s - \lambda_s)\,\mathrm{d}W_s,$$
$$\frac{\mathrm{d}Q^\kappa}{\mathrm{d}P} := \mathcal{E}\Big(-\int \kappa\,\mathrm{d}W\Big)_T.$$

Note that $\mathcal{K} = \mathcal{K}^{(m)}$ is here a set of $\mathbb{R}^m$-valued processes. The next result follows fairly directly from Lemma 4.14 and Theorem 4.5, but spelling out all details is rather tedious and gives no new insights; hence we only outline the argument. We apply Theorem 4.5 to (4.56) with $\tilde{\epsilon} := \frac{\gamma\epsilon}{2}$, $\tilde{B} := (W, W^{\perp})'$, $\tilde{n} := m + n$,

$$\tilde{\Lambda} := \frac{1}{\gamma}\begin{pmatrix} \epsilon^{-1} I_{m\times m} & 0 \\ 0 & I_{n\times n} \end{pmatrix}, \ \tilde{\alpha} := -\tilde{\Lambda}\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \text{ and } \tilde{\chi} := -\Big(\frac{1}{2\gamma\epsilon} + \frac{1}{2\gamma}\Big)|\lambda|^2.$$

This gives $\delta_t^{\min}(\tilde{\Lambda}) = 1/\gamma$, and now we obtain from Lemma 4.14 and (4.17) in Theorem 4.5 for $\tilde{\mathcal{K}} := \mathcal{K}^{(m)} \times \mathcal{K}^{(n)}$ the following result.

Proposition 4.15. *We have*

$$\check{\Gamma}_t = -\operatorname*{ess\,inf}_{\epsilon\in]0,1]}\operatorname*{ess\,inf}_{\kappa\in\mathcal{K}^{(m)}} \log E_{Q^\kappa}[\exp(-\gamma H_t^{\kappa,\epsilon})|\mathcal{G}_t]^{1/\gamma}. \tag{4.57}$$

By picking arbitrary $\kappa \in \mathcal{K}^{(m)}$ and $\epsilon \in]0,1]$, the representation (4.57) allows us to get lower bounds for $\check{\Gamma}_t$, and hence also for V_t^H by Lemma 4.13. Note that Q^κ is a martingale measure for S only for $\kappa = \lambda$. In that case, Q^κ equals the minimal martingale measure $\hat{P}$, and we get from (4.57) that

$$\check{\Gamma}_t \geq -\log E_{\hat{P}}\bigg[\exp\bigg(-\gamma H - \frac{1}{2}\int_t^T |\lambda_s|^2 \,\mathrm{d}s\bigg)\bigg|\mathcal{G}_t\bigg]^{1/\gamma}.$$

4.5.2 Projecting onto incompleteness

This short section exploits the projection result from Section 4.3.2 to give an upper bound for V_0^H. For any process Z, we denote by $\mathbb{F}^Z = (\mathcal{F}_s^Z)_{0\leq s\leq T}$ the P-augmented filtration generated by Z. In this section, Z^o stands for the optional projection of Z onto the filtration $\mathbb{F}^{W^\perp}$ *under the minimal martingale measure* $\hat{P}$, i.e., $Z_\tau^o = E_{\hat{P}}[Z_\tau|\mathcal{F}_\tau^{W^\perp}]$ for any $\mathbb{F}^{W^\perp}$-stopping time τ.

Proposition 4.16. *For any $s \in [0,T]$, V^H satisfies*

$$\big(\log(-V^H)\big)_s^o \geq \log E_{\hat{P}}\bigg[\exp\bigg(-E_{\hat{P}}\big[\gamma H\big|\mathcal{F}_T^{W^\perp}\big] - \frac{1}{2}\int_s^T (|\lambda|^2)_r^o \,\mathrm{d}r\bigg)\bigg|\mathcal{F}_s^{W^\perp}\bigg].$$

Proof. Using (4.51), we can rewrite (4.55) in the form

$$\check{\Gamma}_s = H - \int_s^T \Big(\frac{\gamma}{2}\big|\check{Z}_r\big|^2 - \frac{1}{2\gamma}|\lambda_r|^2\Big)\,\mathrm{d}r + \int_s^T \hat{Z}_r \,\mathrm{d}\hat{W}_r + \int_s^T \check{Z}_r \,\mathrm{d}W_r^\perp.$$

By Remark 4.8, we have $\check{\Gamma}^o \leq \bar{\Gamma}$ where $(\bar{\Gamma}, \bar{Z})$ solves the BSDE

$$\bar{\Gamma}_s = E_{\hat{P}}\big[H\big|\mathcal{F}_T^{W^\perp}\big] - \int_s^T \Big(\frac{\gamma}{2}\big|\bar{Z}_r\big|^2 - \frac{1}{2\gamma}\big(|\lambda|^2\big)_r^o\Big)\,\mathrm{d}r + \int_s^T \bar{Z}_r \,\mathrm{d}W_r^\perp$$

for $0 \leq s \leq T$. A direct calculation shows similarly to (4.5) that

$$\bar{\Gamma}_s = -\frac{1}{\gamma} \log E_{\hat{P}}\left[\exp\left(-E_{\hat{P}}\left[\gamma H \middle| \mathcal{F}_T^{W^\perp}\right] - \frac{1}{2}\int_s^T \left(|\lambda|^2\right)_r^o \,\mathrm{d}r\right)\middle|\mathcal{F}_s^{W^\perp}\right],\ 0 \leq s \leq T,$$

which concludes the proof since $V^H = -\exp\bigl(-\gamma\check{\Gamma}\bigr)$ by Lemma 4.13. □

Proposition 4.16 gives an upper bound for V_0^H and thus also for h_0, but these bounds are rather rough. In the next section, we show how additional measurability assumptions can be exploited to derive other bounds via the symmetrisation result of Section 4.3.3.

4.5.3 Symmetrising a nontradable claim

Recall that $\mathbb{F}^Z = \left(\mathcal{F}_s^Z\right)_{0 \leq s \leq T}$ denotes the P-augmented filtration generated by a process Z. We recall W and Y from (4.48) and write for brevity

$$\mathbb{W} = (\mathcal{W}_s)_{0\leq s\leq T} \text{ for } \mathbb{F}^W, \quad \mathbb{Y} = (\mathcal{Y}_s)_{0\leq s\leq T} \text{ for } \mathbb{F}^Y, \quad \hat{\mathbb{Y}} = \left(\hat{\mathcal{Y}}_s\right)_{0\leq s\leq T} \text{ for } \mathbb{F}^{\hat{Y}}.$$

If $R\lambda$ is $\mathbb{Y}$-predictable, then $\hat{Y}$ from (4.51) is $\mathbb{Y}$-adapted and hence $\hat{\mathbb{Y}} \subseteq \mathbb{Y}$. In general, however, none of the above three filtrations contains any other. We study two cases which were introduced in Section 2.4.1 in a setting with one-dimensional W and Y.

Cases. *We consider one of the following two situations:*

(I) $H \in L^\infty(\mathcal{Y}_T, P)$, *$\lambda$ is $\mathbb{Y}$-predictable, and R is $\mathbb{Y}$-predictable.*

(II) $H \in L^\infty\bigl(\hat{\mathcal{Y}}_T, P\bigr)$, *$\lambda$ is $\mathbb{F}^{S,\hat{Y}}$-predictable, and λ is $\mathbb{W}$-predictable.*

Each case reflects a situation where the payoff H is driven by Y $\bigl(\text{or } \hat{Y}\bigr)$, whereas hedging can only be done in S which is imperfectly correlated with Y $\bigl(\text{or } \hat{Y}\bigr)$. Direct hedging in the underlying of H may be impossible for two basic reasons: In case (I), its driver is not traded at all (e.g., a volatility or a consumer price index), whereas in case (II), it is traded in principle but not tradable for our investor, due to legal, liquidity, practicability, cost or other reasons. We refer to Section 2.4.1 for a thorough explanation and motivation of the assumptions in cases (I) and (II).

We focus in this section on case (I) and first relate V^H to a BSDE of the form (4.2). A similar result for case (II) is given in Proposition 4.23 in the Appendix. Recall from (4.50) that $\Lambda := \frac{1}{\gamma}(I - RR')^{-1}$.

Proposition 4.17. *In case (I), the BSDE*

$$\Gamma_s = H - \int_s^T \Big(\frac{1}{2} Z_r' \Lambda_r^{-1} Z_r - Z_r' R_r \lambda_r - \frac{1}{2\gamma}|\lambda_r|^2\Big) \, \mathrm{d}r + \int_s^T Z_r \, \mathrm{d}Y_r \tag{4.58}$$

for $0 \le s \le T$ *has a unique solution* (Γ, Z) *where* Γ *is a real-valued bounded continuous* $(\mathbb{Y}, P)$*-semimartingale and* Z *is an* $\mathbb{R}^n$*-valued* $\mathbb{Y}$*-predictable process such that* $\int_0^T |Z_s|^2 \, \mathrm{d}s < \infty$ *almost surely. Moreover,* $V^H = -\exp(-\gamma\Gamma)$.

Proposition 4.17 shows in particular that V^H is $\mathbb{Y}$-adapted in case (I). This generalises Remark 3.3 of Ankirchner et al. [2] who made the same observation in a Markovian setting. It also shows that the distortion power $\delta^{\hat{H}}$ in Theorem 2.9 can be chosen $\mathbb{Y}$-adapted.

Proof. The BSDE (4.58) can be brought into the form (4.2) by defining

$$B := Y, \quad \mathbb{F} := \mathbb{Y}, \quad \chi := -\frac{1}{2}\lambda'\Big(\frac{1}{\gamma} I + R'\Lambda R\Big)\lambda \quad \text{and} \quad \alpha := -\Lambda R \lambda, \tag{4.59}$$

and so (4.58) has a unique solution (Γ, Z) by Lemma 4.2. Using (4.48) then shows that $\big(\Gamma, R'Z, \sqrt{I - RR'}Z\big)$ solves (4.55), and

$$E_P\Big[\int_0^T \big(Z_s' R_s R_s' Z_s + Z_s'(I - R_s R_s') Z_s\big) \, \mathrm{d}s\Big] = E_P\Big[\int_0^T |Z_s|^2 \, \mathrm{d}s\Big] < \infty,$$

since $\int Z \, \mathrm{d}Y \in BMO(\mathbb{Y}, P)$ by Lemma 4.2. Moreover, $V^H = -\exp(-\gamma\Gamma)$ by uniqueness for (4.55). For later use, note that plugging (4.59) into (4.13) gives

$$\begin{aligned} H_t^\kappa = H &+ \frac{1}{2\gamma}\int_t^T |\lambda_s|^2 \, \mathrm{d}s - \frac{1}{2}\int_t^T \big(\kappa_s' \Lambda_s \kappa_s - (R_s\lambda_s)'\Lambda_s(R_s\lambda_s)\big) \, \mathrm{d}s \\ &- \int_t^T \Lambda_s(\kappa_s - R_s\lambda_s) \, \mathrm{d}Y_s. \end{aligned} \tag{4.60}$$

□

The key point for rewriting the description of V^H from (4.55) in Lemma 4.13 to (4.58) in Proposition 4.17 is that the latter BSDE has the form (4.2); and this reformulation, by working in the filtration $\mathbb{F}^Y$ instead of $\mathbb{G} = \mathbb{F}^{(W,W^\perp)}$, is possible thanks to the measurability conditions imposed by case (I). We could now apply to (4.58) all the results of Section 4.3, but we focus here on symmetrisation via Theorem 4.11. However, we also briefly mention in the next remarks some consequences of the probability change via Theorem 4.5 and the projection via Theorem 4.7.

Remark 4.18. **1)** Theorem 4.5 applied to the BSDE (4.58) generalises Theorem 2.9, which corresponds to the choice $\kappa = R\lambda$. In that case, H_t^κ from (4.60) simplifies to $H_t^{R\lambda} = H + \frac{1}{2\gamma}\int_t^T |\lambda_s|^2 \,\mathrm{d}s$, $P^{R\lambda}$ is the projection onto $\mathcal{Y}_T$ of the minimal martingale measure $\hat{P}$ in (4.52), and $\delta_t^{\hat{H}}$ from Theorem 2.9 is linked to $\delta_t^{R\lambda,-H}$ from Theorem 4.5 via $\delta_t^{\hat{H}} = \gamma\delta_t^{R\lambda,-H}$. The freedom in Theorem 4.5 of choosing κ arbitrarily allows one to obtain other bounds. Note from (4.60) that $\kappa = R\lambda$ is special because only with this choice, H_t^κ has no $\mathrm{d}Y$-integral in addition to H. So the minimality of $\hat{P}$ in the original sense corresponds to the minimality of $H_t^{R\lambda}$ in the sense that it only differs from H by the terminal value of a finite variation process.

2) Theorem 3.12 is the general semimartingale analogue of Theorem 4.5 applied to (4.58), with slightly different assumptions.

3) Proposition 4.17 starts with an optimisation problem in a financial market and relates this to the solution of a BSDE. In the opposite direction, one could also start with a BSDE and link its solution to an optimisation problem in an artificially constructed financial market. For the BSDE (4.2) with fixed $(H, \Lambda, \alpha, \chi)$ as in Section 4.2.1, we can define

$$\gamma := \sup_{s\in[t,T]} \big\| \max \operatorname{spec}(\Lambda_s^{-1}) \big\|_{L^\infty} + 1, \quad R := \sqrt{I - \frac{1}{\gamma}\Lambda^{-1}}, \quad \lambda := -R^{-1}\Lambda^{-1}\alpha,$$
$$\tilde{H} := H + \int_t^T \chi_s \,\mathrm{d}s + \frac{1}{2}\int_t^T \lambda_s'\Big(\frac{1}{\gamma}I + R_s'\Lambda_s R_s\Big)\lambda_s \,\mathrm{d}s, \quad m := n.$$

If we construct with these parameters a model as in Section 4.4, then Proposition 4.17 yields $\Gamma_t = -\frac{1}{\gamma}\log\big(-V_t^{\tilde{H}}\big)$.

4) Theorem 4.7 gives an upper bound for the solution of (4.58) in terms of a solution to a projected, lower-dimensional BSDE. Combining this with the above remark shows that projecting the optimisation problem relates to constructing a lower-dimensional artificial market. $\diamondsuit$

Applying Theorem 4.5 to the BSDE (4.58) yields bounds for V_t^H which depend directly on the claim H. If we also use symmetrisation via Theorem 4.11, we obtain bounds depending on a symmetrisation of H.

For any $\mathbb{Y}$-predictable $\mathcal{S}^n$-valued process Λ, there exist a $\mathbb{Y}$-predictable O valued in $\mathrm{O}(n)$ and a $\mathbb{Y}$-predictable diagonal matrix $D = \operatorname{diag}(D^{11}, \dots, D^{nn})$ with $\Lambda = O'DO$. For a bounded $\mathbb{Y}$-predictable process κ, we define a process $Y^{\kappa,O}$ null at 0 by $\mathrm{d}Y^{\kappa,O} = O(\mathrm{d}Y + \kappa \,\mathrm{d}s)$, and we set $\mathbb{Y}^{\kappa,O} = \big(\mathcal{Y}_s^{\kappa,O}\big)_{0\le s\le T} := \mathbb{F}^{Y^{\kappa,O}}$. For the next result, we work on Wiener space with coordinate process $Y^{\kappa,O}$ and use the notations of Sections 4.3.1 and 4.3.3 with $B := Y^{\kappa,O}$ and χ, α given by (4.59).

Proposition 4.19. *Write $\Lambda = O'DO$ and fix a bounded $\mathbb{Y}$-predictable process κ. In case (I), assume that D is $\mathbb{Y}^{\kappa,O}$-predictable and H_t^κ in (4.60) is $\mathcal{Y}_T^{\kappa,O}$-measurable, and set $H_t^{\kappa,\mathrm{Sym}} := \frac{1}{n!}\sum_{u\in\mathrm{Perm}} H_t^\kappa \circ U_t$. Then we have*

$$V_t^H \le -E_{P^\kappa}\big[\exp\big(-H_t^{\kappa,\mathrm{Sym}}/d_t\big)\big|\mathcal{Y}_t^{\kappa,O}\big]^{\gamma d_t} \quad a.s., \tag{4.61}$$

where

$$d_t := \sup_{s\in[t,T]} \left\| \frac{1}{n}\sum_{j=1}^n \sup_{u\in\mathrm{Perm}} \big(D_s^{jj}\circ U_t\big)\right\|_{L^\infty}.$$

Proof. By Proposition 4.17, $V_t^H = -\exp(-\gamma\Gamma_t)$ where (Γ, Z) solves (4.58). If $\big(\tilde\Gamma, \tilde Z\big)$ solves for $0 \le s \le T$ the BSDE

$$\tilde\Gamma_s = H_t^\kappa - \frac{1}{2}\int_s^T \big(O_r\tilde Z_r\big)' D_r^{-1}\big(O_r\tilde Z_r\big)\,\mathrm{d}r + \int_s^T \big(O_r\tilde Z_r\big)\,\mathrm{d}Y_r^{\kappa,O}, \tag{4.62}$$

combining (4.62), (4.60) and (4.58) on $[\![0,T]\!]$ shows $Z = \tilde Z - \Lambda(\kappa - R\lambda)$ and

$$\Gamma = \tilde\Gamma - \frac{1}{2}\int_t^{t\vee\cdot} \big(|\lambda|^2/\gamma - (\kappa - R\lambda)'\Lambda(\kappa + R\lambda)\big)\,\mathrm{d}s + \int_t^{t\vee\cdot} \Lambda(\kappa - R\lambda)\,\mathrm{d}Y$$

so that in particular $\Gamma_t = \tilde\Gamma_t$. Now we apply Theorem 4.11 to the BSDE (4.62) and obtain (4.61) from (4.47), except for one detail: The payoff H_t^κ from (4.60) is not bounded, as Theorem 4.11 requires. But a closer look shows that H_t^κ differs from a bounded payoff only by $\int_t^T \Lambda_s(\kappa_s - R_s\lambda_s)\,\mathrm{d}Y_s$, and since this Y-integrand is bounded, the arguments from Theorem 4.11 still go through. □

If we choose $\kappa = R\lambda$ in Proposition 4.19, the random variable H_t^κ in (4.60) simplifies to $H_t^{R\lambda} = H + \frac{1}{2\gamma}\int_t^T |\lambda_s|^2\,\mathrm{d}s$ and the resulting upper bound

$$V_t^H \le -E_{\hat P}\big[\exp\big(-H_t^{R\lambda,\mathrm{Sym}}/d_t\big)\big|\mathcal{Y}_t^{R\lambda,O}\big]^{\gamma d_t} \quad \text{a.s.}$$

can be written under the minimal martingale measure $\hat P$ from (4.52). In general, the bound of Proposition 4.19 differs from the upper bound in Theorem 4.5 in two respects. On the one hand, the expectation in (4.61) is distorted by d_t which depends on the average eigenvalue of (the permuted) D, whereas $\delta_t^{\max}$ from Theorem 4.5 reflects the maximal eigenvalue of D. We have $d_t \le \delta_t^{\max}$ and in the multidimensional case $n > 1$, there can be a big difference between d_t and $\delta_t^{\max}$ so that the bound of Proposition 4.19 may significantly improve that of Theorem 4.5. But on the other hand, the bound of Proposition 4.19 depends on the symmetrised claim $H_t^{\kappa,\mathrm{Sym}}$ instead of H_t^κ, which may make it worse. It depends on the concrete situation which of the two impacts is stronger and whether Proposition 4.19 or Theorem 4.5 gives the better bound. For $n = 1$, the bounds coincide.

In practice, the claim H often has symmetry properties (e.g., if it is the sum of individual assets); then $H_t^{\kappa,\mathrm{Sym}}$ does not differ much from H_t^κ, and the bounds of Proposition 4.19 can be much better than those from Theorem 4.5. We illustrate the above discussion in the next simple example.

Example 4.20. Take $m = \dim W = 1$ and $n = \dim Y = 2$. We assume that instantaneous correlations between W and Y are given by $R = (\rho^1, \rho^2)'$ for two constants $\rho^1, \rho^2 \in\,]-1, 1[$ with $0 \neq |\rho^1|^2 + |\rho^2|^2 < 1$. By (4.50), we have

$$\Lambda = \frac{1}{\gamma}(I - RR')^{-1} = \frac{1}{\gamma}\begin{pmatrix} 1 - |\rho^1|^2 & -\rho^1\rho^2 \\ -\rho^1\rho^2 & 1 - |\rho^2|^2 \end{pmatrix}^{-1},$$

which can be written as $\Lambda = O'DO$ for

$$D = \frac{1}{\gamma}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-|\rho^1|^2-|\rho^2|^2} \end{pmatrix} \quad \text{and} \quad O = \frac{1}{\sqrt{|\rho^1|^2 + |\rho^2|^2}}\begin{pmatrix} \rho^2 & -\rho^1 \\ \rho^1 & \rho^2 \end{pmatrix}.$$

We assume that $\lambda = \frac{\mu}{\sigma}$ is constant, and we consider a claim of the form $H = q^1 Y_T^1 + q^2 Y_T^2 = q' Y_T$ for a constant $q = (q^1, q^2)' \in \mathbb{R}^2 \setminus \{0\}$. In this simple setting, V^H can be explicitly determined. Indeed, writing $H = q'Y_t + \int_t^T q \,\mathrm{d}Y_s$, plugging this into (4.60) and choosing $\kappa = \Lambda^{-1} q + R\lambda$ leads to

$$H_t^\kappa = q'Y_t + \frac{1}{2}\big(\lambda^2/\gamma - 2\lambda R' q - q'\Lambda^{-1} q\big)(T - t),$$

and because this is $\mathcal{Y}_t$-measurable, we get $V_t^H = -\exp(-\gamma H_t^\kappa)$. But note that this works only because H is of the special form $H = \int_0^T q \,\mathrm{d}Y$ and q, R and λ are deterministic.

Although V_t^H is explicitly known here, we next also compare the bounds from Theorem 4.5 and Proposition 4.19 for the special choice $\kappa = R\lambda$. We choose this κ since it does not depend on H and also has nice consequences, as explained after Proposition 4.19; and we compute the bounds despite their non-optimality since they are explicit and illustrative. Applying Theorem 4.5 for $\kappa = R\lambda$ to the BSDE (4.58) gives with an easy computation for the indifference value in (4.54) the upper bound

$$\begin{aligned} h_0 &\leq -\log E_{P^{R\lambda}}[\exp(-H/\delta_0^{\max})]^{\delta_0^{\max}} \\ &= -\frac{\gamma T}{2}\big(1 - |\rho^1|^2 - |\rho^2|^2\big)\big(|q^1|^2 + |q^2|^2\big) - T\lambda(q^1\rho^1 + q^2\rho^2), \end{aligned} \tag{4.63}$$

where $\delta_0^{\max} = \max \operatorname{spec}(\Lambda) = \frac{1}{\gamma(1-|\rho^1|^2-|\rho^2|^2)}$. To apply Proposition 4.19 with $\kappa = R\lambda$, we have to symmetrise with respect to $Y_s^{R\lambda,O} = OY_s + OR\lambda s$, $0 \le s \le T$. The symmetrised claim is

$$\begin{aligned} H_0^{R\lambda,\mathrm{Sym}} &= \frac{1}{2}(q^1,q^2)O'Y_T^{R\lambda,O} + \frac{1}{2}(q^1,q^2)O'\begin{pmatrix} (Y_T^{R\lambda,O})^2 \\ (Y_T^{R\lambda,O})^1 \end{pmatrix} - T\lambda(q^1\rho^1 + q^2\rho^2) \\ &= \frac{1}{2}(\tilde{q}^1 + \tilde{q}^2)\big((Y_T^{R\lambda,O})^1 + (Y_T^{R\lambda,O})^2\big) - T\lambda(q^1\rho^1 + q^2\rho^2), \end{aligned}$$

where

$$\begin{pmatrix} \tilde{q}^1 \\ \tilde{q}^2 \end{pmatrix} := O\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \frac{1}{\sqrt{|\rho^1|^2 + |\rho^2|^2}}\begin{pmatrix} \rho^2 q^1 - \rho^1 q^2 \\ \rho^1 q^1 + \rho^2 q^2 \end{pmatrix},$$

and so Proposition 4.19 and (4.54) yield

$$\begin{aligned} h_0 &\le -\log E_{P^{R\lambda}}\big[\exp\big(-H_0^{R\lambda,\mathrm{Sym}}/d_0\big)\big]^{d_0} \\ &= -\frac{\gamma T}{2}\big(1 - |\rho^1|^2 - |\rho^2|^2\big)\frac{(\tilde{q}^1 + \tilde{q}^2)^2}{2 - |\rho^1|^2 - |\rho^2|^2} - T\lambda(q^1\rho^1 + q^2\rho^2), \end{aligned}$$

where $d_0 = \frac{1}{2}\operatorname{tr}(\Lambda) = \frac{1-|\rho^1|^2/2-|\rho^2|^2/2}{\gamma(1-|\rho^1|^2-|\rho^2|^2)}$. Due to the symmetry of the model, we can interchange ρ^1 and ρ^2 and, simultaneously, q^1 and q^2. This leads to

$$\begin{aligned} h_0 \le &-T\lambda(q^1\rho^1 + q^2\rho^2) - \gamma T\big(1 - |\rho^1|^2 - |\rho^2|^2\big) \\ &\times \frac{\max\big\{\big(\rho^1(q^1 - q^2) + \rho^2(q^1 + q^2)\big)^2, \big(\rho^2(q^2 - q^1) + \rho^1(q^1 + q^2)\big)^2\big\}}{2(2 - |\rho^1|^2 - |\rho^2|^2)(|\rho^1|^2 + |\rho^2|^2)}, \end{aligned} \tag{4.64}$$

which is a better bound for h_0 than (4.63) if and only if

$$1 < \frac{\max\big\{\big(\rho^1(q^1 - q^2) + \rho^2(q^1 + q^2)\big)^2, \big(\rho^2(q^2 - q^1) + \rho^1(q^1 + q^2)\big)^2\big\}}{(|q^1|^2 + |q^2|^2)(2 - |\rho^1|^2 - |\rho^2|^2)(|\rho^1|^2 + |\rho^2|^2)}. \tag{4.65}$$

We assume without loss of generality that $q^1 \neq 0$. Then $q^2 = cq^1$ for some $c \in \mathbb{R}$, and a calculation shows that (4.65) is equivalent to

$$(\rho^1,\rho^2) \notin \Big(\overline{\mathbb{D}}_{\frac{1}{\sqrt{2}}}(c^-,c^+) \cup \overline{\mathbb{D}}_{\frac{1}{\sqrt{2}}}(-c^-,-c^+)\Big) \cap \Big(\overline{\mathbb{D}}_{\frac{1}{\sqrt{2}}}(c^+,-c^-) \cup \overline{\mathbb{D}}_{\frac{1}{\sqrt{2}}}(-c^+,c^-)\Big), \tag{4.66}$$

where $c^\pm := \frac{1 \pm c}{2\sqrt{1+|c|^2}}$ and $\overline{\mathbb{D}}_{\frac{1}{\sqrt{2}}}(z)$ denotes the closed disk of radius $1/\sqrt{2}$ centered at $z \in \mathbb{R}^2$. Note that $|c^-|^2 + |c^+|^2 = 1/2$ so that the centers of all four disks in (4.66) lie on a circle of radius $1/\sqrt{2}$ centered at the origin. Figure 4.1 below shows in green the area on which (4.66) holds and in red its complement in the unit disk. In the green area, the symmetrised bound

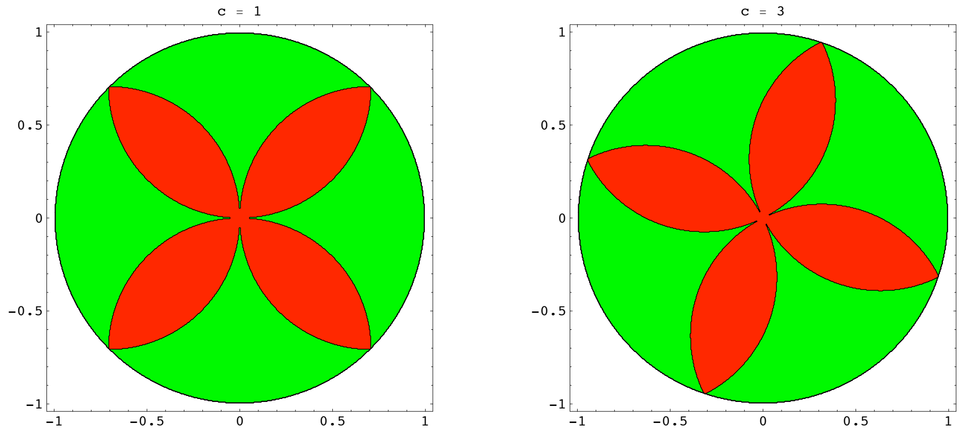

Figure 4.1: Graphical visualisation of (4.66) for $c = 1$ (left panel) and $c = 3$ (right panel) with ρ^1 on the horizontal and ρ^2 on the vertical axis. We have $c^- = 0$, $c^+ = 1/\sqrt{2}$ (left panel) and $c^- = -1/\sqrt{10}$, $c^+ = 2/\sqrt{10}$ (right panel).

(4.64) is better than (4.63), and vice versa in the red area. The green area amounts to $2/\pi \approx 63.66\,\%$ of the total surface of the unit disk. In principle, the bigger $|\rho^1|^2 + |\rho^2|^2$ is and the nearer (ρ^1, ρ^2) is to one of the points (c^-, c^+), $(-c^-, -c^+)$, $(c^+, -c^-)$ or $(-c^+, c^-)$, the more likely it is that (ρ^1, ρ^2) is in the green area and the symmetrised bound is better. This reflects the idea that if H is more symmetric with respect to $Y^{R\lambda,O}$ and the eigenvalues of Λ differ a lot, then making everything symmetric will achieve more than only squeezing the eigenvalues together. $\diamondsuit$

4.6 Appendix: Auxiliary results

Lemma 4.21. *The function $f(A, z) = \frac{1}{2} z' A^{-1} z$ in (4.3) is jointly convex.*

Proof. It is enough to show that, for fixed $z, y \in \mathbb{R}^n$ and $A, F \in \mathcal{S}^n$,

$$z' A^{-1} z + y' F^{-1} y \geq (z + y)'(A + F)^{-1}(z + y). \tag{4.67}$$

We first note that there is $C \in \mathrm{GL}(n)$ such that $C'AC = I$ and $D := C'FC$ is diagonal. Indeed, $A = U'U$ for some $U \in \mathrm{GL}(n)$, and $(U^{-1})'FU^{-1}$ is symmetric; so there exists $V \in \mathrm{O}(n)$ with $V'(U^{-1})'FU^{-1}V$ diagonal, and $C := U^{-1}V$ will do. Thus (4.67) is equivalent to

$$|\check{z}|^2 + \check{y}' D^{-1} \check{y} \geq (\check{z} + \check{y})'(I + D)^{-1}(\check{z} + \check{y}), \quad \check{z} := C'z \;\text{ and }\; \check{y} := C'y,$$

or, with $D = \mathrm{diag}(D^{11}, \ldots, D^{nn})$, to

$$\sum_{j=1}^{n} \big(|\check{z}^j|^2 + |\check{y}^j|^2 / D^{jj} \big) \geq \sum_{j=1}^{n} \frac{|\check{z}^j + \check{y}^j|^2}{1 + D^{jj}}.$$

But the last relation is true because for $j = 1, \dots, n$, we have

$$\big(|\breve{z}^j|^2 + |\breve{y}^j|^2/D^{jj}\big) - \frac{|\breve{z}^j + \breve{y}^j|^2}{1 + D^{jj}} = \frac{\big|\breve{z}^j\sqrt{D^{jj}} - \breve{y}^j/\sqrt{D^{jj}}\big|^2}{1 + D^{jj}} \geq 0.$$

$\square$

Lemma 4.22. *In the setting of Section 4.3.3, we have*

$$\Big(\int Z \,\mathrm{d}B\Big) \circ U_t = \int (Z \circ U_t) \,\mathrm{d}B^u. \tag{4.68}$$

for any predictable process Z on Wiener space with $\int_0^T |Z_s|^2 \,\mathrm{d}s < \infty$ a.s.

Proof. By Itô's representation theorem, any local martingale is of the form $c + \int \beta \,\mathrm{d}B$ for a constant c and a predictable process β with $\int_0^T |\beta_s|^2 \,\mathrm{d}s < \infty$ a.s. Therefore, (4.68) is equivalent to

$$\Big\langle \Big(\int Z \,\mathrm{d}B\Big) \circ U_t, \int \beta \,\mathrm{d}B \Big\rangle = \Big\langle \int (Z \circ U_t) \,\mathrm{d}B^u, \int \beta \,\mathrm{d}B \Big\rangle \tag{4.69}$$

for any predictable β with $\int_0^T |\beta_s|^2 \,\mathrm{d}s < \infty$ a.s. To prove (4.69), we note first that $P \circ U_t^{-1} = P$ by the invariance of Wiener measure under orthogonal transformations, and thus

$$E[X \circ U_t] = E[X] \quad \text{for all } X \in L^1. \tag{4.70}$$

This implies that the (local) martingale property is invariant under U_t, i.e., for an adapted integrable process M, we have

$$M \text{ is a (local) martingale} \iff M \circ U_t \text{ is a (local) martingale.} \tag{4.71}$$

Indeed, if τ is a stopping time and $M_{\tau\wedge\cdot}$ is a martingale, then $\tau \circ U_t$ is a stopping time and we have for any $s \in [0, T]$ and $A \in \mathcal{F}_s$ that

$$\begin{aligned} E\big[(M \circ U_t)_{(\tau\circ U_t)\wedge T}\mathbb{1}_A\big] &= E\big[\big(M_{\tau\wedge T}\mathbb{1}_{U_t^{-1}(A)}\big) \circ U_t\big] = E\big[M_{\tau\wedge T}\mathbb{1}_{U_t^{-1}(A)}\big] \\ &= E\big[M_{\tau\wedge s}\mathbb{1}_{U_t^{-1}(A)}\big] = E\big[(M \circ U_t)_{\tau\wedge s}\mathbb{1}_A\big] \end{aligned}$$

by (4.70), using also that $U_t^{-1}(A) \in \mathcal{F}_s$. This gives "$\Longrightarrow$" in (4.71), and "$\Longleftarrow$" follows by symmetry.

We are now ready to prove (4.69). Its left-hand side equals

$$\Big\langle \Big(\int Z \,\mathrm{d}B\Big) \circ U_t, \int \beta \,\mathrm{d}B \Big\rangle = \int \mathrm{d}\Big\langle \Big(\int Z \,\mathrm{d}B\Big) \circ U_t, B \Big\rangle \beta, \tag{4.72}$$

and by (4.71) we have

$$\begin{aligned}\left\langle\left(\int Z\,\mathrm{d}B\right)\circ U_t, B\right\rangle &= \left\langle\left(\int Z\,\mathrm{d}B\right)\circ U_t, \big((\mathbb{1}_{[\![0,t]\!]}I + \mathbb{1}_{]\!]t,T]\!]}u^{-1})B\big)\circ U_t\right\rangle \\ &= \left\langle \int Z\,\mathrm{d}B, (\mathbb{1}_{[\![0,t]\!]}I + \mathbb{1}_{]\!]t,T]\!]}u^{-1})B\right\rangle\circ U_t \\ &= \int_0^{t\wedge\cdot} Z'\,\mathrm{d}s + \left(\int_t^{t\vee\cdot} Z'(u^{-1})'\,\mathrm{d}s\right)\circ U_t.\end{aligned}$$

Since $\left(\int Z'(u^{-1})'\,\mathrm{d}s\right)\circ U_t = \int (Z\circ U_t)'u\,\mathrm{d}s$, we obtain from (4.72) that

$$\begin{aligned}\left\langle\left(\int Z\,\mathrm{d}B\right)\circ U_t, \int \beta\,\mathrm{d}B\right\rangle &= \int \big(\mathbb{1}_{[\![0,t]\!]}Z'\beta + \mathbb{1}_{]\!]t,T]\!]}(Z\circ U_t)'u\beta\big)\,\mathrm{d}s \\ &= \left\langle \int (Z\circ U_t)\,\mathrm{d}B^u, \int \beta\,\mathrm{d}B\right\rangle,\end{aligned}$$

which shows (4.69) and concludes the proof. □

The following result is the analogue of Proposition 4.17 for case (II).

Proposition 4.23. *Under the assumptions of case (II) in Section 4.5.3 with the additional requirement that R is $\hat{\mathbb{Y}}$-predictable, the BSDE*

$$\Gamma_s = H - \int_s^T \frac{1}{2} Z_r'\Lambda_r^{-1}Z_r\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}\hat{Y}_r, \quad 0\le s\le T \tag{4.73}$$

has a unique solution (Γ, Z) where Γ is a real-valued bounded continuous $\big(\hat{\mathbb{Y}}, \hat{P}\big)$-semimartingale and Z is an $\mathbb{R}^n$-valued $\hat{\mathbb{Y}}$-predictable process such that $\int_0^T |Z_s|^2\,\mathrm{d}s < \infty$ almost surely. Moreover, for any $s\in[0,T]$,

$$V_s^H = -\exp\left(-\gamma\Gamma_s - \frac{1}{2}E_{\hat{P}}\left[\int_s^T |\lambda_r|^2\,\mathrm{d}r\,\middle|\,\mathcal{W}_s\right]\right) \quad a.s. \tag{4.74}$$

Proof. This follows the same idea as Proposition 4.17, using additionally that the mean-variance tradeoff $\int_0^T |\lambda_r|^2\,\mathrm{d}r$ is in case (II) attainable by trading in S. In more detail, we replace P in Section 4.2 by $\hat{P}$ and set

$$B := \hat{Y}, \quad \mathbb{F} := \hat{\mathbb{Y}}, \quad \alpha := 0 \quad\text{and}\quad \chi := 0$$

to bring (4.73) into the form (4.2). By Lemma 4.2, (4.73) has a unique solution (Γ, Z), and $\int Z\,\mathrm{d}\hat{Y}$ is in $BMO\big(\hat{\mathbb{Y}}, \hat{P}\big)$. Since $R\lambda$ is bounded, $\int Z\,\mathrm{d}Y$ is a $(\mathbb{G}, P)$-martingale, and because Γ is bounded, we obtain from (4.73) that $E_P\big[\frac{1}{2}\int_0^T Z_s'\Lambda_s^{-1}Z_s\,\mathrm{d}s\big] < \infty$, which implies $E_P\big[\int_0^T |Z_s|^2\,\mathrm{d}s\big] < \infty$ due to (4.49).

To deal with the term involving λ, we use Itô's representation theorem as in Lemma 1.6.7 of Karatzas and Shreve [39] and obtain a $\mathbb{W}$-predictable process $\eta = (\eta_s)_{0\le s\le T}$ with $E_{\hat{P}}\big[\int_0^T |\eta_s|^2\,\mathrm{d}s\big] < \infty$ and

$$\begin{aligned}\frac{1}{2\gamma}\int_0^T |\lambda_s|^2\,\mathrm{d}s &= \frac{1}{2\gamma}E_{\hat{P}}\left[\int_0^T |\lambda_s|^2\,\mathrm{d}s\right] + \int_0^T \eta_s\,\mathrm{d}\hat{W}_s \\ &= \frac{1}{2\gamma}E_{\hat{P}}\left[\int_0^T |\lambda_s|^2\,\mathrm{d}s\right] + \int_0^T \eta_s'\lambda_s\,\mathrm{d}s + \int_0^T \eta_s\,\mathrm{d}W_s. \qquad (4.75)\end{aligned}$$

Here we use that λ is $\mathbb{W}$-predictable in case (II), recalling that $\mathbb{W} = \mathbb{F}^W$. As λ is bounded, $\int \eta\,\mathrm{d}\hat{W}$ is in $BMO\big(\mathbb{G},\hat{P}\big)$ and so $\int \eta\,\mathrm{d}W$ is in $BMO(\mathbb{G},P)$ by Theorem 3.6 of Kazamaki [40]. For the solution (Γ, Z) of (4.73), we set

$$\big(\check{\Gamma},\hat{Z},\check{Z}\big) := \left(\Gamma + \frac{1}{2\gamma}E_{\hat{P}}\left[\int_\cdot^T |\lambda_s|^2\,\mathrm{d}s\middle|\mathcal{W}_\cdot\right], -\eta + R'Z, \sqrt{I-RR'}Z\right)$$

and calculate

$$\begin{aligned}\mathrm{d}\check{\Gamma}_s &= \mathrm{d}\Gamma_s + \frac{1}{2\gamma}\mathrm{d}\left(E_{\hat{P}}\left[\int_0^T |\lambda_r|^2\,\mathrm{d}r\middle|\mathcal{W}_s\right] - \int_0^s |\lambda_r|^2\,\mathrm{d}r\right) \\ &= \mathrm{d}\Gamma_s + \eta_s'\lambda_s\,\mathrm{d}s + \eta_s\,\mathrm{d}W_s - \frac{1}{2\gamma}|\lambda_s|^2\,\mathrm{d}s \\ &= \left(\frac{\gamma}{2}\big|\check{Z}_s\big|^2 - \hat{Z}_s'\lambda_s - \frac{1}{2\gamma}|\lambda_s|^2\right)\mathrm{d}s - \hat{Z}_s\,\mathrm{d}W_s - \check{Z}_s\,\mathrm{d}W_s^\perp, \quad 0\le s\le T\end{aligned}$$

by (4.48), (4.51) and (4.75). Hence $\big(\check{\Gamma},\hat{Z},\check{Z}\big)$ solves (4.55) and we also have

$$\begin{aligned}&E_P\left[\int_0^T \big((\eta_s + R_s'Z_s)'(\eta_s + R_s'Z_s) + Z_s'(I - R_sR_s')Z_s\big)\,\mathrm{d}s\right] \\ &\le 3E_P\left[\int_0^T |Z_s|^2\,\mathrm{d}s\right] + 2E_P\left[\int_0^T |\eta_s|^2\,\mathrm{d}s\right] < \infty.\end{aligned}$$

Finally, (4.74) follows from the uniqueness of solutions to (4.55). □

Chapter 5

A convergence result for BSDEs

This chapter yields an explicitly computable sequence that converges to the indifference value in a two-dimensional Brownian model with stochastic correlation.

5.1 Introduction

In a basic model, the financial market consists of a risk-free bank account and a stock S driven by a Brownian motion W. The contingent claim H to be valued via exponential utility indifference depends on another Brownian motion Y, which has instantaneous correlation ρ with W. If ρ is deterministic and constant in time, an explicit formula for the indifference value h_t is available from Tehranchi [56]. However, for general (stochastic and/or time-dependent) ρ, only bounds and a structurally explicit but not an explicit formula[3] for h_t are known; see Chapter 2.

Our starting point to study this problem with general ρ is the well-known characterisation of $(h_t)_{0\leq t\leq T}$ via a backward stochastic differential equation (BSDE). We deduce that if ρ is piecewise constant in time, we can obtain an explicit formula for h_t in the same way as for constant ρ, just by considering iteratively the BSDE on intervals where ρ is constant. For general ρ, the idea is to approximate ρ pointwise by a sequence $(\rho^n)_{n\in\mathbb{N}}$ of piecewise constant processes and to replace ρ by ρ^n in the BSDE so that the solutions have an explicit form. These solutions then converge to (a transform of) h_t by a convergence result for quadratic BSDEs, which we prove in a general continuous filtration. We thus have an explicitly known sequence which converges to h_t.

[3] Recall that according to the footnote 1 on page 6, a structurally explicit formula is defined as a formula which is explicit in principle, but, unlike an explicit formula, some parameters are not directly given in terms of the input parameters of the model.

The only point left is whether we can approximate ρ pointwise by a sequence $(\rho^n)_{n\in\mathbb{N}}$ of piecewise constant processes. To handle this issue, we first restrict our study to the case where ρ is deterministic. We show that the above approximation of ρ and thus that of h_t work in a general way if ρ is Riemann integrable. A bounded real-valued function is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous, which is satisfied for every correlation function in practice, except for "pathological examples". We also give such a counterexample where the approximation of h_t indeed fails. For general stochastic ρ, we prove that the approximation of h_t works if ρ has left-continuous paths.

This chapter is structured as follows. In Section 5.2, we state convergence results for quadratic BSDEs in a general continuous filtration. Section 5.3 gives a first application of these results. We show that the indifference value in a general continuous filtration with trading constraints enjoys a continuity property in the constraints. The results on the indifference valuation in a Brownian setting are contained in Section 5.4. We lay out the model and prove some preliminary results in Section 5.4.1. We then study in Section 5.4.2 the approximation of the indifference value h_t by applying the convergence results of Section 5.2. Section 5.4.3 shows a continuity property of h_t in the correlation ρ. Finally, the Appendix contains the proofs of the convergence results of Section 5.2.

5.2 Convergence results

The financial applications in the subsequent sections are based on convergence results for quadratic BSDEs in the setting of Morlais [46]. We first recall this framework and then state the main convergence theorem.

We work on a finite time interval $[0,T]$ for a fixed $T > 0$, *and we fix* $t \in [0,T]$ *throughout this section.* Let $\big(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{0\le s\le T}, P\big)$ be a filtered probability space satisfying the usual assumptions with $\mathcal{F} = \mathcal{F}_T$. We assume that $\mathbb{F}$ *is continuous*, i.e., all local martingales are continuous. We fix an $\mathbb{R}^d$-valued local martingale $M = (M_s)_{0\le s\le T}$ and take a nondecreasing and bounded process D $\big($e.g., $D = \arctan\big(\sum_{j=1}^d \langle M^j\rangle\big)\big)$ such that $\mathrm{d}\langle M\rangle = m\,m'\,\mathrm{d}D$ for an $\mathbb{R}^{d\times d}$-valued predictable process m.

Let us consider, for $0 \le s \le T$, the BSDE

$$\Gamma_s = H + \int_s^T f(r, Z_r)\,\mathrm{d}D_r + \frac{\beta}{2}\big(\langle N\rangle_T - \langle N\rangle_s\big) - \int_s^T Z_r\,\mathrm{d}M_r - (N_T - N_s), \quad (5.1)$$

where $f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable $\big(\mathcal{P}$ denotes the σ-field of all predictable sets on $\Omega \times [0,T]$ and $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d\big)$, $\beta \in \mathbb{R}$ is a constant and H is a bounded random variable. A *solution* of (5.1) is a triple (Γ, Z, N) satisfying (5.1), where Γ is a real-valued *bounded* continuous semimartingale, Z is an $\mathbb{R}^d$-valued predictable process with $E\big[\int_0^T |m_s Z_s|^2 \,\mathrm{d}D_s\big] < \infty$ and N is a real-valued square-integrable martingale null at 0 and strongly orthogonal to M.

Theorem 5.1. *Let $(f^n, \beta^n, H^n)_{n=1,2,\dots,\infty}$ be a sequence of $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable real-valued mappings, constants, and random variables uniformly bounded in L^∞, such that*

(i) there exist a nonnegative predictable κ^1 with $\big\| \int_0^T \kappa^1_s \,\mathrm{d}D_s \big\|_{L^\infty} < \infty$ and a constant c^1 such that

$$|f^n(s,z)| \leq \kappa^1_s + c^1 |m_s z|^2 \quad \text{for all } s \in [0,T],\ z \in \mathbb{R}^d \tag{5.2}$$

and $n = 1, \dots, \infty$;

(ii) there exist a nonnegative predictable κ^2 with $\big\| \int_0^T |\kappa^2_s|^2 \,\mathrm{d}D_s \big\|_{L^\infty} < \infty$ and a constant c^2 such that

$$\big|f^n(s,z^1) - f^n(s,z^2)\big| \leq c^2 \big(\kappa^2_s + |m_s z^1| + |m_s z^2|\big) \big|m_s(z^1 - z^2)\big|$$

for all $s \in [0,T]$, $z^1, z^2 \in \mathbb{R}^d$ and $n = 1, \dots, \infty$;

(iii) $\lim_{n\to\infty} \beta^n = \beta^\infty$, $\lim_{n\to\infty} H^n = H^\infty$ a.s. and for $(P \otimes D)$-almost all $(\omega, s) \in [\![t,T]\!]$, $\lim_{n\to\infty} f^n(s,z)(\omega) = f^\infty(s,z)(\omega)$ for all $z \in \mathbb{R}^d$.

Then there exist unique solutions (Γ^n, Z^n, N^n) to the BSDE (5.1) with parameters (f^n, β^n, H^n) for $n = 1, \dots, \infty$, and Γ^n_t converges to Γ^∞_t a.s. as $n \to \infty$. Moreover, $\sup_{s \in [t,T]} |\Gamma^n_s - \Gamma^\infty_s| \to 0$ as $n \to \infty$ in probability and in L^p, $1 \leq p < \infty$.

The proofs of Theorem 5.1 and of the next corollary are presented in the Appendix.

Corollary 5.2. *Suppose in addition to the assumptions of Theorem 5.1 that*

(iv) H^n converges to H^∞ in L^∞ as $n \to \infty$;

(v) there exist sequences $(\underline{a}^n)_{n\in\mathbb{N}}$ and $(\overline{a}^n)_{n\in\mathbb{N}}$ of deterministic functions which converge to 1 uniformly on $[t,T]$ (up to a $(P \otimes D)$-nullset) such that $f^n = \underline{a}^n \underline{f} + \overline{a}^n \overline{f}$ for every $n = 1, \dots, \infty$, where $\underline{f}$, $\overline{f}$ satisfy (5.2) with f^n replaced by $\underline{f}$, $\overline{f}$.

Then we have $\sup_{s\in[t,T]}|\Gamma^n_s-\Gamma^\infty_s|\to 0$ *in* L^∞ *as* $n\to\infty$ *and there even exists a constant* $K>0$ *such that for all* $n\in\mathbb{N}$,

$$\left\|\sup_{s\in[t,T]}|\Gamma^n_s-\Gamma^\infty_s|\right\|_{L^\infty}\le K\big(\|\underline{a}^n-1\|_{L^\infty(P\otimes D)}+\|\overline{a}^n-1\|_{L^\infty(P\otimes D)}+|\beta^n-\beta^\infty|+\|H^n-H^\infty\|_{L^\infty}\big). \tag{5.3}$$

Further, $\int Z^n\,\mathrm{d}M\to\int Z^\infty\,\mathrm{d}M$ *and* $N^n\to N^\infty$ *on* $[\![t,T]\!]$ *in* BMO *as* $n\to\infty$.

In the literature on BSDEs, convergence results are also called stability results. The main differences between Theorem 2.8 of Kobylanski [42] and our Theorem 5.1 are the following: Kobylanski [42] works in a Brownian setting and imposes locally uniform convergence on the generators, whereas our Theorem 5.1 is stated in a general continuous filtration and for generators $(f^n)_{n=1,\dots,\infty}$ that converge only pointwise. Moreover, the generators in Kobylanski's Theorem 2.8 can unlike ours also depend on Γ^n, and there is also an $L^2(P\otimes\mathrm{Leb})$-convergence result for $(Z^n)_{n=1,\dots,\infty}$, which is, however, less strong than the BMO-convergence in Corollary 5.2. Another convergence result in a Brownian setting is Proposition 7 of Briand and Hu [12], which gives convergence of the moments of $\exp\big(\sup_{s\in[t,T]}|\Gamma^n_s-\Gamma^\infty_s|\big)$ and $\big(\int_t^T|Z^n_s-Z^\infty_s|^2\,\mathrm{d}s\big)^{1/2}$ for unbounded terminal conditions if the generators are convex.

For a general continuous filtration, a convergence result for an exponential transformation of the BSDE (5.1) is available from Lemma 3.3 and Remark 3.4 of Morlais [46]. Lemma 3.3 serves in [46] as an auxiliary result to show existence of a solution to (5.1) with a more general generator f which can also depend on Γ. The proof of the existence result first establishes a one-to-one correspondence between solutions to (5.1) and those to a simpler BSDE which results from an exponential transformation of the original BSDE. Lemma 3.3 is then used in proving existence of a solution to the simpler BSDE. Due to the one-to-one correspondence between solutions to the original and to the simpler BSDEs, Lemma 3.3 gives also a convergence result for the original BSDE, as Morlais remarks. In particular, its application to (5.1) needs that $\exp(\beta^n H^n)$ and a certain transform of f^n are nondecreasing in n, and it yields $E\big[\sup_{s\in[t,T]}|\mathrm{e}^{\beta^n\Gamma^n_s}-\mathrm{e}^{\beta^\infty\Gamma^\infty_s}|\big]\to 0$, which is equivalent to $\sup_{s\in[t,T]}|\Gamma^n_s-\Gamma^\infty_s|\to 0$ in L^1 for $\beta^\infty\neq 0$; the equivalence can be shown using

$$\min\{\mathrm{e}^x,\mathrm{e}^y\}|x-y|\le|\mathrm{e}^x-\mathrm{e}^y|\le\max\{\mathrm{e}^x,\mathrm{e}^y\}|x-y|,\quad x,y\in\mathbb{R},$$

and that Γ^n is uniformly bounded in $n=1,\dots,\infty$ by Lemma 3.1 of Morlais [46]. In contrast to Morlais [46], who proves existence and uniqueness of solutions to (5.1), we focus on convergence questions and work in the proof of Theorem 5.1 directly with the BSDE (5.1) instead of doing first an exponential transformation. Standard BSDE comparison techniques and the application of BMO-theory enable us to prove in the Appendix the a.s. convergence of Γ_t^n under weak assumptions.

5.3 Indifference valuation under convergent constraints

Before we restrict our study to a Brownian setting, we give in this section a first application of Theorem 5.1 in a financial context. We work within the framework of Section 5.2 with a continuous filtration $\mathbb{F}$. Recall that M is a local martingale and $\mathrm{d}\langle M\rangle = m\,m'\,\mathrm{d}D$. We suppose that almost surely, the matrix m_s is invertible for every $s \in [0,T]$. The financial market consists of a risk-free bank account yielding zero interest and d risky assets whose price process $S = (S_s)_{0\le s\le T}$ is given by

$$\frac{\mathrm{d}S_s^j}{S_s^j} = \mathrm{d}M_s^j + \sum_{i=1}^d \lambda_s^i\,\mathrm{d}\langle M^j, M^i\rangle_s, \quad 0 \le s \le T,\ S_0^j > 0 \quad \text{for } j = 1,\dots,d,$$

where λ is a predictable process which satisfies $\big\|\int_0^T |m_s\lambda_s|^2\,\mathrm{d}D_s\big\|_{L^\infty} < \infty$, i.e., the mean-variance tradeoff process is bounded. Let H be a bounded random variable, interpreted as a contingent claim or payoff due at time T, and let $C \subseteq \mathbb{R}^d$ be a closed set with $0 \in C$. We assume that our investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. Starting at time t with bounded $\mathcal{F}_t$-measurable capital x_t, she runs a self-financing strategy $\pi = (\pi_s)_{t\le s\le T}$ valued in C so that her wealth at time $s \in [t,T]$ is $X_s^{x_t,\pi} = x_t + \int_t^s \sum_{j=1}^d \frac{\pi_r^j}{S_r^j}\,\mathrm{d}S_r^j$, where π^j represents the amount invested in S^j, $j = 1,\dots,d$. The set $\mathcal{A}_t^C$ of *C-admissible strategies* on $[t,T]$ consists of all predictable $\mathbb{R}^d$-valued processes $\pi = (\pi_s)_{t\le s\le T}$ which satisfy a.s., $\pi_s \in C$ for all $s \in [t,T]$, $E\big[\int_t^T |m_s\pi_s|^2\,\mathrm{d}D_s\big] < \infty$ and are such that $\exp(-\gamma X_s^{x_t,\pi})$, $t \le s \le T$, is of class (D). We define $V_t^{H,C}(x_t)$ by

$$\begin{aligned} V_t^{H,C}(x_t) &:= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t^C} E\big[U(X_T^{x_t,\pi} + H)\big|\mathcal{F}_t\big] \\ &= \mathrm{e}^{-\gamma x_t}\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t^C} E\big[-\exp\big(-\gamma X_T^{0,\pi} - \gamma H\big)\big|\mathcal{F}_t\big] \end{aligned} \tag{5.4}$$

so that $V_t^{H,C}(x_t)$ is the maximal expected utility the investor can achieve by starting at time t with initial capital x_t, using some C-admissible strategy π, and receiving H at time T. For ease of notation, we write

$$V_t^{H,C}(x_t) = \mathrm{e}^{-\gamma x_t} V_t^{H,C}(0) =: \mathrm{e}^{-\gamma x_t} V_t^{H,C}. \tag{5.5}$$

Viewed over time t, $V^{H,C}$ is then the *dynamic value process* for the stochastic control problem associated to exponential utility maximisation.

The time t *indifference (buyer) value* $h_t^{H,C}(x_t)$ for H is implicitly defined by

$$V_t^{0,C}(x_t) = V_t^{H,C}\big(x_t - h_t^{H,C}(x_t)\big). \tag{5.6}$$

This says that the investor is indifferent between solely trading with initial capital x_t, versus trading with initial capital $x_t - h_t^{H,C}(x_t)$ but receiving H at T. By (5.5),

$$h_t^{H,C}(x_t) = h_t^{H,C} = \frac{1}{\gamma}\log\frac{V_t^{0,C}}{V_t^{H,C}} \tag{5.7}$$

does not depend on x_t.

The following proposition can be seen as a kind of continuity result for $V_t^{H,C}$ and $h_t^{H,C}$ in (H,C).

Proposition 5.3. *Let H^n, $n = 1,2,\dots,\infty$, be uniformly bounded random variables with $\lim_{n\to\infty} H^n = H^\infty$ a.s., and let C^n, $n = 1,2,\dots,\infty$, be closed subsets of $\mathbb{R}^d$ which contain zero and are such that ($P\otimes D$)-a.e.,*

$$\lim_{n\to\infty}\inf_{y\in C^n}|m(y-z)| = \inf_{y\in C^\infty}|m(y-z)| \quad \textit{for all } z\in\mathbb{R}^d. \tag{5.8}$$

Then $\lim_{n\to\infty} V_t^{H^n,C^n} = V_t^{H^\infty,C^\infty}$ and $\lim_{n\to\infty} h_t^{H^n,C^n} = h_t^{H^\infty,C^\infty}$ a.s., and there exist continuous versions V^{H^n,C^n} and h^{H^n,C^n}, $n = 1,\dots,\infty$, such that

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{s\in[t,T]}\big|V_s^{H^n,C^n} - V_s^{H^\infty,C^\infty}\big| = 0 \textit{ in probab. and in } L^p,\ 1\le p<\infty,\\
&\lim_{n\to\infty}\sup_{s\in[t,T]}\big|h_s^{H^n,C^n} - h_s^{H^\infty,C^\infty}\big| = 0 \textit{ in probab. and in } L^p,\ 1\le p<\infty.
\end{aligned} \tag{5.9}$$

Proof. Fix $n\in\{1,\dots,\infty\}$. By Theorem 4.1 of Morlais [46], there is a version V^{H^n,C^n} such that $V^{H^n,C^n} = -\exp(\gamma\Gamma^n)$, where (Γ^n, Z^n) is the solution of (5.1) with $\beta := \gamma$, H replaced by $-H^n$ and with generator f^n given by

$$f^n(s,z) := \inf_{y\in C^n}\left(\frac{\gamma}{2}\Big|m_s\Big(y - z - \frac{1}{\gamma}\lambda_s\Big)\Big|^2\right) - (m_s z)'(m_s\lambda_s) - \frac{1}{2\gamma}|m_s\lambda_s|^2$$

for $s\in[0,T]$ and $z\in\mathbb{R}^d$. Remarks 2.3 and 2.4 of Morlais [46] and (5.8) imply that the assumptions (i)–(iii) of Theorem 5.1 are satisfied, which yields

$$\lim_{n\to\infty}\Gamma_t^n = \Gamma_t^\infty \text{ a.s. and } \lim_{n\to\infty}\sup_{s\in[t,T]}|\Gamma_s^n - \Gamma_s^\infty| = 0 \text{ in probab. and in } L^p. \tag{5.10}$$

Therefore, we obtain $\lim_{n\to\infty} V_t^{H^n,C^n} = \lim_{n\to\infty} -\mathrm{e}^{\gamma\Gamma_t^n} = -\mathrm{e}^{\gamma\Gamma_t^\infty} = V_t^{H^\infty,C^\infty}$ a.s. and analogously $\lim_{n\to\infty} V_t^{0,C^n} = V_t^{0,C^\infty}$ a.s., so $\lim_{n\to\infty} h_t^{H^n,C^n} = h_t^{H^\infty,C^\infty}$

a.s. by (5.7). Because we have

$$\begin{aligned}\sup_{s\in[t,T]}\left|h_s^{H^n,C^n}-h_s^{H^\infty,C^\infty}\right| &= \frac{1}{\gamma}\sup_{s\in[t,T]}\left|\log\frac{V_s^{0,C^n}}{V_s^{0,C^\infty}}-\log\frac{V_s^{H^n,C^n}}{V_s^{H^\infty,C^\infty}}\right|\\ &\le \frac{1}{\gamma}\sup_{s\in[t,T]}\left|\log\frac{V_s^{0,C^n}}{V_s^{0,C^\infty}}\right|+\sup_{s\in[t,T]}|\Gamma_s^n-\Gamma_s^\infty|,\end{aligned}$$

we obtain (5.9) from (5.10) and its analogue with (H^n, C^n) replaced by $(0, C^n)$. We also have $\lim_{n\to\infty}\sup_{s\in[t,T]}\left|\mathrm{e}^{\gamma\Gamma_s^n}-\mathrm{e}^{\gamma\Gamma_s^\infty}\right| = 0$ in probab. and in L^p, since

$$\sup_{s\in[t,T]}\left|\mathrm{e}^{\gamma\Gamma_s^n}-\mathrm{e}^{\gamma\Gamma_s^\infty}\right|\le\gamma\exp\left(\gamma\left\|\sup_{n=1,\dots,\infty}\sup_{s\in[t,T]}|\Gamma_s^n|\right\|_{L^\infty}\right)\sup_{s\in[t,T]}|\Gamma_s^n-\Gamma_s^\infty|$$

and Γ^n is uniformly bounded by Lemma 3.1 of Morlais [46]. This concludes the proof because $V^{H^n,C^n}=-\exp(\gamma\Gamma^n)$ for a version V^{H^n,C^n}. □

Remark 5.4. **1)** The condition (5.8) can be rephrased as follows. Define a time-dependent random inner product $\langle\cdot,\cdot\rangle_m$ by $\langle x,y\rangle_m := x'm'my$ for x, y in $\mathbb{R}^d$ and denote by d_m the induced metric, i.e., $d_m(x,y) := \langle x-y, x-y\rangle_m$ for $x, y \in \mathbb{R}^d$. Then $\langle\cdot,\cdot\rangle_m$ is the standard scalar product on $\mathbb{R}^d$ after a basis transformation by m^{-1}. Defining $d_m(x,C) := \inf_{y\in C} d_m(x,y)$ for a closed set $C\subseteq\mathbb{R}^d$, the condition (5.8) is equivalent to $\lim_{n\to\infty} d_m(x,C^n) = d_m(x,C^\infty)$ for all $x\in\mathbb{R}^d$. This means that the sets $(C^n)_{n\in\mathbb{N}}$ are Wijsman convergent to C^∞ with respect to the metric d_m. A survey on Wijsman convergence, which is a weaker notion than convergence in the Hausdorff metric, is provided by Beer [6].

2) We have used an exponential utility function $U(x) = -\exp(-\gamma x)$, $x\in\mathbb{R}$, for a fixed $\gamma>0$. By applying Theorems 4.4 and 4.7 of Morlais [46], analogous results can be derived for the value process related to power utility $U(x) = x^\gamma/\gamma$, $x>0$, for a fixed $\gamma\in\,]0,1[$, and to logarithmic utility $U(x)=\log x$, $x>0$, when there is no claim, i.e., $H=0$. ◊

5.4 Indifference valuation in a Brownian setting

We now apply the convergence Theorem 5.1 to the indifference valuation in a Brownian setting with variable correlation. We first introduce in Section 5.4.1 the model and explain the problem. We then apply Theorem 5.1 in Sections 5.4.2 and 5.4.3 to give convergence results for the indifference value and the dynamic value process.

5.4.1 Model setup and preliminary results

We work on a finite time interval $[0,T]$ for a fixed $T > 0$, *and we fix* $t \in [0,T]$ *throughout this section.* On a complete filtered probability space $\big(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_s)_{0\le s\le T}, P\big)$, we have two independent one-dimensional $(\mathbb{G}, P)$-Brownian motions Y and $Y^\perp$. We denote by $\mathbb{Y} = (\mathcal{Y}_s)_{0\le s\le T}$ the P-augmented filtration generated by Y. Let W be a $(\mathbb{G}, P)$-Brownian motion with instantaneous correlation ρ to Y so that

$$\mathrm{d}W_s = \rho_s\,\mathrm{d}Y_s + \sqrt{1-\rho_s^2}\,\mathrm{d}Y_s^\perp, \quad 0 \le s \le T. \tag{5.11}$$

Our financial market consists of a risk-free bank account yielding zero interest and a traded risky asset S with dynamics

$$\mathrm{d}S_s = S_s\mu_s\,\mathrm{d}s + S_s\sigma_s\,\mathrm{d}W_s, \quad 0 \le s \le T,\ S_0 > 0;$$

the drift μ and the (positive) volatility σ are $\mathbb{G}$-predictable. We set $\lambda := \frac{\mu}{\sigma}$ and assume that $\int_0^T \lambda_s^2\,\mathrm{d}s$ is bounded. The processes

$$\hat{W} := W + \int \lambda\,\mathrm{d}s \quad\text{and}\quad \hat{Y} := Y + \int \rho\lambda\,\mathrm{d}s$$

are Brownian motions under the minimal martingale measure $\hat{P}$ given by

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \mathcal{E}\Big(-\int \lambda\,\mathrm{d}W\Big)_T := \exp\Big(-\int_0^T \lambda_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda_s^2\,\mathrm{d}s\Big). \tag{5.12}$$

In contrast to Section 5.3, the investor here can trade in S without constraints. He starts at time t with bounded $\mathcal{G}_t$-measurable capital x_t and runs a self-financing strategy $\pi = (\pi_s)_{t\le s\le T}$ so that his wealth at time $s \in [t,T]$ is

$$X_s^{x_t,\pi} = x_t + \int_t^s \frac{\pi_r}{S_r}\,\mathrm{d}S_r = x_t + \int_t^s \pi_r\sigma_r\,\mathrm{d}\hat{W}_r, \tag{5.13}$$

where π represents the amount invested in S. For a bounded random variable H, we define $V_t^H(x_t)$ like $V_t^{H,C}(x_t)$ in (5.4) with $\mathcal{G}_t$ instead of $\mathcal{F}_t$ and $\mathcal{A}_t^C$ replaced by $\mathcal{A}_t$ which consists of all $\mathbb{G}$-predictable real-valued processes $\pi = (\pi_s)_{t\le s\le T}$ which satisfy $\int_t^T \pi_s^2\sigma_s^2\,\mathrm{d}s < \infty$ a.s. and are such that

$$\exp(-\gamma X_s^{x_t,\pi}),\ t \le s \le T,\ \text{is of class } (D) \text{ on } (\Omega, \mathcal{G}_T, \mathbb{G}, P). \tag{5.14}$$

The dynamic value process V^H and the indifference value h_t^H are defined analogously to (5.5) and (5.6). From (5.7), we see that once we can calculate V_t^H and V_t^0, we also know h_t^H. So our focus lies on studying V_t^H.

We always impose without further mention the *standing assumption* that

$$H \in L^\infty(\mathcal{Y}_T, P) \text{ and } \lambda,\ \rho \text{ are } \mathbb{Y}\text{-predictable.} \tag{5.15}$$

This reflects a situation where the payoff H is driven by Y, whereas hedging can only be done in S which is in general imperfectly correlated with Y. We refer to Section 2.4.1 for a thorough explanation and motivation of the standing assumption (5.15), which corresponds to case (I). $\big($For case (II), results analogous to those in Section 5.4.2 can be derived if ρ is predictable for the filtration generated by $\hat{Y}$.$\big)$

We next state a BSDE characterisation for V^H.

Lemma 5.5. *The BSDE*

$$\Gamma_s = H - \int_s^T \Big(\frac{1}{2}\gamma(1-\rho_r^2)Z_r^2 - Z_r\rho_r\lambda_r - \frac{\lambda_r^2}{2\gamma}\Big)\,\mathrm{d}r + \int_s^T Z_r\,\mathrm{d}Y_r \tag{5.16}$$

for $0 \le s \le T$ has a unique solution (Γ, Z) where Γ is a real-valued bounded continuous $(\mathbb{Y}, P)$-semimartingale and Z is a $\mathbb{Y}$-predictable process such that $E_P\big[\int_0^T Z_s^2\,\mathrm{d}s\big] < \infty$. Moreover, there exists a continuous version V^H (which we always use in the sequel) such that $V^H = -\exp(-\gamma\Gamma)$.

Lemma 5.5 is essentially well known. In particular, Proposition 4.17 gives a multidimensional version. However, two assumptions of that proposition are not satisfied in our setting; $\mathbb{G}$ is not necessarily generated by W and a Brownian motion orthogonal to W, and $|\rho|$ is not bounded away from 1. Instead of adapting the proof of Proposition 4.17, we give the complete argument.

Proof of Lemma 5.5. Existence and uniqueness of a solution (Γ, Z) of (5.16) follow from Theorem 5.1 with $\mathbb{F} := \mathbb{Y}$, $M := -Y$ and

$$f(s,z) := -\frac{1}{2}\gamma(1-\rho_s^2)z^2 + z\rho_s\lambda_s + \frac{\lambda_s^2}{2\gamma} \quad \text{for } s \in [0,T] \text{ and } z \in \mathbb{R}.$$

$\big($Since any $\mathbb{Y}$-martingale orthogonal to Y is constant, we can choose in (5.1) $\beta \in \mathbb{R}$ arbitrarily.$\big)$ Moreover, Proposition 7 of Mania and Schweizer [44] and its proof yield that $\int Z\,\mathrm{d}Y$ is in both $BMO(\mathbb{Y}, P)$ and $BMO(\mathbb{G}, P)$.

To establish the result, it remains to show $V_t^H = -\exp(-\gamma\Gamma_t)$. A simple calculation based on (5.13) and (5.16) yields for $\pi \in \mathcal{A}_t$ that

$$\begin{aligned}\exp\big(-\gamma X_s^{0,\pi}\big) &= \exp(\gamma\Gamma_s - \gamma\Gamma_t)\frac{\mathcal{E}\big(\int \gamma Z\,\mathrm{d}Y - \int \gamma\pi\sigma\,\mathrm{d}W\big)_s}{\mathcal{E}\big(\int \gamma Z\,\mathrm{d}Y - \int \gamma\pi\sigma\,\mathrm{d}W\big)_t} \\ &\qquad \times \exp\bigg(\frac{1}{2}\int_t^s (\gamma\rho_r Z_r + \lambda_r - \gamma\pi_r\sigma_r)^2\,\mathrm{d}r\bigg) \qquad (5.17) \\ &\ge \exp(\gamma\Gamma_s - \gamma\Gamma_t)\frac{\mathcal{E}\big(\int \gamma Z\,\mathrm{d}Y - \int \gamma\pi\sigma\,\mathrm{d}W\big)_s}{\mathcal{E}\big(\int \gamma Z\,\mathrm{d}Y - \int \gamma\pi\sigma\,\mathrm{d}W\big)_t}, \quad t \le s \le T.\end{aligned}$$

Therefore, if $\int \pi\sigma \, \mathrm{d}W \in BMO(\mathbb{G}, P)$, we obtain

$$E_P\Big[\exp\big(-\gamma X_T^{0,\pi} - \gamma H\big)\Big|\mathcal{G}_t\Big] \geq \exp(-\gamma\Gamma_t), \tag{5.18}$$

since the stochastic exponential of a continuous BMO-martingale is a true martingale by Theorem 2.3 of Kazamaki [40]. By a localisation argument and (5.14), we have (5.18) for every $\pi \in \mathcal{A}_t$, which implies $V_t^H \leq -\exp(-\gamma\Gamma_t)$. Equality in (5.18) holds for $\pi = \pi^\star := \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma}$. Since $\exp\big(-\gamma X^{0,\pi^\star}\big)$ is by (5.17) the product of a bounded process and a $(\mathbb{G}, P)$-martingale, it is of class (D) on $(\Omega, \mathcal{G}_T, \mathbb{G}, P)$; hence $\pi^\star \in \mathcal{A}_t$ and $V_t^H = -\exp(-\gamma\Gamma_t)$. $\square$

Although V^H is given in terms of the solution of (5.16), there is no computable formula available for V_t^H unless ρ is deterministic and constant in time. While the methods in Chapters 2 and 4 give bounds for V_t^H, we approximate in the subsequent sections V_t^H by approaching ρ by piecewise constant processes. Let us denote by Ξ the set of all processes q of the form

$$q = q^1 \mathbb{1}_{\{\tau_0\}} + \sum_{j=1}^n q^j \mathbb{1}_{]\!]\tau_{j-1},\tau_j]\!]}, \quad \text{for } t = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T,$$

where τ_j is a $\mathbb{Y}$-stopping time and q^j is a $\mathcal{Y}_{\tau_{j-1}}$-measurable random variable valued in $]-1,1[$. We call $(q^j, \tau_j)_{j=1,\dots,n}$ a *characterising sequence* of q.

Proposition 5.6. *Let q be a bounded $\mathbb{Y}$-predictable process. The BSDE*

$$\Gamma_s^q = H - \int_s^T \Big(\frac{1}{2}\gamma(1-q_r^2)|Z_r^q|^2 - Z_r^q\rho_r\lambda_r - \frac{\lambda_r^2}{2\gamma}\Big)\,\mathrm{d}r + \int_s^T Z_r^q \,\mathrm{d}Y_r \tag{5.19}$$

for $0 \leq s \leq T$ has a unique solution (Γ^q, Z^q) (in the sense of Lemma 5.5).
1) If $q \in \Xi$ with characterising sequence $(q^j, \tau_j)_{j=1,\dots,n}$, then

$$\mathrm{e}^{-\gamma\Gamma_t^q} = E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[E_{\hat{P}}\left[\mathrm{e}^{\hat{H}(1-|q^n|^2)}\Big|\mathcal{Y}_{\tau_{n-1}}\right]^{\frac{1-|q^{n-1}|^2}{1-|q^n|^2}}\Big|\mathcal{Y}_{\tau_{n-2}}\right]^{\frac{1-|q^{n-2}|^2}{1-|q^{n-1}|^2}}\cdots\Big|\mathcal{Y}_t\right]^{\frac{1}{1-|q^1|^2}}, \tag{5.20}$$

where $\hat{H} := -\gamma H - \frac{1}{2}\int_t^T \lambda_s^2 \,\mathrm{d}s$.
2) If $|q| \geq |\rho|$ $(P\otimes\mathrm{Leb})$-almost everywhere, then $V^H \leq -\exp(-\gamma\Gamma^q)$.
3) If $|q| \leq |\rho|$ $(P\otimes\mathrm{Leb})$-almost everywhere, then $V^H \geq -\exp(-\gamma\Gamma^q)$.

If ρ itself is in Ξ, Proposition 5.6 gives explicit formulas for V_t^H and h_t^H by choosing $q = \rho$ and using (5.7). For general ρ, the idea is to find a sequence $(q^n)_{n\in\mathbb{N}}$ in Ξ which converges pointwise to ρ. The solutions $\Gamma_t^{q^n}$ of (5.19) with $q = q^n$ have the explicit form (5.20) and converge a.s. to the solution $\Gamma_t = \Gamma_t^\rho$

of (5.16) by Theorem 5.1. We thus obtain an explicitly known sequence converging a.s. to V_t^H. The only open point, which we treat in Section 5.4.2, is whether we can find a sequence $(q^n)_{n\in\mathbb{N}}$ in Ξ which converges pointwise to ρ.

Note that the right-hand side of (5.20) is *not* the value of V_t^H in a model with correlation q instead of ρ. Comparing (5.19) with (5.16), we see that only the ρ in front of $|Z|^2$ is replaced by q; the ρ in the term linear in Z is kept. This implies that the measure used in the iterated expectations in (5.20) is $\hat{P}$, which does not depend on q — a property desired for the above-mentioned approximation of V_t^H, since we prefer to take always the same explicitly known measure in calculating the conditional expectations. If we replace ρ in (5.19) by q, the solution of the BSDE is linked to the value of V_t^H when ρ is replaced by q. In Section 5.4.3, we deduce from this a continuity property of V_t^H in ρ.

Parts 2) and 3) of Proposition 5.6 can be seen as a monotonicity property of V_t^H. However, since ρ still appears in (5.19), we cannot simply say that V_t^H is monotonic in $|\rho|$. This has already been pointed out in Section 2.5 by saying that V_t^H is monotonic in $|\rho|$ only when the measure $\hat{P}$ from (5.12), which depends via W on ρ, is kept fixed. Proposition 2.14 gives a result analogous to parts 2) and 3) of Proposition 5.6 when $|q|$ and $|\rho|$ can be separated by a constant. Proposition 5.6 shows that this additional assumption is superfluous and thus generalises Proposition 2.14 as announced in Remark 2.15.

Proof of Proposition 5.6. Like in the proof of Lemma 5.5, (5.19) has a unique solution (Γ^q, Z^q) and $\int Z^q\,\mathrm{d}Y \in BMO(\mathbb{Y}, P)$. Theorem 3.6 of Kazamaki [40] yields $\int Z\,\mathrm{d}\hat{Y}$, $\int Z^q\,\mathrm{d}\hat{Y} \in BMO\big(\mathbb{Y}, \hat{P}\big)$ for the solution (Γ, Z) of (5.16), and as a consequence, their stochastic exponentials are true martingales.

To prove 1), we fix $j \in \{1,\dots,n\}$ and write (5.19), for $\tau_{j-1} \le s \le \tau_j$ as

$$\Gamma_s^q = \Gamma_{\tau_j}^q + \frac{1}{2\gamma}\int_s^{\tau_j} \lambda_r^2\,\mathrm{d}r - \frac{1}{2}\gamma(1-|q^j|^2)\int_s^{\tau_j} |Z_r^q|^2\,\mathrm{d}r + \int_s^{\tau_j} Z_r^q\,\mathrm{d}\hat{Y}_r,$$

which implies

$$\begin{aligned}&\mathrm{e}^{-\gamma(1-|q^j|^2)\Gamma_{\tau_{j-1}}^q}\exp\left(\gamma(1-|q^j|^2)\int_{\tau_{j-1}}^{\tau_j} Z_r^q\,\mathrm{d}\hat{Y}_r - \frac{1}{2}\gamma^2(1-|q^j|^2)^2\int_{\tau_{j-1}}^{\tau_j}|Z_r^q|^2\,\mathrm{d}r\right)\\ &= \exp\left(-\gamma(1-|q^j|^2)\Gamma_{\tau_j}^q - \frac{1-|q^j|^2}{2}\int_{\tau_{j-1}}^{\tau_j}\lambda_r^2\,\mathrm{d}r\right).\end{aligned}$$

Taking $\big(\mathcal{Y}_{\tau_{j-1}}, \hat{P}\big)$-conditional expectations and logarithms yields

$$\Gamma_{\tau_{j-1}}^q = -\frac{1}{\gamma(1-|q^j|^2)}\log E_{\hat{P}}\left[\exp\left(-\gamma\Gamma_{\tau_j}^q - \frac{1}{2}\int_{\tau_{j-1}}^{\tau_j}\lambda_r^2\,\mathrm{d}r\right)^{1-|q^j|^2}\middle|\mathcal{Y}_{\tau_{j-1}}\right].$$

Using this argument iteratively for $j = n, \dots, 1$ results in (5.20).

To prove 2), we subtract (5.16) from (5.19), which gives

$$\begin{aligned}\Gamma_s^q - \Gamma_s &= \frac{1}{2}\gamma \int_s^T \big((1-\rho_r^2)|Z_r|^2 - (1-q_r^2)|Z_r^q|^2\big)\,\mathrm{d}r + \int_s^T (Z_r^q - Z_r)\,\mathrm{d}\hat{Y}_r \\ &\geq \frac{1}{2}\gamma \int_s^T (1-\rho_r^2)\big(|Z_r|^2 - |Z_r^q|^2\big)\,\mathrm{d}r + \int_s^T (Z_r^q - Z_r)\,\mathrm{d}\hat{Y}_r \\ &= \int_s^T (Z_r^q - Z_r)\,\big(\mathrm{d}\hat{Y}_r - \kappa_r\,\mathrm{d}r\big), \qquad 0 \leq s \leq T, \end{aligned} \tag{5.21}$$

with $\kappa := \frac{1}{2}\gamma(1-\rho^2)(Z^q + Z)$. The $BMO\big(\mathbb{Y}, \hat{P}\big)$-property of $\int Z\,\mathrm{d}\hat{Y}$ and $\int Z^q\,\mathrm{d}\hat{Y}$ implies that $\int \kappa\,\mathrm{d}\hat{Y}$ is in $BMO\big(\mathbb{Y}, \hat{P}\big)$, and by Theorem 3.6 of Kazamaki [40], the process $\int (Z^q - Z)\,\big(\mathrm{d}\hat{Y} - \kappa\,\mathrm{d}r\big)$ is thus also a $BMO\big(\mathbb{Y}, \hat{P}'\big)$-martingale for the probability measure $\hat{P}'$ given by $\frac{\mathrm{d}\hat{P}'}{\mathrm{d}\hat{P}} := \mathcal{E}\big(\int \kappa\,\mathrm{d}\hat{Y}\big)_T$. Taking $\big(\mathcal{Y}_s, \hat{P}'\big)$-conditional expectations in (5.21) yields $\Gamma_s^q \geq \Gamma_s$ for any $s \in [0,T]$, which gives $V^H = -\exp(-\gamma\Gamma) \leq -\exp(-\gamma\Gamma^q)$ by Lemma 5.5 and the continuity of Γ and Γ^q. The proof of 3) goes analogously to 2). □

5.4.2 Approximating the indifference value

As explained after Proposition 5.6, the question whether V_t^H is the a.s. limit of an explicitly known sequence boils down to whether it is possible to find a sequence $(q^n)_{n\in\mathbb{N}}$ in Ξ which converges pointwise to ρ. We start with the case where ρ is deterministic, give afterwards a counterexample to the possibility of approximating V_t^H, and conclude with a result for general (stochastic) ρ.

Deterministic correlation

The approximation of ρ with piecewise constant processes is reminiscent of the construction of the Riemann integral. We recall that a bounded function $g : [t,T] \to \mathbb{R}$ is called *Riemann integrable* if there exists $J \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| J - \sum_{j=1}^n g(s_j)(t_j - t_{j-1}) \right| < \epsilon$$

for every partition $(t_0, \dots, t_n)$ of $[t,T]$ with $\max_{1\leq j\leq n}(t_j - t_{j-1}) < \delta$ and every choice of $s_j \in [t_{j-1}, t_j]$.

The following result, which is shown on page 29 of Lebesgue [43], is known as Lebesgue's theorem.

Lemma 5.7. *A bounded function $g:[t,T]\to\mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[t,T]$.*

We now come to the convergence result for V_t^H when ρ is deterministic.

Theorem 5.8. *Assume that ρ is deterministic, Riemann integrable and valued in $]-1,1[$, and recall $\hat{H}=-\gamma H-\frac{1}{2}\int_t^T\lambda_s^2\,\mathrm{d}s$. Then for every sequence $(t_0^n,\dots,t_{\ell_n}^n)_{n\in\mathbb{N}}$ of partitions of $[t,T]$ with $\lim_{n\to\infty}\big(\max_{1\le j\le\ell_n}(t_j^n-t_{j-1}^n)\big)=0$ and every choice of $s^j\in[t_{j-1}^n,t_j^n]$ (the dependence of s^j on n is omitted for notational reasons),*

$$-E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[E_{\hat{P}}\left[\mathrm{e}^{\hat{H}(1-\rho^2_{s^{\ell_n}})}\middle|\mathcal{Y}_{t^n_{\ell_n-1}}\right]^{\frac{1-\rho^2_{s^{\ell_n-1}}}{1-\rho^2_{s^{\ell_n}}}}\middle|\mathcal{Y}_{t^n_{\ell_n-2}}\right]^{\frac{1-\rho^2_{s^{\ell_n-2}}}{1-\rho^2_{s^{\ell_n-1}}}}\cdots\middle|\mathcal{Y}_t\right]^{\frac{1}{1-\rho^2_{s^1}}}\tag{5.22}$$

converges to V_t^H a.s. If $(\nu_n)_{n\in\mathbb{N}}=(t_0^n,\dots,t_{\ell_n}^n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[t,T]$ with $\nu_n\subseteq\nu_{n+1}$, $n\in\mathbb{N}$, and $\lim_{n\to\infty}\big(\max_{1\le j\le\ell_n}(t_j^n-t_{j-1}^n)\big)=0$, then

$$-E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[E_{\hat{P}}\left[\mathrm{e}^{\hat{H}(1-\rho^2_{n,\ell_n})}\middle|\mathcal{Y}_{t^n_{\ell_n-1}}\right]^{\frac{1-\rho^2_{n,\ell_n-1}}{1-\rho^2_{n,\ell_n}}}\middle|\mathcal{Y}_{t^n_{\ell_n-2}}\right]^{\frac{1-\rho^2_{n,\ell_n-2}}{1-\rho^2_{n,\ell_n-1}}}\cdots\middle|\mathcal{Y}_t\right]^{\frac{1}{1-\rho^2_{n,1}}}$$

with $\rho_{n,j}:=\inf_{s\in[t^n_{j-1},t^n_j]}|\rho_s|$ (or $\rho_{n,j}:=\sup_{s\in[t^n_{j-1},t^n_j]}|\rho_s|$) is a nondecreasing (or nonincreasing) sequence which converges to V_t^H a.s.

Proof. Fix $n\in\mathbb{N}$ and let (Γ^{q^n},Z^{q^n}) be the solution of the BSDE (5.19) with $q=q^n:=\sum_{j=1}^{\ell_n}\rho_{s^j}\mathbb{1}_{]t^n_{j-1},t^n_j]}$. By 1) of Proposition 5.6, $-\exp\big(-\gamma\Gamma_t^{q^n}\big)$ equals (5.22), and we show that this converges to V_t^H a.s. Because ρ is Riemann integrable, Lemma 5.7 yields $\lim_{n\to\infty}|q_s^n|=|\rho_s|$ for a.a. $s\in[t,T]$. From Theorem 5.1 and Lemma 5.5 follows that $-\exp\big(-\gamma\Gamma_t^{q^n}\big)$ converges to V_t^H a.s.

The second part of Theorem 5.8 follows analogously, with q^n replaced by $\sum_{j=1}^{\ell_n}\rho_{n,j}\mathbb{1}_{]t^n_{j-1},t^n_j]}$, using additionally parts 2) and 3) of Proposition 5.6. □

Let us mention two straightforward generalisations of Theorem 5.8. The convergence still works if ρ itself is not Riemann integrable, but ρ equals Lebesgue-almost everywhere a Riemann integrable function $\tilde{\rho}$. One simply replaces ρ by $\tilde{\rho}$ in Theorem 5.8, and uses $V_t^H=-\exp(-\gamma\Gamma_t^\rho)=-\exp\big(-\gamma\Gamma_t^{\tilde{\rho}}\big)$ a.s. for the solutions (Γ^ρ,Z^ρ) and $(\Gamma^{\tilde{\rho}},Z^{\tilde{\rho}})$ of the BSDE (5.19) with $q=\rho$ and $q=\tilde{\rho}$, respectively. An example for such a pair of ρ and $\tilde{\rho}$ is $\rho=\frac{1}{2}\mathbb{1}_{\mathbb{Q}\cap[t,T]}$ and $\tilde{\rho}=0$.

In the first part of Theorem 5.8, one can easily get rid of the restriction that ρ is valued in $]-1,1[$. To this end, one replaces $|\rho_{s^j}|$ by $|\rho_{s^j}|\wedge(1-1/n)$ in (5.22), and uses for the proof that $\sum_{j=1}^{\ell_n}|\rho_{s^j}|\wedge(1-1/n)\mathbb{1}_{]t^n_{j-1},t^n_j]}$ converges pointwise to $|\rho|$ since the correlation ρ is valued in $[-1,1]$. The same procedure works for the second part of Theorem 5.8, but the sequence of iterated expectations with $\rho_{n,j} := \sup_{s\in[t^n_{j-1},t^n_j]} |\rho_s| \wedge (1-1/n)$ instead of $\rho_{n,j} := \sup_{s\in[t^n_{j-1},t^n_j]} |\rho_s|$ is no longer nonincreasing.

Further comments on Theorem 5.8 are given in the next remark.

Remark 5.9. 1) One can show that the a.s. convergence of (5.22) to V^H_t holds uniformly with respect to the partitions. In more detail, we denote by $g_t(\varDelta^n, \vec{s}^{\,n})$ the random variable given by the iterated conditional expectation in (5.22), where the pair $(\varDelta^n, \vec{s}^{\,n}) = \big((t^n_0,\dots,t^n_{\ell_n}),(s^1,\dots,s^{\ell_n})\big)$ is called a tagged partition of $[t,T]$ with mesh $|\varDelta^n|$. The first part of Theorem 5.8 yields $\lim_{n\to\infty} g_t(\varDelta^n, \vec{s}^{\,n}) = V^H_t$ a.s. In the Appendix, we sketch the proof of the more general result

$$\lim_{\epsilon\searrow 0} \operatorname*{ess\,sup}_{(\varDelta,\vec{s}):\,|\varDelta|<\epsilon} \big|g_t(\varDelta,\vec{s}) - V^H_t\big| = 0 \quad \text{a.s.}, \tag{5.23}$$

where the essential supremum is taken over all tagged partitions $(\varDelta,\vec{s})$ of $[t,T]$ with mesh $|\varDelta| < \epsilon$.

2) For q valued in $[-1,1]$, the generator of (5.19) is concave in Z^q_r, and we can apply to $(-\Gamma^q, Z^q)$ the convergence result of Briand and Hu [12]. Recall (Γ^{q^n}, Z^{q^n}) from the proof of Theorem 5.8, and let (Γ, Z) be the solution of (5.16). Proposition 7 of [12] yields for every $p \geq 1$ that

$$\lim_{n\to\infty} E_P\left[\exp\Big(\sup_{s\in[t,T]} \big|\Gamma^{q^n}_s - \Gamma_s\big|\Big)^p + \Big(\int_t^T \big|Z^{q^n}_s - Z_s\big|^2\,\mathrm{d}s\Big)^{p/2}\right] = 0. \tag{5.24}$$

From the proof of Lemma 5.5, we have that the optimiser $\pi^\star$ for V^H_t is given by $\pi^\star = \frac{\rho}{\sigma} Z + \frac{\lambda}{\gamma\sigma}$. If σ is uniformly bounded away from zero, (5.24) implies for every $p \geq 1$ that $\lim_{n\to\infty} E_P\Big[\big(\int_t^T \big|\frac{\rho_s}{\sigma_s} Z^{q^n}_s + \frac{\lambda_s}{\gamma\sigma_s} - \pi^\star_s\big|^2\,\mathrm{d}s\big)^{p/2}\Big] = 0$.

3) Assume that ρ is deterministic and the one-sided limits $\lim_{r\nearrow s}\rho_r$ for all $s \in \,]t,T]$ and $\lim_{r\searrow s}\rho_r$ for all $s\in[t,T[$ exist. This condition is a bit more restrictive than the assumption of Theorem 5.8. Fix $n\in\mathbb{N}$ and define by

$$t^n_0 := t, \quad t^n_j := \inf\Big\{s > t^n_{j-1} : \Big|\rho_s - \lim_{r\searrow t^n_{j-1}} \rho_r\Big| > 1/n\Big\} \wedge T, \quad j\in\mathbb{N},$$

a partition of $[t,T]$, noting that there is $\ell_n\in\mathbb{N}$ such that $t^n_{\ell_n} = T$ by a compactness argument. For every $s^j \in \,]t^n_{j-1}, t^n_j[$, $q^n = \sum_{j=1}^{\ell_n} \rho_{s^j} \mathbb{1}_{]t^n_{j-1},t^n_j]}$ converges to ρ in $L^\infty(\mathrm{Leb},[t,T])$ as $n\to\infty$. For this uniform convergence, Corollary 5.2 yields convergence results in addition to that from Theorem 5.8. $\lozenge$

A counterexample

We have seen that V_t^H is the a.s. limit of an explicitly known sequence if ρ equals almost everywhere a Riemann integrable function. In particular, the choice of a nondecreasing sequence of partitions in Theorem 5.8 allows us to approximate V_t^H from above and below. We give here an example of a correlation process which is not almost everywhere equal to a Riemann integrable function and where indeed the approximations of V_t^H in the sense of Theorem 5.8 from above and below are not possible.

We take for simplicity $t = 0$, $T = 1$, $\gamma = 1$, $\mu \equiv 0$ and $\sigma \equiv 1$. Let $C \subseteq [0,1]$ be the "fat" Cantor set with Lebesgue measure $1/2$. This set, which is also known as Smith-Volterra-Cantor set, is constructed iteratively as follows: Start by removing $]3/8, 5/8[$ from the interval $[0,1]$; in the n-th step, remove subintervals of width $1/2^{2n}$ from the middle of each of the 2^{n-1} intervals. If we continue like this, C consists of all points in $[0,1]$ that are never removed. Because C is the complement of a countable union of open intervals, it is Borel measurable. Moreover, it is well known that C is nowhere dense, yet has Lebesgue measure $1/2$. We assume that the correlation ρ is given by $\rho = \frac{1}{2}\mathbb{1}_{C\cap[0,1/2]} + \frac{1}{2}\mathbb{1}_{C^c\cap]1/2,1]}$ and $H := Y_1$. Since Y_1 is not bounded, we have to adjust slightly the definition of admissible strategies: Instead of (5.14), we impose on $\pi \in \mathcal{A}_0$ that $\big(\exp(-X_s^{0,\pi} - Y_1)\big)_{0\le s\le T}$ is of class (D). (Alternatively, one could approximate Y_1 by bounded random variables like in the example in Section 2.5.) We claim that

$$\sup_{q\in\Xi,|q|\le\rho} \Gamma_0^q \le -15/32 < -\log\big(-V_0^H\big) = -7/16 < -13/32 \le \inf_{q\in\Xi,|q|\ge\rho} \Gamma_0^q,$$

where Γ^q is the solution of (5.19). This means that ρ cannot be approximated by piecewise constant processes from above and below such that the corresponding values converge to V_0^H. We first show $V_0^H = -\exp(7/16)$. For any $\pi \in \mathcal{A}_0$ with bounded $\int_0^T \pi_s^2\,\mathrm{d}s$, we have

$$\begin{aligned}
E_P\big[U\big(X_1^{0,\pi} + H\big)\big] &= -E_P\left[\exp\left(-\int_0^1 \pi_s\,\mathrm{d}W_s - Y_1\right)\right] \\
&= -E_P\left[\exp\left(-\int_0^1 \pi_s\,\mathrm{d}W_s - Y_1 - \frac{1}{2}\left\langle \int \pi\,\mathrm{d}W + Y\right\rangle_1\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2}\int_0^1 \big((\pi_s+\rho_s)^2 + 1 - \rho_s^2\big)\,\mathrm{d}s\right)\right] \\
&\le -E_P\left[\exp\left(-\int_0^1 \pi_s\,\mathrm{d}W_s - Y_1 - \frac{1}{2}\left\langle \int \pi\,\mathrm{d}W + Y\right\rangle_1\right)\right] \\
&\qquad \times \exp\left(\frac{1}{2}\int_0^1 \big(1-\rho_s^2\big)\,\mathrm{d}s\right),
\end{aligned}$$

which implies

$$E_P\big[U\big(X_1^{0,\pi}+H\big)\big] \le -\exp\left(\frac{1}{2}\int_0^1 \big(1-\rho_s^2\big)\,\mathrm{d}s\right) = -\exp(7/16) \tag{5.25}$$

since $\mathrm{Leb}\big(C\cap[0,1/2]\big)=\mathrm{Leb}\big(C^c\cap]1/2,1]\big)=1/4$. Equality in (5.25) holds for $\pi=-\rho\in\mathcal{A}_0$. Because of the class (D) condition on $\big(\exp(-X_s^{0,\pi}-Y_1)\big)_{0\le s\le T}$ for any $\pi\in\mathcal{A}_0$, we obtain $V_0^H=-\exp(7/16)$ by a localisation argument. To prove $\sup_{q\in\Xi,|q|\le\rho}\Gamma_0^q\ge -15/32$, we note that $q\in\Xi$, $|q|\le\rho$ implies $q\equiv 0$ on $[0,1/2]$ since C does not contain any nontrivial intervals. By 3) of Proposition 5.6 with ρ replaced by $\tilde\rho:=\rho\mathbb{1}_{]1/2,1]}=\frac{1}{2}\mathbb{1}_{C^c\cap]1/2,1]}$, we have

$$\sup_{q\in\Xi,|q|\le\rho}\Gamma_0^q\le\Gamma_0^{\tilde\rho},$$

and a calculation similar to (5.25) shows $\Gamma_0^{\tilde\rho}=-15/32$, using that by Lemma 5.5, $-\exp\big(-\Gamma_0^{\tilde\rho}\big)$ equals V_0^H with ρ replaced by $\tilde\rho$. Similarly, we obtain

$$\inf_{q\in\Xi,|q|\ge\rho}\Gamma_0^q\ge\Gamma_0^{\hat\rho}=-13/32,$$

where $\hat\rho:=\rho\mathbb{1}_{[0,1/2]}+\frac{1}{2}\mathbb{1}_{]1/2,1]}=\frac{1}{2}\mathbb{1}_{C\cap[0,1/2]}+\frac{1}{2}\mathbb{1}_{]1/2,1]}$.

Stochastic correlation

When ρ is stochastic, we cannot approximate V_t^H from above and below like in Theorem 5.8. However, we still have a convergence result for V_t^H if ρ is left-continuous.

Theorem 5.10. *Assume that ρ is on $]\!]t,T]\!]$ left-continuous and valued in $]-1,1[$. Then for every sequence $(t=\tau_0^n\le\cdots\le\tau_{\ell_n}^n=T)_{n\in\mathbb{N}}$ of $[t,T]$-valued $\mathbb{Y}$-stopping times with $\lim_{n\to\infty}\big(\max_{1\le j\le\ell_n}(\tau_j^n-\tau_{j-1}^n)\big)=0$ a.s.,*

$$-E_{\hat P}\left[\cdots E_{\hat P}\left[E_{\hat P}\left[\mathrm{e}^{\hat H(1-\rho^2_{\tau^n_{\ell_n-1}})}\Big|\mathcal{Y}_{\tau^n_{\ell_n-1}}\right]^{\frac{1-\rho^2_{\tau^n_{\ell_n-2}}}{1-\rho^2_{\tau^n_{\ell_n-1}}}}\Bigg|\mathcal{Y}_{\tau^n_{\ell_n-2}}\right]^{\frac{1-\rho^2_{\tau^n_{\ell_n-3}}}{1-\rho^2_{\tau^n_{\ell_n-2}}}}\cdots\Bigg|\mathcal{Y}_t\right]^{\frac{1}{1-\rho_t^2}} \tag{5.26}$$

converges to V_t^H a.s.

Proof. Fix $n\in\mathbb{N}$ and let (Γ^{q^n},Z^{q^n}) be the solution of the BSDE (5.19) with $q=q^n:=\sum_{j=1}^{\ell_n}\rho_{\tau^n_{j-1}}\mathbb{1}_{]\!]\tau^n_{j-1},\tau^n_j]\!]}$, which is $\mathbb{Y}$-predictable. By 1) of Proposition 5.6, $-\exp\big(-\gamma\Gamma_t^{q^n}\big)$ equals (5.26). We have $\lim_{n\to\infty}q_s^n(\omega)=\rho_s(\omega)$ for a.a. $(\omega,s)\in[\![t,T]\!]$ by the left-continuity of ρ, and from Theorem 5.1 and Lemma 5.5 it follows that $-\exp\big(-\gamma\Gamma_t^{q^n}\big)$ converges to V_t^H a.s. $\square$

In the same way as Theorem 5.8, one can slightly generalise Theorem 5.10 to the case where ρ equals $(P\otimes\mathrm{Leb})$-a.e. a $(P\otimes\mathrm{Leb})$-a.e. left-continuous process, and one can get rid of the assumption that ρ is valued in $]-1,1[$.

Remark 5.11. The assumption from Section 5.4.1 that $\int_0^T \lambda_s^2\,\mathrm{d}s$ is bounded can be slightly weakened. Theorem 5.10 still holds if $\int \lambda\,\mathrm{d}W \in BMO(\mathbb{G},P)$ and

$$\sup_{s\in[0,T]}\left\|E_P\left[\exp\left(\int_s^T (1+\rho_r^2)\lambda_r^2\,\mathrm{d}r\right)\middle|\mathcal{G}_s\right]\right\|_{L^\infty} < \infty. \tag{5.27}$$

By the John-Nirenberg inequality (see Theorem 2.2 of Kazamaki [40]), (5.27) is satisfied if, for example, the $BMO_2(\mathbb{G},P)$-norm of $\int \lambda\,\mathrm{d}W$ is less than $1/\sqrt{2}$. In the Appendix, we sketch the proof of this slight generalisation of Theorem 5.10. $\diamondsuit$

5.4.3 Continuity of the value process in the correlation

This short section exploits the convergence Theorem 5.1 to show a continuity property of V^H in ρ.

Let us introduce more precise notations by writing (5.11) as

$$\mathrm{d}W_s(\tilde\rho) = \tilde\rho_s\,\mathrm{d}Y_s + \sqrt{1-\tilde\rho_s^2}\,\mathrm{d}Y_s^\perp, \quad 0\le s\le T$$

for a $\mathbb{G}$-predictable process $\tilde\rho$ denoting the instantaneous correlation between the $(\mathbb{G},P)$-Brownian motions $W(\tilde\rho)$ and Y so that $W = W(\rho)$. We replace in all definitions W by $W(\tilde\rho)$ and write $\hat W(\tilde\rho)$, $V^H(\tilde\rho)$, etc. $V^H(\tilde\rho)$ is then the dynamic value process for a stochastic control problem when the correlation between the underlying Brownian motions $W(\tilde\rho)$ and Y is $\tilde\rho$. Note that if we change $\tilde\rho$, only $W(\tilde\rho)$ and all expressions depending on it will change. This is reasonable; clearly H and $\mathbb{Y}$ should not be affected.

Proposition 5.12. *Let $(\rho^n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{Y}$-predictable $[-1,1]$-valued processes which converge pointwise to ρ on $[\![t,T]\!]$. Then $V_t^H(\rho^n)$ converges to $V_t^H = V_t^H(\rho)$ P-a.s. as $n\to\infty$. Moreover, $\sup_{s\in[t,T]}\left|V_s^H(\rho^n) - V_s^H\right| \to 0$ as $n\to\infty$ in P-probability and in $L^p(P)$, $1\le p<\infty$.*

Proof. This follows from Lemma 5.5 and Theorem 5.1, using additionally the same argument as in the last part of the proof of Proposition 5.3 to show the second statement. $\square$

Proposition 5.12 can be generalised to a multidimensional setting where W and Y are stochastically correlated multidimensional Brownian motions. But we give no details since this provides no essential new insights.

5.5 Appendix: Proofs of the convergence results

Proof of Theorem 5.1. By Theorems 2.5 and 2.6 of Morlais [46], there exist unique solutions (Γ^n, Z^n, N^n) to (5.1) with parameters (f^n, β^n, H^n) for $n = 1, \ldots, \infty$. Moreover, Lemma 3.1 of Morlais [46] implies that Γ^n and the $BMO(P)$-norms of $\int Z^n \, \mathrm{d}M$ and N^n are bounded uniformly in $n = 1, \ldots, \infty$. (Theorems 2.5, 2.6 and Lemma 3.1 of [46] do not use the assumption in Section 2.1 of [46] that a.s., the matrix $m_s m_s'$ is invertible for every $s \in [0, T]$.)

We now subtract (5.1) with parameters $(f^\infty, \beta^\infty, H^\infty)$ from that with parameters (f^n, β^n, H^n) for a fixed $n \in \mathbb{N}$ to obtain, for $0 \le s \le T$,

$$
\begin{aligned}
&\Gamma^n_s - \Gamma^\infty_s \\
&= H^n - H^\infty + \int_s^T \big(f^n(r, Z^n_r) - f^\infty(r, Z^\infty_r)\big)\,\mathrm{d}D_r - \int_s^T (Z^n_r - Z^\infty_r)\,\mathrm{d}M_r \\
&\quad + \frac{\beta^n}{2}\big(\langle N^n\rangle_T - \langle N^n\rangle_s\big) - \frac{\beta^\infty}{2}\big(\langle N^\infty\rangle_T - \langle N^\infty\rangle_s\big) - \int_s^T \mathrm{d}(N^n - N^\infty)_r \\
&= \frac{\beta^n - \beta^\infty}{2}\big(\langle N^\infty\rangle_T - \langle N^\infty\rangle_s\big) - \int_s^T (Z^n_r - Z^\infty_r)\big(\mathrm{d}M_r - \mathrm{d}\langle M\rangle_r g^n_r\big) \\
&\quad + H^n - H^\infty - \int_s^T \Big(\mathrm{d}(N^n - N^\infty)_r - \frac{\beta^n}{2}\,\mathrm{d}\langle N^n - N^\infty, N^n + N^\infty\rangle_r\Big) \\
&\quad + \int_s^T \big(f^n(r, Z^\infty_r) - f^\infty(r, Z^\infty_r)\big)\,\mathrm{d}D_r,
\end{aligned}
\tag{5.28}
$$

where g^n is defined for $0 \le s \le T$ by

$$
g^n_s := \begin{cases} \frac{f^n(s,Z^n_s) - f^n(s,Z^\infty_s)}{|m(Z^n_s - Z^\infty_s)|^2}(Z^n_s - Z^\infty_s) & \text{if } |m(Z^n_s - Z^\infty_s)| \neq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Due to the assumption (ii) of the theorem, $\int g^n \, \mathrm{d}M$ is in $BMO(P)$ and its $BMO(P)$-norm is uniformly bounded since the $BMO(P)$-norm of $\int Z^n \, \mathrm{d}M$ is bounded uniformly in $n = 1, \ldots, \infty$. Therefore, taking conditional expectations in (5.28) under the probability measure Q^n given by

$$
\frac{\mathrm{d}Q^n}{\mathrm{d}P} := \mathcal{E}\Big(\int g^n \, \mathrm{d}M + \frac{\beta^n}{2}(N^n + N^\infty)\Big)_T \tag{5.29}
$$

yields

$$
\begin{aligned}
\Gamma^n_s - \Gamma^\infty_s = \frac{\beta^n - \beta^\infty}{2} &E_{Q^n}\big[\langle N^\infty\rangle_T - \langle N^\infty\rangle_s \big| \mathcal{F}_s\big] + E_{Q^n}[H^n - H^\infty | \mathcal{F}_s] \\
&+ E_{Q^n}\Big[\int_s^T \big(f^n(r, Z^\infty_r) - f^\infty(r, Z^\infty_r)\big)\,\mathrm{d}D_r \Big| \mathcal{F}_s\Big].
\end{aligned}
\tag{5.30}
$$

Because the convergent sequence $(\beta^n)_{n=1,\dots,\infty}$ is bounded, the $BMO(P)$-norm of $\tilde{M}^n := \int g^n \,\mathrm{d}M + \frac{\beta^n}{2}(N^n + N^\infty)$, which equals the stochastic logarithm of the P-density process of Q^n, is uniformly bounded in $n = 1, \dots, \infty$. Therefore, Theorem 3.6 of Kazamaki [40] and continuity in s of $E_{Q^n}[\langle N^\infty\rangle_T | \mathcal{F}_s]$ and $\langle N^\infty\rangle_s$ imply

$$\left\| \sup_{n=1,\dots,\infty} \sup_{s\in[0,T]} E_{Q^n}[\langle N^\infty\rangle_T - \langle N^\infty\rangle_s | \mathcal{F}_s] \right\|_{L^\infty(P)} < \infty, \tag{5.31}$$

and hence

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| \frac{\beta^n - \beta^\infty}{2} E_{Q^n}\big[\langle N^\infty\rangle_T - \langle N^\infty\rangle_s \big| \mathcal{F}_s\big] \right| = 0 \quad \text{in } L^\infty(P). \tag{5.32}$$

Since the $BMO(P)$-norm of $\tilde{M}^n$ is uniformly bounded in n, there exist by Theorem 3.1 of Kazamaki [40] $p > 1$ and a constant C_p, which both do not depend on n, such that for all $n = 1, \dots, \infty$

$$E_{Q^n}\left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{1/(p-1)} \middle| \mathcal{F}_s \right] = E_P\left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{p/(p-1)} \middle| \mathcal{F}_s \right] \le C_p. \tag{5.33}$$

Recall the constant c^1 from the assumption (i) of the theorem and set

$$\alpha := \frac{1}{2pc^1 \| \int Z^\infty \,\mathrm{d}M \|^2_{BMO_2(P)} + 1}. \tag{5.34}$$

Using $\alpha x \le \mathrm{e}^{\alpha x} - 1$ for $x \in \mathbb{R}$, the Hölder inequality and (5.33) yields

$$\begin{aligned}
&E_{Q^n}\left[\int_s^T \big| f^n(r, Z^\infty_r) - f^\infty(r, Z^\infty_r) \big| \,\mathrm{d}D_r \middle| \mathcal{F}_s \right] \\
&\le \frac{1}{\alpha} E_{Q^n}\left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{1/p} \left(\frac{\mathcal{E}(\tilde{M}^n)_s}{\mathcal{E}(\tilde{M}^n)_T} \right)^{1/p} \left(\mathrm{e}^{\alpha \int_s^T |f^n(r,Z^\infty_r) - f^\infty(r,Z^\infty_r)| \,\mathrm{d}D_r} - 1 \right) \middle| \mathcal{F}_s \right] \\
&\le \frac{1}{\alpha} |C_p|^{(p-1)/p} E_P\left[\left(\mathrm{e}^{\alpha \int_s^T |f^n(r,Z^\infty_r) - f^\infty(r,Z^\infty_r)| \,\mathrm{d}D_r} - 1 \right)^p \middle| \mathcal{F}_s \right]^{1/p}.
\end{aligned} \tag{5.35}$$

By the assumption (i), we have

$$\begin{aligned}
&\left(\mathrm{e}^{\alpha \int_s^T |f^n(r,Z^\infty_r) - f^\infty(r,Z^\infty_r)| \,\mathrm{d}D_r} - 1 \right)^p \\
&\le \exp\left(p\alpha \int_s^T \big| f^n(r, Z^\infty_r) - f^\infty(r, Z^\infty_r) \big| \,\mathrm{d}D_r \right) \\
&\le \exp\left(2p\alpha \left\| \int_0^T \kappa^1_r \,\mathrm{d}D_r \right\|_{L^\infty(P)} \right) \exp\left(2pc^1 \alpha \int_0^T |m_r Z^\infty_r|^2 \,\mathrm{d}D_r \right),
\end{aligned}$$

and the last expression is P-integrable by the definition (5.34) of α and the John-Nirenberg inequality; see Theorem 2.2 of Kazamaki [40]. Therefore, dominated convergence and (5.35) imply that for $s \in [t,T]$,

$$\lim_{n\to\infty} E_{Q^n}\left[\int_s^T \left|f^n(r,Z_r^\infty) - f^\infty(r,Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] = 0 \quad P\text{-a.s.} \tag{5.36}$$

Similarly to (5.35), we obtain

$$E_{Q^n}\left[|H^n - H^\infty|\middle|\mathcal{F}_s\right] \le |C_p|^{(p-1)/p} E_P\left[\left(\mathrm{e}^{|H^n-H^\infty|}-1\right)^p\middle|\mathcal{F}_s\right]^{1/p}, \tag{5.37}$$

which again converges P-a.s. to zero by dominated convergence. Therefore, (5.30), (5.32) and (5.36) give $\lim_{n\to\infty}|\Gamma_s^n - \Gamma_s^\infty| = 0$ P-a.s. for every $s \in [t,T]$.

We obtain from (5.35) and the maximum inequality for martingales that, for any $\epsilon \ge 0$,

$$\begin{aligned}
&\epsilon^p P\left[\sup_{s\in[t,T]} E_{Q^n}\left[\int_s^T \left|f^n(r,Z_r^\infty) - f^\infty(r,Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] \ge \frac{1}{\alpha}|C_p|^{(p-1)/p}\epsilon\right] \\
&\le \epsilon^p P\left[\sup_{s\in[t,T]} E_P\left[\left(\mathrm{e}^{\alpha\int_s^T |f^n(r,Z_r^\infty)-f^\infty(r,Z_r^\infty)|\,\mathrm{d}D_r}-1\right)^p\middle|\mathcal{F}_s\right]^{1/p} \ge \epsilon\right] \\
&\le \epsilon^p P\left[\sup_{s\in[t,T]} E_P\left[\left(\mathrm{e}^{\alpha\int_t^T |f^n(r,Z_r^\infty)-f^\infty(r,Z_r^\infty)|\,\mathrm{d}D_r}-1\right)^p\middle|\mathcal{F}_s\right] \ge \epsilon^p\right] \\
&\le E_P\left[\left(\mathrm{e}^{\alpha\int_t^T |f^n(r,Z_r^\infty)-f^\infty(r,Z_r^\infty)|\,\mathrm{d}D_r}-1\right)^p\right],
\end{aligned} \tag{5.38}$$

which converges to zero as $n \to \infty$ by dominated convergence. Analogously, we obtain from (5.37) that

$$\epsilon^p P\left[\sup_{s\in[t,T]} E_{Q^n}\left[|H^n - H^\infty|\middle|\mathcal{F}_s\right] \ge |C_p|^{(p-1)/p}\epsilon\right] \le E_P\left[\left(\mathrm{e}^{|H^n-H^\infty|}-1\right)^p\right], \tag{5.39}$$

which also converges to zero as $n \to \infty$ by dominated convergence. All in all, (5.30), (5.32), (5.38) and (5.39) show that $\sup_{s\in[t,T]}|\Gamma_s^n - \Gamma_s^\infty|$ converges in P-probability to zero, and also convergence in $L^p(P)$, $1 \le p < \infty$, follows since Γ^n is bounded uniformly in n. □

Remark 5.13. To prove (5.36), one can also apply directly the energy inequalities instead of using the John-Nirenberg inequality. In fact, taking $\ell \in \mathbb{N}$ with $\ell \ge p$, we obtain from (5.33) and the Hölder inequality that

$$\begin{aligned}
&E_{Q^n}\left[\int_s^T \left|f^n(r,Z_r^\infty) - f^\infty(r,Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] \\
&\le |C_p|^{(p-1)/p} E_P\left[\left(\int_s^T \left|f^n(r,Z_r^\infty) - f^\infty(r,Z_r^\infty)\right| \mathrm{d}D_r\right)^\ell \middle| \mathcal{F}_s\right]^{1/\ell}.
\end{aligned} \tag{5.40}$$

By the assumption (i), we have

$$
\begin{aligned}
&\left(\int_s^T \left|f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)\right| \mathrm{d}D_r\right)^\ell \\
&\le \left(2\int_0^T \kappa_r^1 \,\mathrm{d}D_r + 2c^1 \int_0^T |m_r Z_r^\infty|^2 \,\mathrm{d}D_r\right)^\ell \\
&= 2^\ell \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\int_0^T \kappa_r^1 \,\mathrm{d}D_r\right)^{\ell-j} |c^1|^j \left\langle \int Z^\infty \,\mathrm{d}M \right\rangle_T^j,
\end{aligned}
$$

which is P-integrable since $\left\| \int_0^T \kappa_r^1 \,\mathrm{d}D_r \right\|_{L^\infty(P)} < \infty$ and

$$
E_P\left[\left\langle \int Z^\infty \,\mathrm{d}M \right\rangle_T^j\right] \le j! \left\| \int Z^\infty \,\mathrm{d}M \right\|_{BMO_2(P)}^{2j} < \infty, \quad j \in \mathbb{N}
$$

by the energy inequalities; see the corollary to Theorem 4 of Kikuchi [41]. Dominated convergence and (5.40) now imply (5.36). $\Diamond$

Proof of Corollary 5.2. To show (5.3), it is by (5.30) and (5.31) enough to prove the existence of a constant $K > 0$ such that

$$
\begin{aligned}
&\sup_{s\in[t,T]} E_{Q^n}\left[\int_s^T \left|f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] \\
&\le K\big(\|\underline{a}^n - 1\|_{L^\infty(P\otimes D)} + \|\overline{a}^n - 1\|_{L^\infty(P\otimes D)}\big), \qquad n \in \mathbb{N}.
\end{aligned}
$$

But the assumption (v) implies

$$
\begin{aligned}
&\sup_{s\in[t,T]} E_{Q^n}\left[\int_s^T \left|f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] \\
&\le \|\underline{a}^n - 1\|_{L^\infty(P\otimes D)} \sup_{s\in[t,T]} E_{Q^n}\left[\int_s^T \left|\underline{f}(r, Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right] \\
&\quad + \|\overline{a}^n - 1\|_{L^\infty(P\otimes D)} \sup_{s\in[t,T]} E_{Q^n}\left[\int_s^T \left|\overline{f}(r, Z_r^\infty)\right| \mathrm{d}D_r \middle| \mathcal{F}_s\right],
\end{aligned}
$$

and the conditional expectations are bounded in $L^\infty(P)$ uniformly in $n \in \mathbb{N}$ and $s \in [t, T]$ by an argument similar to (5.31). So (5.3) is established, and since its right-hand side converges to zero by the assumptions (iii)–(v), we have $\sup_{s\in[t,T]} |\Gamma_s^n - \Gamma_s^\infty| \to 0$ in $L^\infty(P)$.

To show that $\int Z^n \,\mathrm{d}M \to \int Z^\infty \,\mathrm{d}M$ and $N^n \to N^\infty$ on $[\![t, T]\!]$ in $BMO(P)$, we apply Itô's formula between a stopping time τ valued in $[\![t, T]\!]$ and T, and use (5.28) to obtain

$$
\begin{aligned}
&\exp(H^n - H^\infty) - \exp(\Gamma^n_\tau - \Gamma^\infty_\tau)\\
&= -\int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}(Z^n_s - Z^\infty_s)\big(\mathrm{d}M_s - \mathrm{d}\langle M\rangle_s g^n_s\big)\\
&\quad - \int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\Big(\mathrm{d}(N^n - N^\infty)_s - \frac{\beta^n}{2}\,\mathrm{d}\langle N^n - N^\infty, N^n + N^\infty\rangle_s\Big)\\
&\quad + \int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\Big(\big(f^n(s, Z^\infty_s) - f^\infty(s, Z^\infty_s)\big)\,\mathrm{d}D_s + \frac{1}{2}(\beta^n - \beta^\infty)\,\mathrm{d}\langle N^\infty\rangle_s\Big)\\
&\quad + \frac{1}{2}\int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\big(|m_s(Z^n_s - Z^\infty_s)|^2\,\mathrm{d}D_s + \mathrm{d}\langle N^n - N^\infty\rangle_s\big)
\end{aligned}
$$

so that

$$
\begin{aligned}
&E_{Q^n}\bigg[\int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\big(|m_s(Z^n_s - Z^\infty_s)|^2\,\mathrm{d}D_s + \mathrm{d}\langle N^n - N^\infty\rangle_s\big)\bigg|\mathcal{F}_\tau\bigg]\\
&= 2E_{Q^n}\big[\exp(H^n - H^\infty) - \exp(\Gamma^n_\tau - \Gamma^\infty_\tau)\big|\mathcal{F}_\tau\big]\\
&\quad - 2E_{Q^n}\bigg[\int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\big(f^n(s, Z^\infty_s) - f^\infty(s, Z^\infty_s)\big)\,\mathrm{d}D_s\bigg|\mathcal{F}_\tau\bigg]\\
&\quad + (\beta^\infty - \beta^n)E_{Q^n}\bigg[\int_\tau^T \mathrm{e}^{\Gamma^n_s - \Gamma^\infty_s}\,\mathrm{d}\langle N^\infty\rangle_s\bigg|\mathcal{F}_\tau\bigg]
\end{aligned}
$$

for Q^n defined by (5.29). Using $|\mathrm{e}^x - \mathrm{e}^y| \le \max\{\mathrm{e}^x, \mathrm{e}^y\}|x - y|$ for $x, y \in \mathbb{R}$ and that $\Gamma^n - \Gamma$ is bounded uniformly in n, there is a constant k such that

$$
\begin{aligned}
&E_{Q^n}\bigg[\int_\tau^T |m_s(Z^n_s - Z^\infty_s)|^2\,\mathrm{d}D_s\bigg|\mathcal{F}_\tau\bigg] + E_{Q^n}\big[\langle N^n - N^\infty\rangle_T - \langle N^n - N^\infty\rangle_\tau\big|\mathcal{F}_\tau\big]\\
&\le k\|H^n - H^\infty\|_{L^\infty(P)} + k\bigg\|\sup_{s\in[t,T]}|\Gamma^n_s - \Gamma^\infty_s|\bigg\|_{L^\infty(P)}\\
&\quad + kE_{Q^n}\bigg[\int_\tau^T \big|f^n(s, Z^\infty_s) - f^\infty(s, Z^\infty_s)\big|\,\mathrm{d}D_s\bigg|\mathcal{F}_\tau\bigg]\\
&\quad + k|\beta^\infty - \beta^n|E_{Q^n}\big[\langle N^\infty\rangle_T - \langle N^\infty\rangle_\tau\big|\mathcal{F}_\tau\big].
\end{aligned}
$$

Similarly to the first claim, this implies

$$
\begin{aligned}
&\sup_\tau\bigg\|E_{Q^n}\bigg[\int_\tau^T |m_s(Z^n_s - Z^\infty_s)|^2\,\mathrm{d}D_s\bigg|\mathcal{F}_\tau\bigg]\bigg\|_{L^\infty(Q^n)} \to 0 \quad \text{and}\\
&\sup_\tau\Big\|E_{Q^n}\big[\langle N^n - N^\infty\rangle_T - \langle N^n - N^\infty\rangle_\tau\big|\mathcal{F}_\tau\big]\Big\|_{L^\infty(Q^n)} \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

and then $\int Z^n\,\mathrm{d}M \to \int Z^\infty\,\mathrm{d}M$ and $N^n \to N^\infty$ on $[\![t, T]\!]$ in $BMO(P)$ as $n \to \infty$ by Theorem 3.6 of Kazamaki [40]. □

Sketch of the proof of (5.23) in Remark 5.9.1. To check (5.23), one first deduces from Lemma 5.7 that $\lim_{\epsilon\searrow 0}\sup_{(\Delta,\vec{s}):|\Delta|<\epsilon}\big|q_r^{(\Delta,\vec{s})}-\rho_r\big|=0$ for Leb-a.a. $r\in[t,T]$, where one sets $q^{(\Delta,\vec{s})}:=\sum_{j=1}^{\ell}\rho_{s^j}\mathbb{1}_{]t_{j-1},t_j]}$ for any tagged partition $(\Delta,\vec{s})=\big((t_0,\dots,t_\ell),(s^1,\dots,s^\ell)\big)$. Then one slightly generalises Theorem 5.1 in the sense that this uniform convergence of $q^{(\Delta,\vec{s})}$ in $(\Delta,\vec{s})$ implies that the corresponding solutions $\Gamma_t^{q^{(\Delta,\vec{s})}}$ of (5.19) converge a.s. to $\Gamma_t^{\rho}=-\frac{1}{\gamma}\log\big(-V_t^H\big)$ uniformly in $(\Delta,\vec{s})$. In fact, one needs only to generalise (5.36), which goes similarly to (5.35) by dominated convergence. Now one deduces (5.23) from

$$\begin{aligned}&\lim_{\epsilon\searrow 0}\ \operatorname*{ess\,sup}_{(\Delta,\vec{s}):\,|\Delta|<\epsilon}\Big|-\frac{1}{\gamma}\log\big(-g_t(\Delta,\vec{s})\big)+\frac{1}{\gamma}\log\big(-V_t^H\big)\Big|\\ &=\lim_{\epsilon\searrow 0}\ \operatorname*{ess\,sup}_{(\Delta,\vec{s}):\,|\Delta|<\epsilon}\Big|\Gamma_t^{q^{(\Delta,\vec{s})}}+\frac{1}{\gamma}\log\big(-V_t^H\big)\Big|=0\qquad\text{a.s.}\end{aligned}$$

similarly to the last part of the proof of Proposition 5.3, using that $-g_t(\Delta,\vec{s})$ is bounded away from zero by $\mathrm{e}^{-\|\hat{H}\|_{L^\infty(P)}}$ uniformly in $(\Delta,\vec{s})$. □

Sketch of the proof of Remark 5.11. From Proposition 3 of Briand and Hu [12] and Proposition 7 and Theorem 8 of Mania and Schweizer [44], one deduces that for a $[-1,1]$-valued $\mathbb{Y}$-predictable process q, the BSDE (5.19) still has a unique solution (Γ^q,Z^q) where Γ^q is a real-valued bounded continuous $(\mathbb{Y},P)$-semimartingale and Z^q is a $\mathbb{Y}$-predictable process such that $E_P\big[\int_0^T|Z_s^q|^2\,\mathrm{d}s\big]<\infty$. Furthermore, $\int Z^q\,\mathrm{d}Y$ is in both $BMO(\mathbb{Y},P)$ and $BMO(\mathbb{G},P)$, and the BMO-norms are bounded uniformly with respect to the $[-1,1]$-valued q. Now one can proceed like in Lemma 5.5 and Proposition 5.6 to obtain $V^H=-\exp(-\gamma\Gamma^\rho)$ and (5.20). The argument is finished by applying Theorem 5.1, using that, under the assumption of uniform boundedness of the $BMO(\mathbb{F},P)$-norms of $\int Z^n\,\mathrm{d}M$ and N^n, the convergence result can also be shown if in the assumptions (i) and (ii), one only has $\sup_\tau\big\|E_P\big[\int_\tau^T\kappa_s^1\,\mathrm{d}D_s\big|\mathcal{F}_\tau\big]\big\|_{L^\infty}<\infty$ and $\sup_\tau\big\|E_P\big[\int_\tau^T|\kappa_s^2|^2\,\mathrm{d}D_s\big|\mathcal{F}_\tau\big]\big\|_{L^\infty}<\infty$ instead of $\big\|\int_0^T\kappa_s^1\,\mathrm{d}D_s\big\|_{L^\infty}<\infty$ and $\big\|\int_0^T|\kappa_s^2|^2\,\mathrm{d}D_s\big\|_{L^\infty}<\infty$, where the suprema are taken over all $\mathbb{F}$-stopping times τ. □

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List of notations

In the subsequent list, the numbers before the colons refer to the chapters; the numbers following the colons refer to the pages.

$\mathcal{A}_t$, $\mathcal{A}_t^H$	sets of admissible trading strategies 2:12, 3:40, 4:103, 5:121 and 124
a.a., a.s., a.e.	almost all, almost sure(ly), almost everywhere
B	P-Brownian motion 4:85
$\mathcal{B}(\mathbb{R}^d)$	Borel σ-field on $\mathbb{R}^d$
BMO	space of martingales with bounded mean oscillation
D	nondecreasing process with $\mathrm{d}\langle M\rangle = m\, m'\, \mathrm{d}D$ 5:118
$\mathcal{E}(N)$	stochastic exponential of a semimartingale N
$\mathbb{F}^Z = \left(\mathcal{F}_t^Z\right)_{0\le t\le T}$	P-augmented filtration generated by a process Z 2:11, 4:106
(f, β, H)	input components of the BSDE (5.1) 5:118
$FER(H)$	fundamental entropy representation of H 3:41
$FER^\star(H)$	$FER(H)$ with additional integrability properties 3:42
$\mathrm{GL}(n)$	set of invertible $(n \times n)$-matrices 4:85
H	contingent claim 1:1, 2:12, 3:39, 4:102, 5:121
h_t	indifference value for H at time $t \in [0, T]$ 1:2, 2:13, 3:40, 4:103, 5:122 and 124
$(H, \Lambda, \alpha, \chi)$	input components of the BSDE (4.2) 4:85
I	identity matrix 2:34, 4:85
$I(Q\|P)$	relative entropy of Q with respect to P 3:40

(J)	condition (J) 3:50
$\mathcal{K}^\delta$	set of processes satisfying a condition depending on $\delta > 0$ 4:89
$(k_t^H)_{0\le t\le T}$	process associated with an $FER(H)$ (N^H, η^H, k_0^H) 3:42
L^p	space of random variables with a finite moment of order $p \in [1,\infty)$
L^∞	space of bounded random variables
$L(S)$	set of predictable S-integrable processes 3:39
Leb	Lebesgue measure on $([0,T], \mathcal{B}[0,T])$
(N^H, η^H, k_0^H)	components of an $FER(H)$ 3:41
$\mathrm{O}(n)$	set of orthogonal $(n\times n)$-matrices 4:85
$\hat{P}$	minimal (local) martingale measure 2:12, 3:43, 4:102, 5:124
$P(N^H)$	probability measure associated with an $FER(H)$ (N^H, η^H, k_0^H) 3:42
P_H	probability measure given by $\frac{\mathrm{d}P_H}{\mathrm{d}P} = \frac{\exp(\gamma H)}{E_P[\exp(\gamma H)]}$ 3:39
$\mathcal{P}$	predictable σ-field on $\Omega \times [0,T]$ 3:80, 5:118
$\mathbb{P}_H^f$, $\mathbb{P}_H^{e,f}$	sets of absolutely and equivalent sigma-martingale measures with finite entropy relative to P_H 3:40
Perm	symmetric group of permutations of length n 4:100
Q_H^E	minimal H-entropy sigma-martingale measure 3:40
R	instantaneous correlation matrix between W and Y 2:33, 4:102
R_p, $R_{L\log L}$	reverse Hölder inequalities 3:50
RCLL	right-continuous with left limits
S_t	price of the traded asset(s) at time $t \in [0,T]$ 1:1, 2:11, 3:39, 4:102, 5:121 and 124
$\mathcal{S}^n$	set of symmetric strictly positive definite $(n\times n)$-matrices 4:85
(SC)	structure condition 3:42
$\mathrm{spec}(A)$	spectrum (set of eigenvalues) of a diagonalisable matrix A 2:34, 4:85

$\mathrm{tr}(A)$	trace (sum of the eigenvalues) of a diagonalisable matrix A 4:85
$U(x)$	exponential utility function evaluated at $x \in \mathbb{R}$, $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$ 1:2, 2:12, 3:39, 4:103, 5:121
$V_t^H(x_t)$	value of an exponential utility maximisation problem 1:1, 2:12, 3:39, 4:103, 5:121 and 124
W	P-Brownian motion driving S 2:11, 3:73, 4:101, 5:124
$W^\perp$	P-Brownian motion orthogonal to W 2:23, 4:101
$\hat{W}$	$\hat{P}$-Brownian motion differing from W by a finite variation process 2:12, 3:75, 4:102, 5:124
$\hat{W}^\perp$	$\hat{P}$-Brownian motion orthogonal to $\hat{W}$ 3:75
$\mathbb{W} = (\mathcal{W}_t)_{0\le t\le T}$	P-augmented filtration generated by W 2:21, 4:107
$\mathcal{W}_H$	set of loss variables in Chapter 3 3:39
x_t	investor's initial capital at time $t \in [0,T]$ 1:1, 2:12, 3:39, 4:103, 5:121 and 124
$X_s^{x_t,\pi}$	investor's wealth at time $s \in [t,T]$ 2:12, 4:103, 5:121 and 5:124
Y	P-Brownian motion correlated with W 2:11, 3:73, 4:102, 5:124
$Y^\perp$	P-Brownian motion orthogonal to Y 2:11, 5:124
$\hat{Y}$	$\hat{P}$-Brownian motion differing from Y by a finite variation process 2:12, 3:75, 4:102, 5:124
$\mathbb{Y} = (\mathcal{Y}_t)_{0\le t\le T}$	P-augmented filtration generated by Y 2:21, 3:73, 4:107, 5:124
$\hat{\mathbb{Y}} = \big(\hat{\mathcal{Y}}_t\big)_{0\le t\le T}$	P-augmented filtration generated by $\hat{Y}$ 2:21, 4:107
γ	investor's absolute risk aversion 1:2, 2:12, 3:39, 4:103, 5:121
(Γ, Z)	solution of the BSDE (4.2) 4:85
(Γ, Z, N)	solution of the BSDE (5.1) 5:119
(Γ, ψ, L)	solution of the BSDE (3.51), (3.52) 3:63
$\delta_t^{\hat{H}}$, δ_t^H	random distortion powers 2:16, 3:57
$\overline{\delta}_t$	upper bound for $\delta_t^{\hat{H}}$ 2:15
$\underline{\delta}_t$	lower bound for $\delta_t^{\hat{H}}$ 2:15

$\delta_t^{\max}$	upper bound for the eigenvalues of Λ 4:89
$\delta_t^{\min}$	lower bound for the eigenvalues of Λ 4:89
Δ	jumps: $\Delta\alpha_t := \alpha_t - \lim_{s\nearrow t}\alpha_s$, $t \in]0,T]$, for any RCLL $\alpha : [0,T] \to \mathbb{R}$
λ	instantaneous Sharpe ratio of the traded asset S 2:11, 3:42, 4:102, 5:124
μ	drift of the traded asset S 2:11, 3:73, 4:102, 5:124
Ξ	set of piecewise constant processes 5:126
ρ	instantaneous correlation between W and Y 2:11, 3:73, 5:124
σ	volatility of the traded asset S 2:11, 3:73, 4:102, 5:124
$(\Omega, \mathcal{F}, \mathbb{F}, P)$ $(\Omega, \mathcal{G}, \mathbb{G}, P)$	filtered probability spaces: 1:1, 3:38, 4:85, 5:118 2:11, 4:101, 5:124

Curriculum vitae

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Name	Christoph Martin Frei
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EDUCATION

Ph.D. Studies **05/2006 – 09/2009**

Ph.D. studies in Mathematics, ETH Zürich.
Topic: Exponential utility indifference valuation.
Supervisor: Professor Martin Schweizer.

Undergraduate Studies **10/2001 – 04/2006**

Diploma with distinction in Mathematics
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ETH Zürich.
Major: Stochastics and mathematical finance.

High School **08/1996 – 01/2001**

Matura at Kantonsschule Hottingen, Zürich.
Major: Economics and law.

EMPLOYMENT

ETH Zürich **since 04/2006**

Research and Teaching Assistant since 04/2006.

Member of NCCR FINRISK research project since 02/2009.

Assistant Editor, Finance and Stochastics 06/2008 – 12/2008.

Credit Suisse, Zürich **since 05/2001**

Controller (20%), External Reporting at Private Banking since 11/2001.

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