Appendices to "Moment Estimators for Autocorrelated Time Series and their Application to Default Correlations"

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A Formula for new estimator of return correlation

For practical purposes, we summarize our estimator for latent intra-segment asset return correlation in the case of a first-order correction term, which typically captures a big part of the bias of the classical estimator; compare Figure 1. Formulas for higher-order correction terms and inter-segment correlation can be found in Section 4.1.

Input:
default rate times series $(p_t)_{t=1,\dots,T}$ for a bucket in a Merton model (see Section 2)

Estimator:

$$\hat{\varrho}_2 = \hat{\varrho}_1 + \frac{g''(\hat{\varrho}_1)}{T(g'(\hat{\varrho}_1))^3}\big(\alpha_0/2 + (1-1/T)\alpha_1\big)$$

Components in estimator:

- classical estimator:

$$\hat{\varrho}_1 = g^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} p_t^2\right)$$

where

$$g(\varrho) = \Phi_2\left(\Phi^{(-1)}\left(\frac{1}{T}\sum_{t=1}^{T} p_t\right), \Phi^{(-1)}\left(\frac{1}{T}\sum_{t=1}^{T} p_t\right); \varrho\right)$$

with $\Phi_2(.,.;\varrho)$ denoting the bivariate normal cumulative distribution function with correlation ϱ,

- derivatives appearing in correction term:

$$g'(\hat{\varrho}_1) = \frac{1}{2\pi\sqrt{1-\hat{\varrho}_1^2}}\exp\Big(-\frac{s^2}{1+\hat{\varrho}_1}\Big),$$
$$g''(\hat{\varrho}_1) = \frac{s^2 + \hat{\varrho}_1(1-2s^2) + s^2\hat{\varrho}_1^2 - \hat{\varrho}_1^3}{2\pi(1-\hat{\varrho}_1^2)^{5/2}}\exp\Big(-\frac{s^2}{1+\hat{\varrho}_1}\Big)$$

with $s = \Phi^{(-1)}\big(\frac{1}{T}\sum_{t=1}^{T} p_t\big)$,

- sample variance and lag-1 sample autocovariance:

$$\alpha_0 = \frac{1}{T}\sum_{t=1}^{T}(p_t^2 - \bar{\mu})^2, \qquad \alpha_1 = \frac{1}{T}\sum_{t=2}^{T}(p_t^2-\bar{\mu})(p_{t-1}^2-\bar{\mu}),$$

where $\bar{\mu} = \frac{1}{T}\sum_{t=1}^{T} p_t^2$.

B Proof of Theorem 3.3

By Taylor's theorem, we can write

$$\tilde{g}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right) = \tilde{g}(\mu) + \tilde{g}'(\mu)\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right) + \frac{\tilde{g}''(\mu)}{2}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^2 + \frac{\tilde{g}'''(\xi)}{6}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^3$$

for some ξ between μ and $\frac{1}{T}\sum_{t=1}^{T} Z_t$. Taking expectations and rearranging terms yield

$$\tilde{g}(\mu) - E\left[\tilde{g}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right)\right] + \frac{\tilde{g}''(\mu)}{2}\text{Var}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right) = -E\left[\frac{\tilde{g}'''(\xi)}{6}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^3\right], \quad \text{(A.1)}$$

using that $E[Z_t] = \mu$ for all t by stationarity. We now analyze the different terms in (A.1). First, we note $\tilde{g}(\mu) = \theta$ and compute

$$\tilde{g}''(\mu) = \left(\frac{1}{g'(g^{-1}(\mu))}\right)' = -\frac{g''(g^{-1}(\mu))(g^{-1})'(\mu)}{(g'(g^{-1}(\mu)))^2} = -\frac{g''(g^{-1}(\mu))}{(g'(g^{-1}(\mu)))^3} = -\frac{g''(\theta)}{(g'(\theta))^3}. \quad \text{(A.2)}$$

Similarly to (9), we can write the variance term as

$$\text{Var}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right) = \frac{1}{T}\text{Var}(Z_1) + \frac{2}{T^2}\sum_{\ell=1}^{T-1}(T-\ell)\text{Cov}(Z_1, Z_{1+\ell}). \quad \text{(A.3)}$$

Finally, we apply Hölder's inequality with $p = 4/3$ and $q = 4$ (which satisfy $1/p + 1/q = 1$) to obtain

$$\begin{aligned} E\left[\left|\tilde{g}'''(\xi)\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^3\right|\right] &\le E\left[|\tilde{g}'''(\xi)|^q\right]^{1/q} E\left[\left|\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^3\right|^p\right]^{1/p} \\ &= E\left[(\tilde{g}'''(\xi))^4\right]^{1/4} E\left[\left(\frac{1}{T}\sum_{t=1}^{T} Z_t - \mu\right)^4\right]^{3/4}, \end{aligned}$$

which concludes the proof in light of (A.1)–(A.3). □