Counterparty Risk in Over-the-Counter Markets

Christoph Frei, Agostino Capponi, and Celso Brunetti*

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Abstract

We study trading and risk management decisions of banks in over-the-counter markets, accounting for two types of risk: payoff risk from loans and counterparty risk from trading activities. Our model provides empirically supported predictions on the structure of the interbank credit default swap (CDS) market: (i) banks with high default probabilities either buy or sell CDS contracts; (ii) because of the counterparty risk friction, payoff risk is only partially shared; and (iii) safe banks act as intermediaries and help diversify counterparty risk. Banks manage their default probabilities to become creditworthy counterparties, but they do so in a socially inefficient way.

Keywords: over-the-counter markets, counterparty risk, risk sharing, negative externalities
JEL Classification: G11, G12, G21

*Frei (corresponding author), cfrei@ualberta.ca, University of Alberta Department of Mathematical and Statistical Sciences; Capponi, ac3827@columbia.edu, Columbia University Department of Industrial Engineering and Operations Research; and Brunetti, celso.brunetti@frb.gov, Board of Governors of the Federal Reserve System. We are grateful for constructive comments by Jennifer Conrad (Managing Editor) and an anonymous referee, as well as Jack Bao, Garth Baughman, Co-Pierre George (discussant), Michael Gordy, Christian Heyerdahl-Larsen (discussant), Julien Hugonnier, Ricardo Lagos, Semyon Malamud, Artem Neklyudov, Pierre-Olivier Weill, and seminar participants at the Board of Governors of the Federal Reserve System, the Swiss Finance Institute at EPFL, the Economic Department at Indiana University, the 2017 Conference of the Northern Finance Association, and the 2018 Annual Meeting of the European Finance Association. We would like to thank Alice Moore and Nathan Heinrich for their excellent assistance in the analysis of data. Christoph Frei thanks the Natural Sciences and Engineering Council of Canada for support under grant RGPIN/2019-04789. The research of Agostino Capponi is supported by a NSF-CMMI: 1752326 CAREER grant.

I Introduction

Counterparty risk is one of the most prominent sources of risk faced by market participants and financial institutions in over-the-counter (OTC) markets. During the global financial crisis of 2007–2008, roughly two thirds of credit related losses were attributed to the market price of counterparty risk, and only about one third to actual default events; see Bank for International Settlements (2011).

Our study investigates the critical role played by counterparty risk in shaping the structure of OTC markets. Banks are subject to payoff risk stemming from loan exposures. Heterogeneity in these exposures incentivizes banks to share their payoff risk. Banks with little risk-sharing needs emerge as intermediaries, and provide buy-sell services to other banks that would otherwise be restricted by trade size limits. However, the counterparty risk friction impairs the sharing of payoff risk and strengthens intermediation. Even in the absence of trade size limits, banks share payoff risk imperfectly. To avoid facing excessive counterparty risk, banks wishing to share payoff risk seek intermediation services from other banks with low default probabilities. We use a proprietary data set of bilateral exposures from the market of credit default swaps (CDSs), and highlight the prominent role of five banks with low default probabilities which act as the main intermediaries in the interbanking market (see the network graph in Figure 2).

We investigate the implications of risk management on trading decisions, characteristics of the intermediaries, and structure of the OTC market. In our framework, there is a finite number of banks, each consisting of a continuum of risk-averse traders. Each bank initially holds a loan portfolio, whose payoff is contingent on the realization of an aggregate credit risk factor.[1] While all banks are exposed to the same risk factor, they have heterogeneous exposures to it. This dispersion in exposures gives banks an incentive to share payoff risk: traders of a bank with higher initial exposure purchase protection in the form of CDSs from traders of banks with lower initial exposures.

Trading is modeled as a two-stages process, as in Atkeson, Eisfeldt, and Weill (2015). First, traders of each bank participate in the decentralized OTC market. When two traders belonging to different banks meet, they negotiate over the terms of a trade, taking into consideration that they are all subject to the same trade size limit. The negotiation process is endogenous and gives rise to equilibrium prices and quantities that depend on banks' loan exposures. Importantly, and unlike in Atkeson et al. (2015), contract prices and traded quantities also account for counterparty risk. When a trader of a bank purchases a contract from a trader of another bank, it pays a bilaterally agreed-upon fee upfront. In exchange, the trader receives the contractually agreed-upon payment if the realization of the aggregate credit risk factor corresponds to a credit event, provided that the bank of its trading counterparty does not default. In the second stage of trading, each bank consolidates the contracts signed by its traders and executes these contracts.

In our model, all banks share their payoff risk through OTC contracts. However, banks with high default probabilities impair the risk-sharing capacity of the market. Because they are not guaranteed to fulfill the obligations towards their counterparties, they do not sell

[1]This factor may be thought of as a proxy for the systematic risk driving the prices of CDS indices, such as the CDX.NA.IG and the Itraxx index.

as much credit protection as they would if they were riskless. A bank with high default probability is active only on one side of the market, i.e., either as a buyer or as a seller, and when it is a seller, it trades the same amount of protection with each bank on the buy side. Safe banks, which in our stylized model have zero default probability, play an important role: they act as intermediaries and sell some of the contracts that they bought from risky banks to other banks. Thus, the safe banks increase the participation rate of banks with high default probabilities by diversifying the counterparty risk in the system. As a consequence of partial risk sharing, payoff risk is distributed more homogeneously among the banks after trading. Because payoff risk stems from the banks' loan exposures, this means that banks' post-trade exposures are closer together than initial exposures. In particular, safe banks have the same post-trade exposure if the trade size limit is big enough, whereas banks with high default probabilities maintain diverse post-trade exposures.

Banks gain fees from intermediation services provided to other banks in the network and manage their payoff risk by entering into CDS contracts. We do not disentangle these two channels of income, i.e., banks do not first intermediate and then hedge their payoff risk. Rather, each bank chooses the optimal trading strategy, which simultaneously identifies the level of intermediation and hedging that is individually optimal. To become a more attractive trading counterparty, each bank engages in costly risk management before trading CDS contracts. We show that an inefficiency arises when each bank manages its default probability to maximize its private certainty equivalent. Because the system-wide benefits of counterparty risk reduction are only partially reflected in bilaterally negotiated prices, banks' decisions on their default probabilities may deviate from the social optimum. Interestingly, there exist circumstances under which banks may act conservatively (e.g., implement stricter risk management strategies) and reduce their default probabilities below the socially optimal level.

The inefficiency of the banks' risk management decisions may be understood as follows. The reduction of default probability is reflected in the bank's certainty equivalent through increased revenues generated from the sale of CDS contracts. In equilibrium, these revenues are based on the banks' marginal valuations, rather than on the banks' total valuations which determine the social optimum. We show that the marginal valuations depend on the banks' default probabilities in a different manner than the total valuations. In particular, if traders of a bank increase their CDS purchases from another bank, counterparty risk becomes more concentrated. As a result of this counterparty concentration, the marginal valuations become more sensitive to changes in default probabilities than the total valuations. Since the revenues of a CDS selling bank depend on the banks' marginal valuations, the bank may be overcompensated for lowering its default probability, compared to the system-wide counterparty risk reduction reflected in the banks' total valuations. We show that such a situation occurs when the buyer has high bargaining power, the payoff risk is high, and the seller incurs low risk management costs. Under these conditions, the selling bank reduces its default probability below the socially optimal level.

The rest of paper is organized as follows. We review related literature in Section II. We develop the model in Section III. We study the equilibrium trading decisions of banks in Section IV. We analyze the normative implications of our model in Section V. Section VI concludes. Proofs of all results are delegated to the Appendix.

II Literature Review

Our main contribution to the literature is the development of a tractable framework to study counterparty risk in OTC markets, along with its implications on banks' risk management and bilateral trading decisions.

Our model implication that high-risk banks engage in imperfect payoff risk sharing is supported by empirical evidence provided by Du, Gadgil, Gordy, and Vega (2016). They find that market participants are more likely to trade with counterparties whose credit quality is high. Their analysis shows that if prices do not adjust materially to the credit risk of the dealers, then trade quantities might adjust, i.e., the market transacts to a smaller extent with weaker dealers. Our model predictions are also consistent with the empirical results of Arora, Gandhi, and Longstaff (2012), who find a statistically significant negative relation between the credit risk of the dealer and the prices at which he sells credit protection. However, they also find that the economic significance of this relation is very small. While in our model changes in banks' default probabilities may lead to small adjustments in contract prices, similarly to their findings, they can nonetheless have large implications on the market structure. Oehmke and Zawadowski (2017) provide evidence consistent with hedging and speculation motives of CDS contracts. While they investigate the relation between CDSs and bonds of firms in different industries, we focus entirely on the interbank CDS market and the role of counterparty risk.

Our framework builds on that developed by Atkeson et al. (2015). A crucial difference is that the sellers of credit protection might default without delivering the full contracted amount. This counterparty risk friction of our model has important ramifications with regard to the equilibrium market structure, and leads to significantly different economic conclusions: First, bilateral trading positions between banks are unique in equilibrium because the counterparty risk of the sellers makes the traded contracts imperfect substitutes. Second, the payoff risk-sharing capacity of the market is impaired, even when the trade size limit is not binding. Finally, intermediaries play the additional role of diversifying counterparty risk, besides increasing the payoff risk-sharing capacity of the market.

The classical setup used to study OTC markets is the search-and-bargaining framework proposed by Duffie, Gârleanu, and Pedersen (2005), which models the trading friction characteristics typical of these markets. Their model was generalized along several dimensions, including the relaxation of the constraint of zero-one units of assets holdings (see Lagos and Rocheteau (2009)), the entry of dealers (see Lagos and Rocheteau (2007)), and investors' valuations drawn from an arbitrary distribution as opposed to being binary (see Hugonnier, Lester, and Weill (2020)). All these studies do not allow for the inclusion of counterparty risk, mainly because the framework cannot keep track of the identities of the counterparties for the continuum of traders.

Our paper is also related to the emerging literature on endogenous OTC networks. Wang (2018) shows that the trading network which emerges endogenously in OTC markets is of the core-periphery type. In his model, intermediaries exploit their central position to balance inventory risk, while in our model they help diversify counterparty risk in the network. Gofman (2014) provides a network model to study the intermediation friction in OTC markets. As in our model, trading decisions and bilateral prices are jointly determined in equilibrium. Traders can only transact if they have a trading relationship and extract a surplus which depends both on the private value of the buyer and on the resale opportunities of the asset. While the focus of his study is on welfare losses due to intermediation frictions, we study the negative externality originating from counterparty concentration. Babus and Hu (2017) consider an infinite-horizon model of endogenous intermediation and analyze two important frictions of OTC markets. The first is the limited commitment of market participants who can renege on due payments, and the second is the opaqueness of OTC markets in which participants have incomplete information on the past behavior of others.

A related branch of literature has studied the incentives behind the formation of interbank loan networks.[2] Farboodi (2017) proposes a model of financial intermediation where profit-maximizing institutions strategically decide on borrowing and lending activities. Her model predicts that banks which make risky investments voluntarily expose themselves to excessive counterparty risk, while banks that mainly provide funding establish connections with a small number of counterparties in the network. A related study by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014-b) analyzes the endogenous formation of interbanking loan networks. They find that banks may overlend in equilibrium and do not spread their lending among a sufficiently large number of potential borrowers, thus creating insufficiently connected financial networks prone to defaults. Different from their settings, our framework captures stylized features of derivatives trading in OTC markets, as opposed to markets for interbanking loans. Meetings between traders are random, and the equilibrium trading patterns are the outcome of bilateral bargaining that accounts for counterparty risk. Parlour and Winton (2013) construct an equilibrium model to assess when banks use loans as opposed to CDSs. They show that, for riskier credits, loan sales typically dominate CDSs as a means for transferring credit risk while the opposite is true for safer credits, and CDSs allow for efficient risk sharing. As we study how counterparty risk affects payoff risk sharing, we focus on how banks manage their default probabilities before participating in the OTC market of CDSs.

III The Model

The model economy consists of a unit continuum of *traders*, who are risk-averse agents. They have constant absolute risk aversion with parameter η, i.e., their utility function is given by $U(x) = -\mathrm{e}^{-\eta x}$. The traders are organized into M *banks*, which are coalitions of traders. All banks are granted access to the same technology to trade swaps.

The banks are heterogeneous in their initial exposures to an aggregate risk factor D, taking binary values 0 (no default) and 1 (default), with $P[D = 1] = q$, where we assume $0 < q < 1$. We denote by ω_i the initial exposure per trader of bank i to the aggregate risk factor. The traders are paired uniformly across the different banks. Therefore, the frequency at which a trader of bank $j \neq i$ is paired with a trader of bank i is proportional to the measure of traders in bank i, which we refer to as the size of bank i. Both the initial exposure ω_i and the size of bank i are exogenously specified and observable to the traders. To

[2] Another branch of literature has studied counterparty risk in an *exogenously* specified network of financial liabilities. The focus of these studies is on how the topology of the network affects the amplification of an initial shock through the network. Relevant contributions in this direction include Eisenberg and Noe (2011), Elliott, Golub, and Jackson (2013), and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014-a).

highlight the primary economic forces at play, in the main body of the paper we present the results for banks of equal size. We restate and prove the results in the Appendix, allowing for heterogeneity in the banks' sizes.

When a trader from bank i meets a trader from bank n, they bargain a contract similar to a CDS. They agree that the trader of bank i sells $\gamma_{i,n}$ contracts to the trader of bank n. If $\gamma_{i,n} > 0$, bank n makes an immediate payment of $\gamma_{i,n} R_{i,n}$, and at the end of the period, bank i makes a payment of $\gamma_{i,n} D$ to bank n if bank i has not defaulted by then; if it has defaulted, there is no payment. For bank i, we denote by A_i the event that the bank defaults with $P[A_i|D=1] = p_i$, where we assume $0 \le p_i < 1$. Because banks will trade contracts of CDS type on the aggregate risk factor D, only the conditional default event $A_i|D=1$ of bank i and not the unconditional default event A_i matters to the trading counterparties of bank i. Therefore, the conditional default probability p_i determines the attractiveness of bank i on the OTC market. We assume that the conditional events $A_i|D=1$ are independent but do not impose that the banks' defaults themselves are independent. In particular, each bank can have different default probabilities depending on the realization of the aggregate risk factor. This setting allows for a dependence structure among the banks' defaults. A special role will be taken by banks with $p_i = 0$. We call such banks *safe*, while banks with $p_i > 0$ are referred to as *risky*.

In summary, the payment at the end of the period is $\gamma_{i,n} D \mathbb{1}_{A_i^\complement}$ from bank i to bank n if $\gamma_{i,n} > 0$, where $A_i^\complement$ denotes the complement of the default event A_i. For the case $\gamma_{i,n} < 0$, the roles of i and n are interchanged. Therefore, the bilateral constraint $\gamma_{i,n} = -\gamma_{n,i}$ holds. We further assume that there is a trade size constraint per trader so that $-k \le \gamma_{i,n} \le k$ for some constant $k > 0$.[3] We call a set of contracts $(\gamma_{i,n})_{i,n=1,\ldots,M}$ *feasible* if both the bilateral constraint $\gamma_{i,n} = -\gamma_{n,i}$ and the trade size constraint $-k \le \gamma_{i,n} \le k$ hold for all $i, n = 1, \ldots, M$. For notational convenience, we will use the abbreviation $\gamma_i := (\gamma_{i,1}, \ldots, \gamma_{i,M})$ to denote the collection of contracts that bank i has with the other banks.

At the end of the trading period, traders of every bank come together and consolidate all their long and short positions. The consolidated per-capita wealth of bank i from its initial exposure and the contracts $\gamma_{i,1}, \ldots, \gamma_{i,M}$ is

$$X_i = \omega_i(1-D) + \sum_{n \ne i} \gamma_{i,n} \big(R_{i,n} - D \mathbb{1}_{A_n^\complement} \mathbb{1}_{\gamma_{i,n}<0} - D \mathbb{1}_{A_i^\complement} \mathbb{1}_{\gamma_{i,n}>0} \big),$$

where

- $\omega_i(1-D)$ is the per-capita payout associated with the initial exposure.
- $\sum_{n \ne i} \gamma_{i,n} R_{i,n}$ is the aggregate net payment received (if positive) or made (if negative) during trading, corresponding to the CDS protection fees.
- $-D \gamma_{i,n} \mathbb{1}_{A_n^\complement} \mathbb{1}_{\gamma_{i,n}<0}$ is the per-capita payment that bank i will receive from bank n. This payment will be executed only if the realization of the aggregate risk factor is $D = 1$

[3] Those limits are an integral part of counterparty credit risk management frameworks: exposures are typically monitored on an on-going basis and adequate risk controls are put in place to reduce the violation of these limits. We refer to Office of the Comptroller of the Currency et al. (2011) for further details, see Section V.I therein.

and bank i net bought protection from bank n ($\gamma_{i,n} < 0$). In this case, bank i will receive $-\gamma_{i,n}$ if bank n does not default $\big($event $A_n^\complement\big)$.

- $D\gamma_{i,n}\mathbb{1}_{A_i^\complement}\mathbb{1}_{\gamma_{i,n}>0}$ is the per-capita payment that bank i will make to bank n. This payment will be executed only if the realization of the aggregate risk factor is $D = 1$ and bank i net sold protection to bank n ($\gamma_{i,n} > 0$). In this case, bank i will pay $\gamma_{i,n}$ if it does not default $\big($event $A_i^\complement\big)$.

IV Equilibrium Decisions and Market Structure

This section studies the trading decisions prevailing at the market equilibrium. In Section IV.A, we develop an explicit expression for a bank's certainty equivalent. In Section IV.B, we establish the existence of such an equilibrium. In Section IV.C, we study the implications of counterparty risk on the banks' post-trade exposures and the market structure. In Section IV.D, we present a numerical example and provide empirical evidence in support of the results of Section IV.C.

IV.A Banks' Certainty Equivalents

We calculate the certainty equivalent x_i of X_i by solving $U(x_i) = E[U(X_i)]$, which yields

$$x_i = \omega_i + \sum_{n\neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_{i,1}, \dots, \gamma_{i,M}), \tag{1}$$

where

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log E\left[\exp\left(\eta D\left(\omega_i + \sum_{n\neq i} y_n\big(\mathbb{1}_{A_i^\complement}\mathbb{1}_{y_n>0} + \mathbb{1}_{A_n^\complement}\mathbb{1}_{y_n<0}\big)\right)\right)\right].$$

The following result gives an explicit formula for Γ^i.

Lemma IV.1. *We have*

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log\left(1 - q + q\mathrm{e}^{\eta\omega_i + \eta f(\sum_{n:y_n\geq 0} y_n, p_i) + \eta \sum_{n:y_n<0} f(y_n, p_n)}\right),$$

where

$$f(y,p) = \frac{1}{\eta}\log\big((1-p)\mathrm{e}^{\eta y} + p\big). \tag{2}$$

For $p > 0$, the functions

$$y \mapsto \Xi(y) := \frac{1}{\eta}\log(1 - q + q\mathrm{e}^{\eta y}) \quad \text{and} \quad y \mapsto f(y,p)$$

are strictly increasing and strictly convex so that the function $\Gamma^i(y_1, \dots, y_M)$ is strictly increasing and convex. If $p_n > 0$, then the function Γ^i, viewed as a function of y_n, is strictly convex on $(-\infty, 0)$. Moreover, the function f satisfies

$$f(y_1, p_1) + f(y_2, p_2) > f(y_1 + y_3, p_1) + f(y_2 - y_3, p_2) \tag{3}$$

for all $y_1 < y_2$, $y_3 \in \big(0, \frac{y_2-y_1}{2}\big]$ and $p_1 \geq p_2$.

The value $f(y,p)$ quantifies how the certainty equivalent of a protection seller (or buyer) changes when it sells y (or buys y if $y<0$) contracts to (from) a protection buyer (seller), where p is the default probability of the bank selling the contracts. If the bank that sells the contracts is safe ($p_i=0$), then $f(y,p_i)=y$ is linear. However, if the bank that is selling the contracts is risky ($p_i>0$), the increase in $f(y,p_i)$ is smaller given that

$$\frac{1}{\eta}\log\big((1-p_i)\mathrm{e}^{\eta y}+p_i\big)\begin{cases}<\frac{1}{\eta}\log\big((1-p_i)\mathrm{e}^{\eta y}+p_i\big)=y & \text{if } y>0\\ >\frac{1}{\eta}\log\big((1-p_i)\mathrm{e}^{\eta y}+p_i\big)=y & \text{if } y<0.\end{cases}$$

The inequality (3) has a very intuitive interpretation. Suppose a bank buys CDS protection from banks 1 and 2 with default probabilities $p_1>p_2$. If the bank were to buy additional protection from bank 1, its certainty equivalent would be lower with respect to the case in which it makes balanced purchases from the two banks.

IV.B Market Equilibrium: Existence and Properties

Because traders are assumed to be small relative to their banks, they maximize the marginal impact of their decisions on their banks' utilities. When a trader of bank i sells protection to a trader of bank n, the cost of risk bearing increases by $\gamma_{i,n}\Gamma^i_{y_n}(\gamma_i)$ for bank i and decreases by $\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n)$ for bank n, where $\Gamma^n_{y_i}(\gamma_n)$ denotes the partial derivative of $\Gamma^n(\gamma_n)$ with respect to the i-th component. Therefore, when traders of banks i and n bargain, their trading surplus is given by

$$\gamma_{i,n}\big(\Gamma^n_{y_i}(\gamma_n)-\Gamma^i_{y_n}(\gamma_i)\big).$$

We assume that (i) all trader pairs from bank i and bank n choose the same trade quantity, and (ii) the terms of a trade in each bilateral meeting are determined via Nash bargaining, with possibly different bargaining powers of the two traders. Assumption (i) guarantees that per-capita trade quantities between banks and the trades of their individual traders coincide. As a consequence of Nash bargaining, the terms of a trade are bilaterally Pareto optimal, hence the above trading surplus is maximized. Therefore, bilateral quantities chosen by traders of two banks i and n are consistent with a perfect equalization of marginal valuations. That is, after consolidation of all trades their banks have the same marginal valuation, i.e., $\Gamma^i_{y_n}(\gamma_i)=\Gamma^n_{y_i}(\gamma_n)$ if this is attainable. However, this outcome may not be achievable for two reasons. First, the quantity of traded contracts needed to achieve equal marginal valuation may be larger than what the trade size limit allows. In this case, the marginal valuations $\Gamma^i_{y_n}(\gamma_i)$ and $\Gamma^n_{y_i}(\gamma_n)$ differ, and traders choose the maximal quantity allowed under the trade size limit, hence $\gamma_{i,n}=k$ if $\Gamma^i_{y_n}(\gamma_i)<\Gamma^n_{y_i}(\gamma_n)$ and $\gamma_{i,n}=-k$ if $\Gamma^i_{y_n}(\gamma_i)>\Gamma^n_{y_i}(\gamma_n)$. The second reason why equal marginal valuation may not be attainable is that traders of bank i may find it beneficial to neither buy from nor sell to traders of bank n, which occurs when $\Gamma^i_{y_n}(\gamma_i)<\Gamma^n_{y_i}(\gamma_n)$ for all $\gamma_{i,n}<0$ and $\Gamma^i_{y_n}(\gamma_i)>\Gamma^n_{y_i}(\gamma_n)$ for all $\gamma_{i,n}>0$. In such a situation, it is optimal not to trade because the trading surplus is negative for any nonzero quantity, and we would expect the marginal valuations to be equal at zero, i.e., $\Gamma^i_{y_n}(\gamma_i)=\Gamma^n_{y_i}(\gamma_n)$ for $\gamma_{i,n}=0$. However, this may not be the case because the marginal valuations may not be defined at zero. The case of zero traded quantities between two banks is very special, because under these circumstances a change in the per-capita trade quantity would alter the direction of counterparty risk exposures for banks i and n: the certainty equivalent of the bank which buys protection is negatively impacted by the increased default probability of its counterparty, while the bank selling protection benefits from its own default, and thus its certainty equivalent is increased. By contrast, the banks' valuations at any quantity other than zero change in a smooth way with respect to the purchased or sold per-capita quantity. Indeed, $\Gamma^i(\gamma_i)$ and $\Gamma^n(\gamma_n)$ are continuously differentiable in the per-capita trade quantities $\gamma_{i,n}$ and $\gamma_{n,i}$ everywhere except at $\gamma_{i,n} = \gamma_{n,i} = 0$. This implies that when the marginal valuations do not exist, the per-capita quantity must be zero. For future reference, we provide a summary of the above discussion: for each bilateral trading relationship between two banks, one of the following four cases arises:

$$
\begin{cases} \Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n), \\ \Gamma^i_{y_n}(\gamma_i) < \Gamma^n_{y_i}(\gamma_n); \text{ in this case, } \gamma_{i,n} = k, \\ \Gamma^i_{y_n}(\gamma_i) > \Gamma^n_{y_i}(\gamma_n); \text{ in this case, } \gamma_{i,n} = -k, \\ \Gamma^i_{y_n}(\gamma_i) \text{ or } \Gamma^n_{y_i}(\gamma_n) \text{ do not exist; in this case, } \gamma_{i,n} = 0. \end{cases} \tag{4}
$$

The unit price $R_{i,n}$ of a CDS is decided via bargaining between a protection seller with bargaining power $\nu \in [0,1]$ and a protection buyer with bargaining power $1-\nu$. Hence,

$$
R_{i,n} = \nu \max\left\{\Gamma^i_{y_n}(\gamma_i), \Gamma^n_{y_i}(\gamma_n)\right\} + (1-\nu)\min\left\{\Gamma^i_{y_n}(\gamma_i), \Gamma^n_{y_i}(\gamma_n)\right\}. \tag{5}
$$

If bank i sells contracts to bank n, it receives a fraction ν of the trading surplus. Indeed, bank i's cost of risk bearing increases by $\gamma_{i,n}\Gamma^i_{y_n}(\gamma_i)$, but it receives a payment $\gamma_{i,n}R_{i,n}$ so that the net effect on bank i is

$$
\begin{aligned}
&-\gamma_{i,n}\Gamma^i_{y_n}(\gamma_i) + \gamma_{i,n}R_{i,n} \\
&= \gamma_{i,n}\big(\nu \underbrace{\max\left\{\Gamma^i_{y_n}(\gamma_i), \Gamma^n_{y_i}(\gamma_n)\right\}}_{=\Gamma^n_{y_i}(\gamma_n) \text{ by (4)}} + (1-\nu)\underbrace{\min\left\{\Gamma^i_{y_n}(\gamma_i), \Gamma^n_{y_i}(\gamma_n)\right\}}_{=\Gamma^i_{y_n}(\gamma_i) \text{ by (4)}} - \Gamma^i_{y_n}(\gamma_i)\big) \\
&= \nu \underbrace{\gamma_{i,n}\big(\Gamma^n_{y_i}(\gamma_n) - \Gamma^i_{y_n}(\gamma_i)\big)}_{\text{trading surplus}}.
\end{aligned}
$$

Because of the translation invariance property of the exponential utility, the relative bargaining power between buyers and sellers does not affect how traded quantities are chosen in equilibrium.

Definition IV.2. *Feasible contracts* $(\gamma_{i,n})_{i,n=1,\ldots,M}$ *build a* market equilibrium *if they are optimal in the sense that they satisfy* (4).

The following result shows that finding a market equilibrium is equivalent to solving a planning problem.

Theorem IV.3. *Feasible contracts* $(\gamma_{i,n})_{i,n=1,\ldots,M}$ *are a market equilibrium if and only if they solve the optimization problem*

$$
\textit{minimize} \sum_{i=1}^{M} \Gamma^i(\gamma_i) \quad \textit{over } \gamma \textit{ subject to } \gamma_{i,n} = -\gamma_{n,i} \textit{ and } -k \le \gamma_{i,n} \le k. \tag{6}
$$

This result follows from the fact that certainty equivalents are quasi-linear so that feasible contracts are a solution to the planning problem if and only if they are Pareto optimal for the banks. Based on the quasi-linearity of certainty equivalents, Atkeson et al. (2015) find that, conditional on entry decisions, the pairwise traded contracts are socially optimal.

In our model, a market equilibrium on the level of the individual traders is thus equivalent to a Pareto optimal allocation for the banks. However, Pareto optimality for banks is only a statement about quantities and does not characterize prices. In our model, prices are determined in each meeting between two traders, as is standard in OTC market models.

Theorem IV.4. *There exists a market equilibrium* $(\gamma_{i,n})_{i,n=1,\dots,M}$*. The per-capita trade quantities* $\gamma_{i,n}$ *are unique for* $p_n > 0$ *and* $\gamma_{i,n} < 0$*, or* $p_i > 0$ *and* $\gamma_{i,n} > 0$*. For every* i*, the value of* $\sum \gamma_{i,n}$ *is unique in equilibrium, where the sum is over* n *such that* $p_n = 0$ *and* $\gamma_{i,n} < 0$*, or* $p_i = 0$ *and* $\gamma_{i,n} > 0$*. In particular, the values of* $\Gamma(\gamma_n)$*'s are uniquely determined for a market equilibrium* $(\gamma_{i,n})_{i,n=1,\dots,M}$*.*

Theorem IV.4 establishes the existence of a market equilibrium, and states that per-capita trade quantities between banks are unique in equilibrium if the bank selling protection is risky. This uniqueness result extends Theorem 1 of Atkeson et al. (2015), where per-capita trade quantities are not unique between banks with the same marginal valuation. As soon as counterparty risk is accounted for in the valuation of a trade, per-capita trade quantities are uniquely pinned down in equilibrium, even for trades between banks with the same marginal valuation. The first reason is that we consider a finite number of large banks, rather than a continuum of infinitesimally small banks like in Atkeson et al. (2015), to keep track of the identities of defaulting banks. As a result, the per-capita trades between banks can change their marginal valuations. The second and deeper reason is that counterparty risk makes CDS contracts purchased from traders of different banks imperfect substitutes. This lack of perfect substitutability is due to counterparty concentration, and persists even if banks were to have the same default probability: because of risk aversion, if a trader buys two CDS contracts, it prefers to choose the two trading counterparties from different banks, rather than purchasing both contracts from traders of the same bank. The situation is different if we only consider safe banks. For example, the size of trades between three safe banks A, B and C could be increased without changing the planning problem (6) if A buys n additional CDS contracts from B, B buys n additional CDS contracts from C, and C buys n additional CDS contracts from A.

The uniqueness of per-capita trade quantities between risky banks implies the uniqueness of trades of their individual traders under our assumption that all trader pairs of two banks choose the same trade quantity. This is because, under this assumption, per-capita trade quantities coincide with the trades of individual traders. If this assumption is relaxed, individual traders could deviate in their traded quantities while maintaining the same per-capita quantity, so that the individual trades would no longer be unique.

IV.C Post-trade Exposures and Intermediation Volume

Denote by $(\gamma_{i,n})_{i,n=1,\dots,M}$ a market equilibrium from Theorem IV.4. The per-capita post-trade exposure Ω_i is given by

$$E[U(-\Omega_i D)] = U(\Gamma^i(\gamma_{i,1},\dots,\gamma_{i,M})) \quad (7)$$

so that traders of bank i would be indifferent between (a) trading and changing the initial exposure (certainty equivalent $\Gamma^i(\gamma_{i,1}, \ldots, \gamma_{i,M})$ on the right-hand side of (7)) and (b) non-trading and maintaining a fictitious initial exposure of Ω_i units to the aggregate risk factor ($-\Omega_i D$ on the left-hand side of (7)). Solving (7) for Ω_i yields

$$\Omega_i = \omega_i + f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n}, p_i\right) + \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n}, p_n), \tag{8}$$

where f is defined in Lemma IV.1. Note that Ω_i accounts for counterparty risk: if bank i and all its counterparties are safe, then Ω_i simplifies to $\omega_i + \sum_{n\neq i} \gamma_{i,n}$, as in Atkeson et al. (2015). Observe also that Ω_i is uniquely determined by Theorem IV.4. If bank i buys $-\gamma_{i,n}$ contracts on average from each trader of a risky bank n, the exposure of bank i is effectively reduced by less than $\gamma_{i,n}$ to adjust for counterparty risk, taking the bank's risk aversion into consideration. Similarly, if bank i is risky and sells $\gamma_{i,n}$ contracts on average to each trader of bank n, then its effective increase in exposure is less than $\gamma_{i,n}$ due to the benefit from not honoring the promised payments if it defaults.

The next result says that the post-trade exposures are increasing and closer together than initial exposures. This result generalizes the first part of Proposition 1 of Atkeson et al. (2015) to our counterparty risk setting, while we will see in our Theorem IV.6 below that the second part of Proposition 1 of Atkeson et al. (2015) takes a quite different form in our model.

Proposition IV.5. *We have the following relations between initial and post-trade exposures:*

1. *If $\omega_i \geq \omega_j$ and $p_i \leq p_j$, then $\Omega_i \geq \Omega_j$.*
2. *If $\omega_i > \omega_j$ and $p_i \geq p_j$, then $\omega_i - \omega_j > \Omega_i - \Omega_j$.*

Proposition IV.5 states that

1. The banks' order in post-trade exposures is the same as that in the initial exposures, provided that their default probabilities are ordered in the opposite direction.
2. Post-trade exposures are closer together than initial exposures if the bank with larger initial exposure is at least as risky as the bank with smaller initial exposure.

To see why conditions on the default probabilities of the banks need to be imposed, consider two banks i and j whose initial exposures $\omega_i > \omega_j$ are smaller than the average initial exposure. Because both banks have initial exposures below the average, they are interested in selling protection and earning the CDS protection fee. These trading motives imply that their post-trade exposures Ω_i and Ω_j are bigger than ω_i and ω_j, respectively. However, if bank i is safer than bank j, it is likely that the other banks will buy a higher amount of protection from bank i so that $\Omega_i - \omega_i > \Omega_j - \omega_j$. This inequality stands in contrast with that in the second statement of Proposition IV.5, noting that $p_i \geq p_j$ does not hold, either. Yet if bank j is safer than bank i, it is likely that the other banks will buy a larger amount of protection from bank j, leading to $\Omega_j > \Omega_i$ even though the initial exposures had the reverse order. We will graphically demonstrate later in Figure 1 that both of these cases can indeed happen so that conditions on the default probabilities in Proposition IV.5

are needed. Building on Proposition IV.5, we give the following theorem which (i) identifies which banks engage in full payoff risk sharing if the trade size limit is big enough, and (ii) characterizes the market structure of trading activities.

Theorem IV.6. *Assume that the trade size limit is not binding, and there are at least two safe banks.*[4] *Then*

1. *There exists the following relation between banks' creditworthiness, initial exposures and post-trade exposures:*
   (a) *All safe banks have the same post-trade exposure, say, $\bar{\Omega}$.*
   (b) *Risky banks with initial exposure above some threshold α also have the same post-trade exposure $\bar{\Omega}$. The threshold α is greater than $\bar{\Omega}$ and depends only on the distribution of initial exposures, but not on the banks' default probabilities.*
   (c) *Risky banks with initial exposure below α will have post-trade exposures strictly smaller than $\bar{\Omega}$.*
2. *Risky banks with initial exposure above α trade as follows:*
   (a) *They do not trade between each other.*
   (b) *They do not sell protection.*
   (c) *Their purchases depend only on their initial exposure, but not on their default probabilities.*
3. *Risky banks with initial exposure below α trade as follows:*
   (a) *They do not trade between each other if their exposures, after trading with other banks but before trading between themselves, are sufficiently close relative to the amount of sold protection.*[5]
   (b) *They do not purchase protection from safe banks or risky banks with initial exposures above α.*
   (c) *They sell the same amount of protection to each safe bank and risky bank with initial exposure above α.*

[4] For payoff risk sharing to take place, the financial system must consist of at least two banks. Because perfect risk sharing is done by safe banks, we consider in the proposition a market environment with at least two safe banks.

[5] To be precise, consider two risky banks i and n that have initial exposures below α. We denote by $\Gamma_i = \sum_{\ell \neq n : \gamma_{i,\ell} \geq 0} \gamma_{i,\ell}$ the sum of contracts that bank i sold to other banks. Assume without loss of generality that $\tilde{\Omega}_i \leq \tilde{\Omega}_n$, where $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ are the exposures of banks i and n, respectively, after trading with other banks but before trading between themselves. If the condition $\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y(\Gamma_i, p_i)}{1-p_i}$ is satisfied for f and Ξ given in Lemma IV.1, banks i and n do not trade between each other. Note that $\frac{f_y(\Gamma_i, p_i)}{1-p_i}$ is increasing in Γ_i and greater than 1, hence the condition is satisfied when $\tilde{\Omega}_n$ and $\tilde{\Omega}_i$ are sufficiently close relative to the protection Γ_i that bank i sold. For the detailed statement, see Theorem A.4 in the Appendix.

Part 1a of the above theorem is consistent with Proposition 3 in Atkeson et al. (2015): if the trade size limit is big enough, safe banks perfectly share their payoff risk and they all end up with the same post-trade exposure. However, the presence of counterparty risk leads to novel insights on the market structure relative to Atkeson et al. (2015), as can be seen from the other parts of the theorem. Risky banks with large initial exposures are active only as buyers on the OTC market. Hence, their default probabilities do not matter to traders of other banks, so that the amount of purchased protection does not depend on their default probabilities (part 2c) and they have the same post-trade exposure as the safe banks (part 1b). By contrast, risky banks with small initial exposures would like to sell credit protection, and do not buy protection (part 3b). These banks will have a lower post-trade exposure than the safe banks (part 1c) for two reasons. Firstly, they sell a smaller amount of credit protection because they are less attractive as trading counterparties due to their non-zero default probabilities. Secondly, as a result of the credit protection sold, they increase their post-trade exposure. However, because they may default and fail to honor obligations, their exposure would be smaller compared to that of a safe bank offering the same amount of protection. Unless the bank's default probability p_i is high, the first effect dominates the second because protection buyers are highly sensitive to the seller's default probability. Although the action of selling protection is the same for a safe and risky bank, the value of sold protection is different: purchasing from a safe bank guarantees complete protection while purchasing from a risky bank is cheaper, but exposes to counterparty risk. If the two banks have very similar credit quality (the default probability p_i is low), buyers are more sensitive to changes in credit quality, and likely switch to the safe bank if p_i increases. By contrast, if the two banks have very different credit quality (the default probability p_i is high), an increase of p_i will have a smaller effect on the buyer's behavior because the price charged by the risky bank is much lower than that charged by the safe bank.

The condition on the non-binding trade size limit in Theorem IV.6 guarantees that safe banks and risky banks with a high initial exposure have enough flexibility to trade and reach a post-trade exposure of $\bar{\Omega}$. If the trade size limit is binding for some but not all of these banks, those with high initial exposures and non-binding trade size limit will still have the same post-trade exposure. Risky banks with low initial exposure will have strictly smaller post-trade exposures regardless of the trade size limit.

To give further insight behind the economic forces at play in Theorem IV.6, we briefly sketch the main steps of the proof, and delegate the full details to the Appendix. If the trade size limit is large enough, all safe banks will perfectly share their payoff risk and have the same post-trade exposure, which we denote by $\bar{\Omega}$ (part 1a of Theorem IV.6). This is similar to Proposition 3 in Atkeson et al. (2015), with the only difference being that in our setting the post-trade exposure of safe banks also depends on the counterparty risk faced when they buy protection from risky banks. We then analyze the post-trade exposures of the risky banks, and show that none of them can have a post-trade exposure greater than $\bar{\Omega}$. This is intuitive because if a bank had a post-trade exposure greater than $\bar{\Omega}$, its traders would buy protection from the safe banks. Therefore, risky banks will have a post-trade exposure that is either exactly $\bar{\Omega}$ or smaller than $\bar{\Omega}$. We can show that there is a threshold α so that risky banks with initial exposure greater than or equal to α have a post-trade exposure equal to $\bar{\Omega}$ and those with initial exposures below α have post-trade exposures smaller than $\bar{\Omega}$. Crucially, this threshold α does not depend on the banks' default probability. Consequently, whether a risky bank will have a post-trade exposure equal to $\bar{\Omega}$ or smaller than $\bar{\Omega}$ depends on its initial exposure, but not on its default probability (parts 1b and 1c).

Consider two risky banks with initial exposures above α, and thus with the same post-trade exposures. Their marginal valuations need to be same, i.e, $\Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n)$, but this cannot hold for any non-zero traded quantity $\gamma_{i,n}$ when the banks are risky and have the same post-trade exposures, as we show in the proof. Intuitively, trading between two risky banks with the same post-trade exposure does not give rise to any trade benefit because of risk aversion: the additional counterparty risk incurred by the buyer outweighs the seller's benefit of defaulting on its obligations (part 2). This argument does not only apply to banks that have the same post-trade exposure, but also to banks with different post-trade exposures as long as the cost of counterparty risk outweighs the potential benefit from sharing payoff risk. This happens when the banks' post-trade exposures are sufficiently close relative to the protection that they sold. Thus, such banks will not trade between each other (part 3a). Finally, safe and risky banks with initial exposures above α have all the same post-trade exposure $\bar{\Omega}$. Therefore, when traders of these banks negotiate with traders of a risky bank whose initial exposure is below α, they all have the same trade incentive (depending on the default probability of the selling bank, but not of their own bank), hence they buy the same quantity in equilibrium (parts 3b & 3c).

An immediate consequence of Theorem IV.6 is the sensitivity of the post-trade exposures to the banks' default probabilities.

Corollary IV.7. *If the trade size limit is big enough, the post-trade exposures of banks with sufficiently high initial exposure are not sensitive to their default probabilities, while the post-trade exposures of banks with small initial exposures are sensitive to their default probabilities.*

The statement in Corollary IV.7 is intuitive. If the payoff risk bearing capacity of the market is not impaired by the presence of trade size limits, banks with sufficiently large initial exposures are protection buyers and thus their own default probabilities do not matter to their counterparties. However, banks with low initial exposures are protection sellers, so their default probabilities matter when other banks decide to trade with them.

We next study which banks endogenously emerge as interbank intermediaries. These banks participate on both sides of the CDS market, as opposed to taking large net positions, either long or short. We consider per-capita gross numbers of sold or purchased contracts, given by $G_i^+ = \sum_{n \neq i} \max\{\gamma_{i,n}, 0\}$ and $G_i^- = \sum_{n \neq i} \max\{-\gamma_{i,n}, 0\}$, respectively. The per-capita intermediation volume of bank i is defined as $I_i = \min\{G_i^+, G_i^-\}$. We call a bank with non-zero intermediation volume an *intermediary.*

From part 2 of Theorem IV.6, we deduce that risky banks with high initial exposures only purchase protection. For each bank i in this group, $G_i^+ = 0$, and thus it is not an intermediary. The situation is more subtle for risky banks with low initial exposures. By part 3b of Theorem IV.6, these banks do not purchase any protection from safe banks and risky banks with high initial exposures. Hence, most of them are active only on the sell side, but some of them may purchase protection from other risky banks that have even lower initial exposures. This happens when their post-trade exposures are not sufficiently close relative to the protection they sold, i.e., they do not satisfy the condition in part 3a of

Theorem IV.6. Therefore, intermediaries are only these banks or safe banks, as summarized in the following corollary.

Corollary IV.8. *Assume that the trade size limit is not binding, and there are at least two safe banks. Let α be the threshold given in Theorem IV.6. Then an intermediary can only be a safe bank or any risky bank whose initial exposure is below α and whose post-trade exposure is not sufficiently close, relative to its protection sold, to that of any other risky bank.*[6]

Specifically, we note that, although all safe banks buy the same amount of protection from a risky bank whose initial exposure is sufficiently low (part 3c of Theorem IV.6), all safe banks still end with the same post-trade exposures (part 1a of Theorem IV.6). Therefore, such safe banks must be active as intermediaries. In our model, intermediation activities are beneficial even when the trade size limit is high and non-binding. This result stands in contrast with Atkeson et al. (2015), where the intermediation activity vanishes if the trade size limit is sufficiently large. This is not the case in our framework due to counterparty risk. Banks avoid buying protection from risky counterparties. By purchasing contracts through intermediaries, part of the counterparty risk is transferred to the intermediaries which in turn benefit from the received CDS protection fees. Because of these transactions, counterparty risk is split between the customer bank and the intermediary, and the concentration of counterparty risk in the OTC market is reduced. Therefore, in our model, the intermediaries have two functions: they help diversify the aggregate level of counterparty risk and, as in Atkeson et al. (2015), they facilitate partial sharing of payoff risk. Compared to safe banks, risky banks are not as attractive for intermediation because they impose counterparty risk on the protection buyer.

IV.D A Numerical Example and Empirical Evidence

We begin the section by constructing an example which illustrates the findings of Theorem IV.6 and Corollary IV.8. We consider an economy consisting of 30 banks, and show in Figure 1 their post-trade exposures, purchases, sales, and intermediation volumes in equilibrium. We first analyze post-trade exposures. We observe from the top left panel of the figure that the dashed and dotted curves hit the solid line at the same point, which means that the initial exposure needed to guarantee that risky banks have the same post-trade exposure does not depend on their default probabilities. This is a consequence of Theorem IV.6, and follows from the fact that the threshold α therein does not depend on the banks' default probabilities. Theorem IV.6 implies that the post-trade exposure of safe banks is higher than their average initial exposure, and this is visually confirmed in Figure 1. We next analyze protection purchases, and display them in the top right panel of Figure 1. The observed patterns are fully consistent with part 3 of Theorem IV.6: risky banks with low initial exposures do not buy protection if their post-trade exposures are sufficiently close, which is the case in this example for all banks with initial exposure less than or equal to 5.

[6] The precise condition for a risky bank n not to be an intermediary is that either it has initial exposure above α, or it has initial exposure below α and satisfies $\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y(\Gamma_i, p_i)}{1-p_i}$ for any risky bank i with $\tilde{\Omega}_i < \tilde{\Omega}_n$, where $\Gamma_i = \sum_{\ell \neq n: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell}$ is the sum of contracts sold by bank i, and $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ are the exposures of banks i and n, respectively, after trading with other banks but before trading between themselves.

Figure 1: **Post-trade exposure and intermediation volume in an example.** An economy consisting of 30 banks: for each initial exposure $1, 2, \ldots, 10$, we consider three banks, respectively with default probabilities $p = 0$, $p = 0.1$ and $p = 0.2$. We set the risk aversion parameter $\eta = 1$, the probability of the binary risk factor $q = 0.1$, and use a non-binding trade size limit. Top left panel: For large enough k, all safe banks and all risky banks with big initial exposures have the same post-trade exposure. The corresponding value 6.19 is higher than the average initial exposure of the safe banks, 5.5 ($= (1 + 2 + \cdots + 10)/10$). Risky banks with $p = 0.2$ (dotted curve) have a smaller post-trade exposure than risky banks with $p = 0.1$ (dashed curve). Top right panel: Risky banks with low initial exposures do not buy protection. The quantities of purchased protection by risky banks with high initial exposure do not depend on the banks' default probabilities (dashed and dotted curves are identical). Bottom left panel: Risky banks with high initial exposures do not sell protection. Bottom right panel: Intermediation is done mostly by safe banks. Among the risky banks, the only ones with a positive intermediation volume are those whose initial exposure equals 6. We verified that all other risky banks satisfy the condition in Footnote 6.

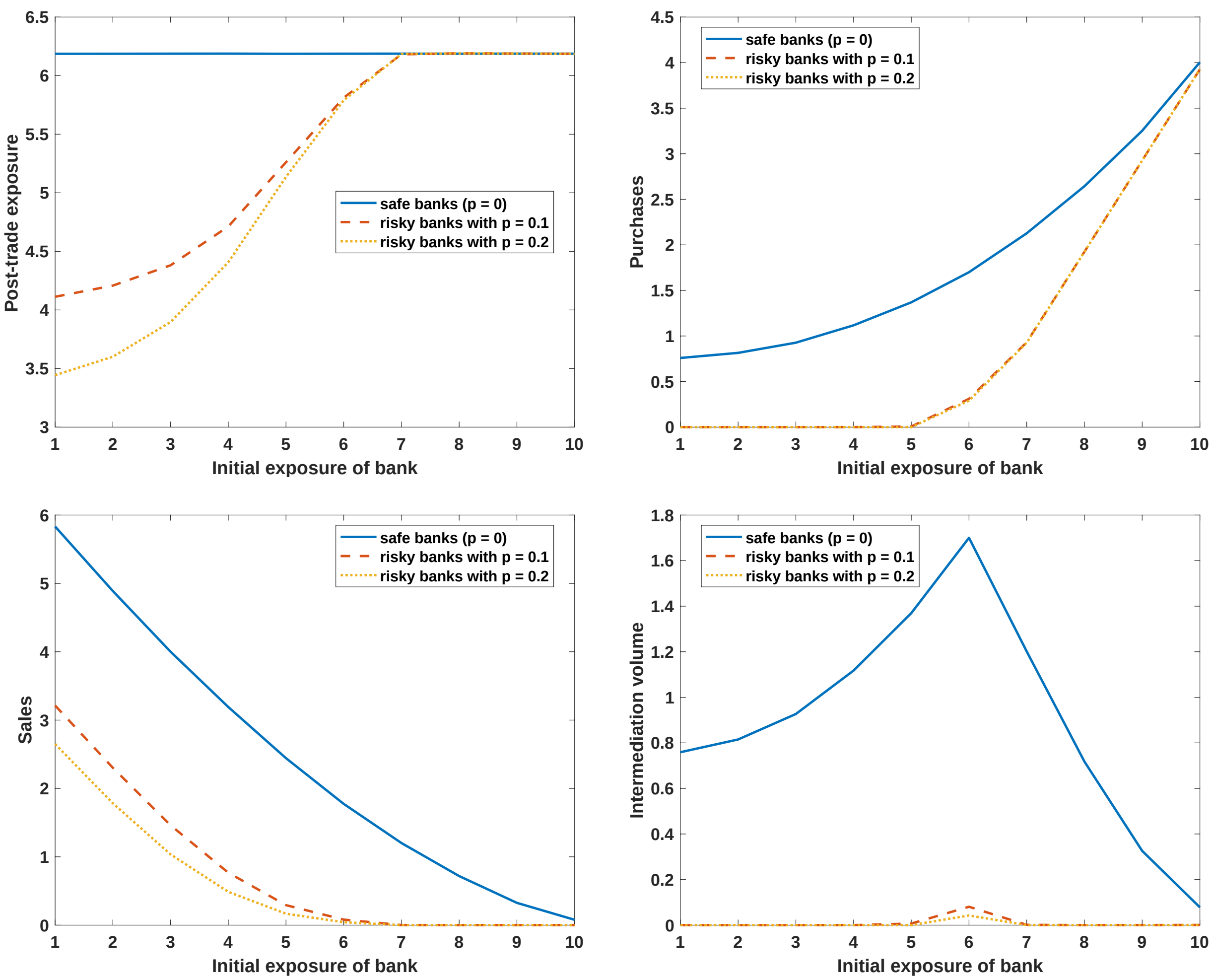
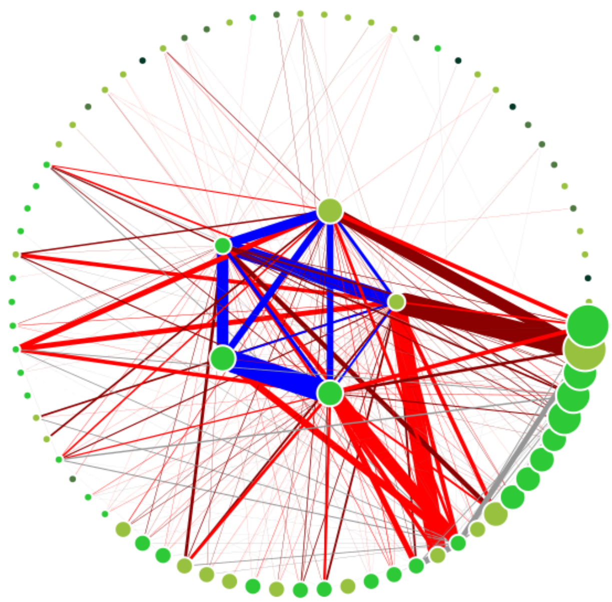


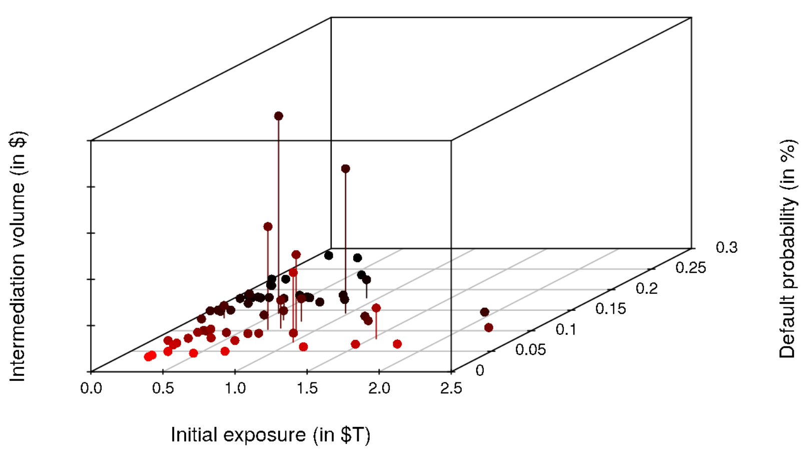
Figure 2: **Network of banks' bilateral CDS exposures.** Each node corresponds to a bank. The inner nodes are the 5 main intermediaries, while the remaining 76 banks are arranged as nodes on an outside circle. Both in the inner area and the outside circle, the nodes are ordered based on the initial exposures from their loan portfolios. These exposures correspond to the sizes of the nodes. The darker a node is, the higher is the default probability of the corresponding bank. The widths of the edges are proportional to the banks' bilateral net CDS volume. We use blue for the CDS volume between two main intermediaries; gray for CDS volume between two banks which are not main intermediaries; light red for CDS protection sold by a main intermediary to another bank; dark red for CDS protection purchased by a main intermediary from another bank.



In regards to protection sales, the graph in the bottom left panel supports the result in part 2 of Theorem IV.6: risky banks with high initial exposures do not sell any credit protection, in contrast to their safe counterparts. Finally, the bottom right panel indicates that almost all intermediation is done by safe banks, as stated in Corollary IV.8. We note that purchases, sales, and intermediation volumes of the safe banks are not uniquely determined and could be increased without changing the post-trade exposures and trading surplus, as explained after Theorem IV.4. Nonetheless, the corresponding quantities for the risky banks as well as the post-trade exposures of all banks are unique in equilibrium. Hence, all curves in the top left panel and the dotted and dashed curves in the other panels are unique.

Corollary IV.8 makes the assumption that there exist banks with zero default probability, which is of course violated in practice. Nonetheless, the prediction of Corollary IV.8 that banks with small default probabilities are the main intermediaries is supported by data from the bilateral CDS market, as shown in Figure 2. We describe the data set and the procedure followed to generate the plot of the intermediation volume in Appendix B. We define the main intermediaries to be those banks that provide at least 5 percent of the total intermediation volume in the market. Using this definition, we obtain 5 intermediaries among the 81 banks in the data set. The selection of these 5 banks is not sensitive to the chosen threshold on the intermediation volume. As it appears from Figure 2, most of the traded CDS volume is either between two of the main intermediaries or between a main intermediary and another bank. The volume of traded CDS contracts between the main intermediaries is high, but each main intermediary has large trading positions with only a few, and not all, other main intermediaries. There is high heterogeneity in the volume of traded CDS contracts for the banks that are not main intermediaries: some banks (primarily those with very small initial exposures) trade a very small volume of CDSs, while others are either large buyers or large sellers of CDSs. Figure 3 further highlights that the main intermediaries are the banks with medium initial exposures and low default probabilities, relative to all banks in the data set. The relation between intermediaries and initial exposures is consistent with a theoretical finding of Atkeson et al. (2015) (see Proposition 2 therein), where the economy consists only of safe banks and the trade size limit is binding.

Figure 3: **Main intermediaries.** Intermediation volume as a function of initial exposures, measured in trillion USD ($T), and default probabilities: each point denotes a bank. The higher the default probability of a bank, the darker the color of the corresponding point.



V Private versus Socially Optimal Default Probabilities

In the previous section, we discussed how banks share payoff risk among themselves. An alternative risk management strategy for a bank is to manage its default probability p_i. As banks are regulated financial institutions, there exists a given maximal value $\bar{p}_i$ that bank i's default probability can take. In order to become a more attractive trading counterparty in the OTC market, bank i can decrease its default probability to $p_i \in [0, \bar{p}_i]$ at a cost $C_i(p_i)$. Depending on its initial exposure, each bank i needs to decide before trading starts how much it is willing to pay to reduce its default probability, i.e., which value of p_i to choose. These decisions also take into consideration the subsequent trading transactions that banks will establish and that are uniquely specified in terms of bilateral prices and quantities (see Theorem IV.4 for details).

We assume that $C_i : [0, \bar{p}_i] \to [0, \infty)$ is a decreasing, convex, and continuous function. Let $p_i \in [0, \bar{p}_i]$ be the decision of bank i. Theorem IV.4 yields that for given $p_1, \dots, p_M$, there exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$. As we focus in this section on the choice of $p_1, \dots, p_M$, we write

$$x_i(p_1, \dots, p_M) := \omega_i + \sum_{n \neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_i) \tag{9}$$

to denote bank i's per-capita certainty equivalent (1) in a market equilibrium.

Lemma V.1. *The value of $x_i(p_1, \dots, p_M)$ is uniquely determined.*

Because each bank chooses individually its default probability, we are looking for a Nash equilibrium.

Definition V.2. *A choice of $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$ is an equilibrium if*

$$x_i(p_1, \dots, p_M) - C_i(p_i) \geq x_i\big(p_1, \dots, p_{i-1}, \tilde{p}_i, p_{i+1}, \dots, p_M\big) - C_i(\tilde{p}_i)$$

for all i and $\tilde{p}_i \in [0, \bar{p}_i]$.

Proposition V.3. *If the cost function C_i is such that*

$$\arg\max_{p_i \in [0, \bar{p}_i]} \big(x_i(p_1, \dots, p_M) - C_i(p_i)\big) \tag{10}$$

is a convex set for each i, then there exists an equilibrium $p_1, \dots, p_M$.

The assumption that the set specified by (10) is convex set means that if $\hat{p}_i$ and p_i^* are maximizers of $x_i(p_1, \dots, p_M) - C_i(p_i)$, then so is any convex combination of $\hat{p}_i$ and p_i^*. Note that, in particular, this assumption is satisfied in the typical case when there is a unique maximizer. The convexity assumption also covers the case where the cost function $C_i(p_i)$ and the certainty equivalent $x_i(p_1, \dots, p_M)$ have the same slope (up to a constant) for values of default probabilities around a maximum.

We consider a social planner who chooses the banks' default probabilities $p_1, \dots, p_M$ and the quantities of traded contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ so to maximize the banks' aggregate certainty equivalent minus the risk management costs. The planner maximizes the objective function

$$\sum_{i=1}^{M} x_i(p_1, \dots, p_M) - \sum_{i=1}^{M} C_i(p_i) \tag{11}$$

over $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$ and $(\gamma_{i,n})_{i,n=1,\dots,M}$ subject to $\gamma_{i,n} = -\gamma_{n,i}$ and $-k \leq \gamma_{i,n} \leq k$, where

- $\sum_{i=1}^{M} x_i(p_1, \dots, p_M)$ is the aggregate certainty equivalent of the banks with default probabilities $p_1, \dots, p_M$.
- $\sum_{i=1}^{M} C_i(p_i)$ is the sum of the costs incurred to reduce the default probabilities to the levels $p_1, \dots, p_M$.

It follows from $\gamma_{i,n} = -\gamma_{n,i}$ and $R_{i,n} = R_{n,i}$ that $\sum_{i=1}^M x_i(p_1, \ldots, p_M) = \sum_{i=1}^M \omega_i - \sum_{i=1}^M \Gamma^i(\gamma_i)$. Therefore, the social planner's optimization problem (11) is equivalent to minimize

$$\sum_{i=1}^M \Gamma^i(\gamma_i) + \sum_{i=1}^M C_i(p_i)$$

over the same optimization variables $p_1 \in [0, \bar{p}_1], \ldots, p_M \in [0, \bar{p}_M]$ and $(\gamma_{i,n})_{i,n=1,\ldots,M}$. Hence, the social planner minimizes the aggregate cost of post-trade payoff risk and of risk management through reduction of default probabilities.

Proposition V.4. *The social planner's optimization problem has a solution.*

We will next analyze when and how the individually optimal banks' risk management choices differ from the social optimum. We compare for each bank i the marginal social value (MSV_i) and marginal private value (MPV_i) of risk management, which are given by

$$MSV_i(p) = -\sum_{n=1}^M \frac{\partial x_n}{\partial p_i}(p) + C_i'(p_i) = \frac{\partial \Gamma^i}{\partial p_i}(\gamma_i, p) + \sum_{n \neq i} \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n, p) + C_i'(p_i),$$
$$MPV_i(p) = -\frac{\partial x_i}{\partial p_i}(p) + C_i'(p_i) = \frac{\partial \Gamma^i}{\partial p_i}(\gamma_i, p) - \sum_{n \neq i} \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i, \gamma_n, p) + C_i'(p_i).$$

Note that γ is fixed in the computation of the partial derivatives. This is justified by the envelop theorem in the form of Corollary 4 in Milgrom and Segal (2002), which we can directly apply because, by our Theorem IV.4, γ is unique for risky banks. In words, MSV_i is the change in the social planner's value given by (11) when bank i reduces its default probability by an infinitesimal amount.[7] To compute MPV_i, we take the negative of the p_i-partial derivative of bank i's certainty equivalent (9) minus its risk management costs. When MSV_i and MPV_i differ, an *externality* in the amount of $MSV_i(p) - MPV_i(p)$ arises.

Theorem V.5. *The externality imposed by bank i on the system equals*

$$MSV_i(p) - MPV_i(p) = \frac{\partial S_i}{\partial p_i}(p) + k(1-\nu)\frac{\partial T_i}{\partial p_i}(p), \tag{12}$$

where

$$S_i(p) := \sum_{n \neq i} \left(\gamma_{i,n} \Gamma^n_{y_i}(\gamma_n, p) + \Gamma^n(\gamma_n, p) \right), \quad T_i(p) := \sum_{n \neq i} \left(\Gamma^i_{y_n}(\gamma_i, p) - \Gamma^n_{y_i}(\gamma_n, p) \right).$$

For small enough p_i and large enough q, we have $\frac{\partial S_i}{\partial p_i}(p) < 0$ and $\frac{\partial T_i}{\partial p_i}(p) \geq 0$ with strict inequality when the trade size limit is binding for at least one bilateral trading relationship.

MPV_i and MSV_i have two terms in common, namely, the p_i-partial derivatives of the bank i's valuation Γ^i and risk management costs C_i. These terms cancel out when computing

[7] We take the negative of the p_i-partial derivative of (11) because increased risk management results in a reduction of p_i.

the difference between MPV_i and MSV_i. However, MPV_i and MSV_i account differently for the effect of a change in p_i on other banks. Such a change contributes directly to MSV_i through the valuations of the other banks, whereas it is reflected only indirectly in MPV_i through the revenues that bank i receives from other banks. Bank i's risk management impacts other banks' revenues in a different way than their valuations, leading to an externality. There are two primary causes of such an externality, which are mathematically captured by the two terms on the right-hand side of (12):

1. The first cause is that the revenues reflected in MPV_i depend on (the derivative of) the *marginal* valuations of the buyer banks, whereas MSV_i accounts for (the derivative of) the banks' *total* valuations. Marginal valuations determine the individual traders' decisions, while the total valuation for a bank depends on the cumulative valuation of its continuum of traders. The marginal and total valuations depend differently on p_i, and their difference is captured by the term $\frac{\partial S_i}{\partial p_i}(p)$, which we call the *coalitional externality*. It arises because traders are organized in large coalitions, namely banks, and changes in a bank's default probability affect the fulfillment of all contracts that traders of a specific bank purchased from the traders of this counterparty bank.

2. The second cause is that the revenues of a protection selling bank generally differ from the marginal valuations of a buyer bank, because the sellers may appropriate part of the trade surplus. They do not coincide if the buyer does not have full bargaining power ($\nu \neq 1$) and there is a trade surplus, which occurs when the trade size limit is binding. This difference is reflected in the term $k(1-\nu)\frac{\partial T_i}{\partial p_i}(p)$, which we call *frictional externality*. It stems from the friction due to trade size limits, and it vanishes when the trade size limit is big enough because then $T_i = 0$.

It is worth noting that two types of externalities arise in our model, while Atkeson et al. (2015) only observe the frictional externality in their analysis of banks' entry decisions to an OTC market. In both models, inefficiencies are due to imperfect sharing of payoff risk in the OTC market. In Atkeson et al. (2015), this arises because the friction from the trade size limit impairs the risk sharing capacity of the market. In our model, in addition to the trade size limit friction, another important reason for imperfect sharing of payoff risk is the presence of counterparty risk. Therefore, we have an additional externality, which does not vanish if the trade size limit is large and not binding. The presence of this externality is also related to the finite number of large banks in our framework, unlike Atkeson et al. (2015) who study a continuum of small banks.

We next discuss the sensitivities of S_i and T_i on p_i. Typically, we would expect a non-negative coalitional externality $\frac{\partial S_i}{\partial p_i}(p) \geq 0$, i.e., a change in p_i would have a smaller impact on bank i's MPV_i than on the other banks' total valuations which contribute to the term MSV_i. This is because MPV_i depends on the other banks only indirectly through the revenues that bank i receives from them. Interestingly, for small enough p_i and large enough q, the theorem states that the coalitional externality is negative. This may be explained in the following way. Because traders are organized in coalitions, changes in a counterparty's default probability affects all traders of the protection purchasing bank. However, the degree to which they are affected depends on how much the traders have purchased collectively: the bigger the amount purchased from a specific bank, the higher is the counterparty concentration risk, and thus the marginal benefit from additional protection purchases will diminish. Correspondingly, a change in the counterparty's default probability affect the valuations differently depending on how much protection has already been purchased, causing a wedge between marginal and total valuations. We show in the proof of Theorem V.5 that, for small enough p_i and large enough q, the marginal valuation is more sensitive to changes in a bank's default probability than the total valuation. This leads to a negative coalitional externality because marginal and total valuations affect MPV_i and MSV_i, respectively. By contrast, the frictional externality $k(1-\nu)\frac{\partial T_i}{\partial p_i}(p)$ is positive, because T_i corresponds to the negative of the trade surplus, which gets smaller when p_i increases.

We can see from (12) that the externality imposed by bank i on the system depends linearly on the seller's bargaining power ν. As ν increases, the size of the externality $MSV_i(p) - MPV_i(p)$ decreases, till becoming negative for $\nu = 1$. If the externality imposed by bank i is negative, it means that the bank chooses a default probability below the socially optimal level.

VI Conclusion

In this paper, we study the impact of counterparty risk on trading decisions and the resulting OTC market structure. We also analyze the incentives behind banks' default risk management decisions, and their misalignment with the socially optimal outcome. Our model predicts that banks share their payoff risk less if the counterparty risk in the market is higher. Negative externalities may arise if protection selling banks are not fairly compensated for their contribution in reducing the system-wide risk exposure. Our results show that banks may reduce their default probabilities below what is socially optimal to benefit from higher fees. These decisions depend on the banks' initial exposures to an aggregate risk factor and on the relative bargaining power of the banks' traders. Intermediaries contribute to social welfare by alleviating the frictions caused by the trade size limit, and more importantly, by increasing counterparty risk diversification.

Our framework can be extended along several directions. A first extension is to construct a model that can capture the dynamic formation of interbank trading relations, taking counterparty risk into consideration. Secondly, our framework can be generalized to include a role for the real economy. In such a model extension, banks might have obligations to the private sector and, additionally, fees charged as the result of a CDS trade may be used to finance the real economy. A third extension is to compare trading decisions when market participants have the choice between bilateral OTC trading, which exposes them to counterparty risk, and centralized trading. In the latter case, the clearinghouse insulates banks from counterparty risk, but they would be required to additionally pay clearing fees. A recent work by Dugast, Üslü, and Weill (2019) studies the welfare implications of central clearing. Their main focus is on the trading capacity and costs of joining the centralized clearing platform. Our proposed work would complement theirs by accounting for the netting and counterparty risk reduction benefits of a clearinghouse. It would also be interesting to analyze counterparty risk in a Walrasian market, where all traders meet together. In such a model, the search friction disappears but the counterparty risk friction remains, so that transaction prices are still negotiated bilaterally.

A Results and their Proofs

This section contains the proofs of our results, which consider the more general case of arbitrary sizes s_i for the banks. When the formulation of the statement is different from that in the case $s_i = 1$, we restate the result.

A.A Proof of Lemma IV.1

Using $P[D = 1] = q$, we compute

$$\begin{aligned}\Gamma^i(y_1, \dots, y_M) &= \frac{1}{\eta} \log E\Big[\exp\Big(\eta D\Big(\omega_i + \sum_{n\neq i} y_n \big(\mathbb{1}_{A_i^\complement} \mathbb{1}_{y_n>0} + \mathbb{1}_{A_n^\complement} \mathbb{1}_{y_n<0} \big) \Big) \Big) \Big] \\ &= \frac{1}{\eta} \log \Big(1 - q + qE\Big[\exp\Big(\eta\omega_i + \eta \sum_{n\neq i} y_n \big(\mathbb{1}_{A_i^\complement|D=1} \mathbb{1}_{y_n>0} + \mathbb{1}_{A_n^\complement|D=1} \mathbb{1}_{y_n<0} \big) \Big) \Big] \Big) \\ &= \frac{1}{\eta} \log \Big(1 - q + q\mathrm{e}^{\eta\omega_i} E\Big[\exp\Big(\eta \mathbb{1}_{A_i^\complement|D=1} \sum_{n\neq i:\, y_n>0} y_n \Big) \Big] \prod_{n\neq i:\, y_n<0} E\big[\exp\big(\eta \mathbb{1}_{A_n^\complement|D=1} y_n \big) \big] \Big).\end{aligned}$$

Using that $p_i = P[A_i|D = 1]$, we obtain

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log \left(1 - q + q\mathrm{e}^{\eta\omega_i} \Big((1 - p_i)\mathrm{e}^{\eta \sum_{n:\, y_n>0} y_n} + p_i \Big) \prod_{n\neq i:\, y_n<0} \big((1 - p_n)\mathrm{e}^{\eta y_n} + p_n \big) \right),$$

which can be brought into the form $\Gamma^i(y_1, \dots, y_M)$ written in the statement of Lemma IV.1.

To show the additional properties of $\Gamma^i(y_1, \dots, y_M)$, we first note that the function Ξ given by

$$\Xi(y) = \frac{1}{\eta} \log \big(1 - q + q\mathrm{e}^{\eta y} \big)$$

is strictly increasing and strictly convex. Indeed, we can calculate

$$\Xi'(y) = \frac{q\mathrm{e}^{\eta y}}{1 - q + q\mathrm{e}^{\eta y}} > 0, \qquad \Xi''(y) = \frac{(1 - q)q\eta\mathrm{e}^{\eta y}}{(1 - q + q\mathrm{e}^{\eta y})^2} > 0.$$

Next, we consider

$$f(y, p) = \frac{1}{\eta} \log \big((1 - p)\mathrm{e}^{\eta y} + p \big)$$

for $p > 0$ and calculate

$$\begin{aligned}(13) \qquad f_y(y, p) &= \frac{(1 - p)\mathrm{e}^{\eta y}}{(1 - p)\mathrm{e}^{\eta y} + p} > 0, \\ f_{yy}(y, p) &= \eta \frac{((1 - p)\mathrm{e}^{\eta y} + p)((1 - p)\mathrm{e}^{\eta y}) - ((1 - p)\mathrm{e}^{\eta y})^2}{((1 - p)\mathrm{e}^{\eta y} + p)^2} = \eta \frac{p(1 - p)\mathrm{e}^{\eta y}}{((1 - p)\mathrm{e}^{\eta y} + p)^2} > 0.\end{aligned}$$

These inequalities show that the function $y \mapsto f(y, p)$ is strictly increasing and strictly convex for $p > 0$. Because $f(y, p)$ either equals y (if $p = 0$) or is strictly increasing and strictly convex (if $p > 0$), we see that $\Gamma^i(y_1, \dots, y_M)$ is strictly increasing, and the statements on convexity of $\Gamma^i(y_1, \dots, y_M)$ now follow from the fact that convexity is maintained under sums and compositions with a convex, nondecreasing function.

Finally, to prove (3), let $y_1 < y_2$, $y_3 \in \left(0, \frac{y_2 - y_1}{2}\right]$ and $p_1 \geq p_2$. We first note that (3) is equivalent to

$$\big((1-p_1)\mathrm{e}^{\eta y_1} + p_1\big)\big((1-p_2)\mathrm{e}^{\eta y_2} + p_2\big) > \big((1-p_1)\mathrm{e}^{\eta(y_1+y_3)} + p_1\big)\big((1-p_2)\mathrm{e}^{\eta(y_2-y_3)} + p_2\big),$$

which can be further simplified to

$$(1-p_1)p_2\mathrm{e}^{\eta y_1} + (1-p_2)p_1\mathrm{e}^{\eta y_2} > (1-p_1)p_2\mathrm{e}^{\eta(y_1+y_3)} + (1-p_2)p_1\mathrm{e}^{\eta(y_2-y_3)}.$$

This inequality follows from

$$a\mathrm{e}^{x_1} + b\mathrm{e}^{x_2} > a\mathrm{e}^{x_1+x_3} + b\mathrm{e}^{x_2-x_3} \tag{14}$$

for all $a \leq b$, $x_1 < x_2$ and $x_3 \in \left(0, \frac{x_2-x_1}{2}\right]$ by choosing

$$a = (1-p_1)p_2, \quad b = (1-p_2)p_1, \quad x_1 = \eta y_1, \quad x_2 = \eta y_2, \quad x_3 = \eta y_3,$$

where we note that $p_1 \geq p_2$, $y_1 < y_2$, and $y_3 \in \left(0, \frac{y_2-y_1}{2}\right]$ imply $a \leq b$, $x_1 < x_2$, and $x_3 \in \left(0, \frac{x_2-x_1}{2}\right]$. The inequality (14) can be seen from the convexity of the exponential function or checked directly by calculating the partial derivative

$$\frac{\partial}{\partial z}(a\mathrm{e}^{x_1+z} + b\mathrm{e}^{x_2-z}) = a\mathrm{e}^{x_1+z} - b\mathrm{e}^{x_2-z} \leq b\mathrm{e}^{x_1+z} - b\mathrm{e}^{x_2-z} < 0$$

for all $z \in \left[0, \frac{x_2-x_1}{2}\right)$. □

A.B Results of Section IV.B and their Proofs

Theorem A.1 (Theorem IV.3)**.** *Feasible contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ are a market equilibrium if and only if they solve the optimization problem*

$$\text{minimize} \sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s) \quad \text{over } \gamma \text{ subject to } \gamma_{i,n} = -\gamma_{n,i} \text{ and } -k \leq \gamma_{i,n} \leq k, \tag{15}$$

where $\gamma_i s := (\gamma_{i,1}s_1, \dots, \gamma_{i,M}s_M)$.

Proof. The Lagrangian function corresponding to (15) is

$$\sum_{i=1}^{M} s_i\Gamma^i(\gamma_i s) - \sum_{i,n=1}^{M} s_i s_n \alpha_{i,n}(\gamma_{i,n} + \gamma_{n,i}) - \sum_{i,n=1}^{M} s_i s_n \underline{\beta}_{i,n}(k - \gamma_{i,n}) - \sum_{i,n=1}^{M} s_i s_n \overline{\beta}_{i,n}(k + \gamma_{i,n}).$$

The optimality conditions are

$$\begin{aligned} &\Gamma^i_{y_n}(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} - \underline{\beta}_{i,n} + \overline{\beta}_{i,n}, \qquad \underline{\beta}_{i,n} \geq 0, \qquad \overline{\beta}_{i,n} \geq 0, \\ &\underline{\beta}_{i,n}(k - \gamma_{i,n}) = 0, \qquad \overline{\beta}_{i,n}(k + \gamma_{i,n}) = 0. \end{aligned} \tag{16}$$

All of them are satisfied for

$$\underline{\beta}_{n,i} = \overline{\beta}_{i,n} = \frac{1}{2} \max\left\{\Gamma^i_{y_n}(\gamma_i s) - \Gamma^n_{y_i}(\gamma_n s), 0\right\}, \quad \alpha_{i,n} + \alpha_{n,i} = \frac{1}{2}\left(\Gamma^i_{y_n}(\gamma_i s) + \Gamma^n_{y_i}(\gamma_n s)\right)$$

if γ satisfies (4) and $\gamma_{i,n} = -\gamma_{n,i}$. This means that if γ is a market equilibrium, it is a solution to (15). Conversely, if γ is a solution to (15), then (16) implies

$$\Gamma^i_{y_n}(\gamma_i s)(k^2 - \gamma^2_{i,n}) = (\alpha_{i,n} + \alpha_{n,i})(k^2 - \gamma^2_{i,n}) = (\alpha_{n,i} + \alpha_{i,n})(k^2 - \gamma^2_{n,i}) = \Gamma^n_{y_i}(\gamma_n s)(k^2 - \gamma^2_{n,i}).$$

This equation shows that if $\gamma_{i,n} \neq \pm k$, we need $\Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s)$. In turn, $\Gamma^i_{y_n}(\gamma_i s) \neq \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = \pm k$. Consider the case $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$ and assume $\gamma_{i,n} = -k$, then $\gamma_{n,i} = k$; it follows from (16) that $\underline{\beta}_{i,n} = 0$, $\overline{\beta}_{n,i} = 0$ and

$$\Gamma^i_{y_n}(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} + \overline{\beta}_{i,n} \geq \alpha_{i,n} + \alpha_{n,i} \geq \alpha_{n,i} + \alpha_{i,n} - \underline{\beta}_{i,n} = \Gamma^n_{y_i}(\gamma_n s),$$

which is a contradiction to $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$. Therefore, $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = k$. By symmetry, $\Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = -k$. This shows that a solution to (15) satisfies (4) and thus is a market equilibrium. □

Theorem A.2 (Theorem IV.4)**.** *There exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\ldots,M}$. The $\gamma_{i,n}$ are unique for $p_n > 0$ and $\gamma_{i,n} < 0$, or $p_i > 0$ and $\gamma_{i,n} > 0$. For every i, the value is the same for $\sum \gamma_{i,n} s_n$ where the sum is over n such that $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. In particular, $\Gamma(\gamma_n s)$ are uniquely determined for a market equilibrium $(\gamma_{i,n})_{i,n=1,\ldots,M}$.*

Proof. We prove first the existence of a market equilibrium. To this end, we will apply Kakutani's fixed-point theorem (see, for example, Corollary 15.3 in Border (1985)). We fix k, set $S = [-k, k]^{M(M-1)/2}$, and define a mapping $\Phi : S \to 2^S$ as follows, where 2^S denotes the power set of S, i.e., the set of all subsets of S. Each element in S corresponds to the lower triangular matrix of $(\gamma_{i,n})_{i,n=1,\ldots,M}$, where we set the diagonal elements γ_{ii} equal to zero and the upper diagonal elements are defined by $\gamma_{i,n} = -\gamma_{n,i}$. Let $\Phi(\gamma)$ consist of all $(\tilde{\gamma}_{i,n})_{i,n=1,\ldots,M}$ that satisfy $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$, $-k \leq \tilde{\gamma}_{i,n} \leq k$, with the further restriction in the following three cases

$$\tilde{\gamma}_{i,n} \begin{cases} = k & \text{if } \Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s), \\ = \gamma_{i,n} & \text{if } \Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s), \\ = 0 & \text{if } \Gamma^i_{y_n}(\gamma_i) \text{ or } \Gamma^n_{y_i}(\gamma_n) \text{ do not exist}, \\ = -k & \text{if } \Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s). \end{cases}$$

Note that these "if" conditions depend on γ and not on $\tilde{\gamma}$. We can see that $\Phi(\gamma)$ is nonempty, compact and convex. To show that Φ has a closed graph, consider a sequence $\left(\gamma^{(m)}, \tilde{\gamma}^{(m)}\right)$ converging to $(\gamma, \tilde{\gamma})$ with $\tilde{\gamma}^{(m)} \in \Phi\left(\gamma^{(m)}\right)$ for all m. Because $\tilde{\gamma}^{(m)} \to \tilde{\gamma}$ and $\tilde{\gamma}^{(m)} \in \Phi\left(\gamma^{(m)}\right)$, we have $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$ and $-k \leq \tilde{\gamma}_{i,n} \leq k$. Moreover, if $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$, we have $\Gamma^i_{y_n}\left(\gamma^{(m)}_i s\right) < \Gamma^n_{y_i}\left(\gamma^{(m)}_n s\right)$ for all m big enough, as $\gamma^{(m)} \to \gamma$. This yields $\tilde{\gamma}^{(m)}_{i,n} = k$ for all m big enough; hence, $\tilde{\gamma}_{i,n} = k$. Similarly, $\Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s)$ implies $\tilde{\gamma}_{i,n} = -k$. The condition is also satisfied for the last case $\Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s)$, as we have already shown $-k \leq \tilde{\gamma}_{i,n} \leq k$.

Therefore, there exists γ with $\Phi(\gamma) = \gamma$ by Kakutani's fixed-point theorem; hence, there is a market equilibrium.

To prove uniqueness, we first apply Theorem IV.3, which says that finding a market equilibrium is equivalent to solving (15). We then write the objective function in (15) as

$$\sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s) = \sum_{i=1}^{M} s_i \Xi\left(\omega_i + f\left(\sum_{n:\gamma_{i,n}s_n \geq 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n}s_n<0} f(\gamma_{i,n}s_n, p_n)\right),$$

where the function Ξ is given in Lemma IV.1. The uniqueness statements now follow from the statements on convexity in Lemma IV.1. □

A.C Results of Section IV.C and their Proofs

Proposition A.3 (Proposition IV.5)**.** *Assume that at least one of the following conditions holds:*

(a) $\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell \geq s_i \max_\ell \gamma_{i,\ell}$, *or*

(b) $\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell \geq s_j \max_\ell \gamma_{j,\ell}$.

We then have the following relations between initial and post-trade exposures:

1. *If* $\omega_i \geq \omega_j$, $p_i \leq p_j$, *and* $s_i \leq s_j$, *then* $\Omega_i \geq \Omega_j$.
2. *If* $\omega_i > \omega_j$, $p_i \geq p_j$, *and* $s_i \geq s_j$, *then* $\omega_i - \omega_j > \Omega_i - \Omega_j$.

Proof. Under conditional independence and for general sizes, the post-trade exposure is given by

$$\Omega_i = \omega_i + f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n}s_n, p_n).$$

We split the proof in several steps, starting with some preparation.

Claim 1a. For two banks i and j, we have

$$\Omega_j > \Omega_i \Longrightarrow \gamma_{j,i} \leq 0. \tag{C1a}$$

Proof of Claim 1a. From Lemma IV.1, it follows that

$$\Gamma^j_{y_i}(\gamma_j s) = \begin{cases} \Xi'(\Omega_j)\eta f_y\big(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n}s_n, p_j\big) & \text{if } \gamma_{j,i} > 0, \\ \Xi'(\Omega_j)\eta f_y(\gamma_{j,i}s_i, p_i) & \text{if } \gamma_{j,i} < 0, \end{cases}$$

with an analogous expression for $\Gamma^i_{y_j}(\gamma_i s)$. If $\gamma_{j,i} > 0$ (and thus $\gamma_{i,j} < 0$), we obtain

$$\begin{aligned} \Gamma^j_{y_i}(\gamma_j s) &= \Xi'(\Omega_j)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n}s_n, p_j\right) \\ &> \Xi'(\Omega_i)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n}s_n, p_j\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \Xi'(\Omega_i)\eta f_y(\gamma_{i,j}s_j, p_j)\\
&= \Gamma^i_{y_j}(\gamma_i s)
\end{aligned}
$$

by strict convexity of Ξ and convexity of $f(., p_j)$ from Lemma IV.1. However, this implies $\gamma_{j,i} = -k$ by (4) in contradiction to the assumption $\gamma_{j,i} > 0$.

Claim 1b. For two banks i and j, we have

$$
\Omega_j > \Omega_i \Longrightarrow \gamma_{j,n} \leq \gamma_{i,n} \text{ or } \gamma_{j,n} = -k \text{ for all } n \text{ with } \Omega_n < \Omega_j. \tag{C1b}
$$

Proof of Claim 1b. We distinguish the following three cases:

- If $\Omega_n \in (\Omega_i, \Omega_j)$, we have $\gamma_{j,n} \leq 0$ and $\gamma_{i,n} \geq 0$ by (C1a) so that $\gamma_{j,n} \leq \gamma_{i,n}$ holds.
- If $\Omega_n < \Omega_i$, we have $\gamma_{j,n} \leq 0$ and $\gamma_{i,n} \leq 0$ by (C1a); thus,

  $$
  \Gamma^j_{y_n}(\gamma_j s) = \Xi'(\Omega_j)\eta f_y(\gamma_{j,n}s_n, p_n), \tag{17}
  $$
  $$
  \Gamma^i_{y_n}(\gamma_i s) = \Xi'(\Omega_i)\eta f_y(\gamma_{i,n}s_n, p_n), \tag{18}
  $$
  $$
  \Gamma^n_{y_j}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y\left(\sum_{\ell:\gamma_{n,\ell}\geq 0} \gamma_{n,\ell}s_\ell, p_n\right) = \Gamma^n_{y_i}(\gamma_n s). \tag{19}
  $$

  Assume that $\gamma_{j,n} \neq -k$, which implies

  $$
  \Gamma^j_{y_n}(\gamma_j s) = \Gamma^n_{y_j}(\gamma_n s) = \Gamma^n_{y_i}(\gamma_n s) \leq \Gamma^i_{y_n}(\gamma_i s)
  $$

  by (4) and (19); thus,

  $$
  1 < \frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n}s_n, p_n)}{f_y(\gamma_{j,n}s_n, p_n)}
  $$

  by (17) and (18). This is only possible if $\gamma_{j,n} < \gamma_{i,n}$.
- If $\Omega_n = \Omega_i$, we argue as in the first item if $\gamma_{i,n} \geq 0$, or as in the second item if $\gamma_{i,n} < 0$.

Note that (C1b) holds regardless of the default probabilities of banks i and j. This is because we are considering banks n with smaller post-trade exposures; thus, banks that are sellers of protection by (C1a) so that the same counterparty risk p_n applies to trades with i and j.

Claim 1c. For two banks i and j, we have

$$
\Omega_j > \Omega_i \Longrightarrow \frac{f_y\left(\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}s_\ell, p_j\right)}{f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right)} < \frac{f_y(\gamma_{n,j}s_j, p_j)}{f_y(\gamma_{n,i}s_i, p_i)} \text{ or } \gamma_{n,i} = -k \text{ for all } n \text{ with } \Omega_n > \Omega_j. \tag{C1c}
$$

Proof of Claim 1c. $\Omega_n > \Omega_j$ implies $\gamma_{j,n} \geq 0$ by (C1a), and thus $\Gamma^j_{y_n}(\gamma_j s) \leq \Gamma^n_{y_j}(\gamma_n s)$. If $\gamma_{n,i} \neq -k$, it follows that $\Gamma^i_{y_n}(\gamma_i s) \geq \Gamma^n_{y_i}(\gamma_n s)$; hence,

$$
\Gamma^n_{y_j}(\gamma_n s) \geq \Gamma^j_{y_n}(\gamma_j s) = \Xi'(\Omega_j)\eta f_y\left(\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}s_\ell, p_j\right) > \Xi'(\Omega_i)\eta f_y\left(\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}s_\ell, p_j\right)
$$

$$= \Gamma^i_{y_n}(\gamma_i s) \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)} \geq \Gamma^n_{y_i}(\gamma_n s) \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)},$$

which shows (C1c), as $\Gamma^n_{y_i}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i} s_i, p_i)$ and $\Gamma^n_{y_j}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,j} s_j, p_j)$.

Claim 1d. For three banks i, j, and n, we have

$$\text{(C1d)} \qquad \Omega_i < \Omega_j = \Omega_n \Longrightarrow \gamma_{j,n} \leq \gamma_{i,n} \text{ or (C1c) holds.}$$

Proof of Claim 1d. If $\gamma_{j,n} \leq 0$, we obtain $\gamma_{j,n} \leq \gamma_{i,n}$, as $\gamma_{i,n} \geq 0$ by (C1a). If $\gamma_{j,n} > 0$, we can argue as (C1c).

We can summarize (C1a)–(C1d) as

$$\text{(C1)} \qquad \Omega_j > \Omega_i \Longrightarrow \begin{cases} \gamma_{j,n} \leq \gamma_{i,n} & \text{for all } \gamma_{j,n} \leq 0, \\ \text{(C1c) holds} & \text{for all } \gamma_{j,n} > 0. \end{cases}$$

Claim 2. For two banks i and j, we have

$$\text{(C2)} \quad \omega_i \geq \omega_j,\ p_j \geq p_i,\ s_j \geq s_i, \text{ and (a), (b) or (c) of the proposition holds} \Longrightarrow \Omega_i \geq \Omega_j.$$

Proof of Claim 2. We prove the claim by contradiction and assume that $\Omega_i < \Omega_j$. This implies $\gamma_{j,n} \leq \gamma_{i,n}$ for all $\gamma_{j,n} \leq 0$ by (C1); hence,

$$\begin{aligned} f\Bigg(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\Bigg) &= \Omega_j - \omega_j - \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n} s_n, p_n) \\ &> \Omega_i - \omega_i - \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n} s_n, p_n) \\ &= f\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\Bigg) \\ &\geq f\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_j\Bigg), \end{aligned}$$

using (8), $p_j \geq p_i$, and that $f(y,p)$ is decreasing in p for $y \geq 0$ because, using definition (2),

$$\text{(20)} \qquad f_p(y,p) = \frac{\partial}{\partial p}\frac{1}{\eta}\log\big((1-p)\mathrm{e}^{\eta y} + p\big) = \frac{-\mathrm{e}^{\eta y} + 1}{\eta((1-p)\mathrm{e}^{\eta y} + p)} < 0 \quad \text{for } y \geq 0.$$

This yields $\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell$, as $y \mapsto f(y, p_j)$ is strictly increasing by Lemma IV.1. This implies that there exists n with $\gamma_{j,n} > \gamma_{i,n} \geq 0$; thus,

$$\text{(21)} \qquad \gamma_{n,j} < \gamma_{n,i} \leq 0 \quad \text{and} \quad \gamma_{n,j} s_j < \gamma_{n,i} s_i$$

because $s_j \geq s_i$ by assumption. Moreover, $\gamma_{j,n} > 0$ implies $\Omega_n \geq \Omega_j$ by (C1a). On the other hand, $\Omega_i < \Omega_j$ implies by (C1c) and (C1d) that $\gamma_{n,i} = -k$ (which stands in contradiction to (21) because $\gamma_{n,j} \geq -k$) or $\gamma_{j,n} \leq \gamma_{i,n}$ (also a contradiction to (21)) or

$$\text{(22)} \qquad \frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)} > \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)}.$$

We will show that (22) contradicts

$$p_j \geq p_i, \quad \sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell \quad \text{and} \quad \gamma_{n,j} s_j < \gamma_{n,i} s_i \tag{23}$$

if one of the conditions (a)–(c) of the proposition holds.

As an auxiliary step, we next analyze the function $p \mapsto \frac{f_y(y_1,p)}{f_y(y_2,p)}$ and show that

$$\frac{\partial}{\partial p} \frac{f_y(y_1,p)}{f_y(y_2,p)} \geq 0 \quad \text{for all } p \in [0,1) \text{ and } y_1 \geq -y_2 \geq 0. \tag{24}$$

To show this, first note that if $p = 0$, then $f_y(y_1,p) = f_y(y_2,p) = 1$ so that $\frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} = 0$. Now assume that $p > 0$. We use (13) and

$$\begin{aligned} f_{yp}(y,p) &= \frac{\partial}{\partial p} \frac{(1-p)\mathrm{e}^{\eta y}}{(1-p)\mathrm{e}^{\eta y} + p} = \frac{((1-p)\mathrm{e}^{\eta y} + p)(-\mathrm{e}^{\eta y}) - ((1-p)\mathrm{e}^{\eta y})(-\mathrm{e}^{\eta y} + 1)}{((1-p)\mathrm{e}^{\eta y} + p)^2} \\ &= \frac{-\mathrm{e}^{\eta y}}{((1-p)\mathrm{e}^{\eta y} + p)^2} \end{aligned}$$

to deduce that

$$\begin{aligned} \frac{\partial}{\partial p} \frac{f_y(y_1,p)}{f_y(y_2,p)} &= \frac{f_y(y_2,p) f_{yp}(y_1,p) - f_{yp}(y_2,p) f_y(y_1,p)}{(f_y(y_2,p))^2} \\ &= \frac{\frac{(1-p)\mathrm{e}^{\eta y_2}}{(1-p)\mathrm{e}^{\eta y_2}+p} \frac{-\mathrm{e}^{\eta y_1}}{((1-p)\mathrm{e}^{\eta y_1}+p)^2} - \frac{-\mathrm{e}^{\eta y_2}}{((1-p)\mathrm{e}^{\eta y_2}+p)^2} \frac{(1-p)\mathrm{e}^{\eta y_1}}{(1-p)\mathrm{e}^{\eta y_1}+p}}{(f_y(y_2,p))^2} \\ &= \frac{-\mathrm{e}^{\eta y_1}\big((1-p)\mathrm{e}^{\eta y_2}\big)\big((1-p)\mathrm{e}^{\eta y_2} + p\big)}{((1-p)\mathrm{e}^{\eta y_1} + p)^2((1-p)\mathrm{e}^{\eta y_2} + p)^2(f_y(y_2,p))^2} \\ &\quad - \frac{-\mathrm{e}^{\eta y_2}\big((1-p)\mathrm{e}^{\eta y_1}\big)\big((1-p)\mathrm{e}^{\eta y_1} + p\big)}{((1-p)\mathrm{e}^{\eta y_1} + p)^2((1-p)\mathrm{e}^{\eta y_2} + p)^2(f_y(y_2,p))^2} \\ &= \frac{\mathrm{e}^{\eta(y_1+y_2)}}{((1-p)\mathrm{e}^{\eta y_1} + p)^2((1-p)\mathrm{e}^{\eta y_2} + p)^2(f_y(y_2,p))^2} \\ &\quad \times \Big((1-p)\mathrm{e}^{\eta y_1}\big(1 - p + p\mathrm{e}^{-\eta y_1}\big) - \big((1-p)\mathrm{e}^{\eta y_2}\big)\big(1 - p + p\mathrm{e}^{-\eta y_2}\big)\Big). \end{aligned}$$

From this, we obtain $\frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} \geq 0$ because

$$(1-p)\mathrm{e}^{\eta y_1}\big(1 - p + p\mathrm{e}^{-\eta y_1}\big) - \big((1-p)\mathrm{e}^{\eta y_2}\big)\big(1 - p + p\mathrm{e}^{-\eta y_2}\big) = (1-p)^2\big(\mathrm{e}^{\eta y_1} - \mathrm{e}^{\eta y_2}\big) \geq 0,$$

using $y_1 \geq y_2$. This concludes the proof of (24).

We now consider each of the two conditions (a) and (b) of the proposition.

Condition (a). We apply (24) choosing $p = p_i$, $y_1 = \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell$, and $y_2 = \gamma_{n,i} s_i$. This implies

$$\frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)} \leq \frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,i} s_i, p_j)} \leq \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)},$$

where we use (23) and the convexity of $y \mapsto f(y, p_j)$ for the second inequality.

Condition (b). This time, we apply (24) choosing $p = p_j$, $y_1 = \sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell$, and $y_2 = \gamma_{n,j} s_j$. We obtain

$$\frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,j} s_j, p_i)} \geq \frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)},$$

where we again use (23) and the convexity of $y \mapsto f(y, p_i)$ for the second inequality.

Under each of the two conditions (a) and (b), we obtain a contradiction to (22). Hence, $\Omega_i < \Omega_j$ cannot hold, which concludes the proof of (C2).

Claim 3. For two banks i and j, we have

$$\omega_i > \omega_j,\ p_j \leq p_i,\ s_j \leq s_i \Longrightarrow \Omega_i - \omega_i < \Omega_j - \omega_j.$$

Proof of Claim 3. We proceed similarly to the proof of (C2). We prove the claim by contradiction and assume that $\Omega_i - \omega_i \geq \Omega_j - \omega_j$. This implies $\Omega_i > \Omega_j$; hence, $\gamma_{i,n} \leq \gamma_{j,n}$ for all $\gamma_{i,n} \leq 0$ by (C1) and $\gamma_{i,j} \leq 0 \leq \gamma_{j,i}$ by (C1a), and thus

$$\begin{aligned} f\Bigg(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\Bigg) &= \Omega_j - \omega_j - \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n} s_n, p_n) \\ &< \Omega_i - \omega_i - \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n} s_n, p_n) \\ &= f\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\Bigg) \\ &\leq f\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_j\Bigg) \end{aligned}$$

using $p_j \leq p_i$ and (20), which yields $\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell < \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell$ because $y \mapsto f(y, p_j)$ is strictly increasing by Lemma IV.1. We conclude the proof in the same way as the proof of (C2) after (21), with i and j interchanged. □

Theorem A.4 (Theorem IV.6)**.** *Assume that the trade size limit is not binding and that there are at least two safe banks. Then*

1. *There exists the following relation between banks' creditworthiness, initial exposures and post-trade exposures:*
   - *(a) All safe banks have the same post-trade exposure, say, $\bar{\Omega}$.*
   - *(b) Risky banks with initial exposure above some level α also have the same post-trade exposure $\bar{\Omega}$. The level α is greater than $\bar{\Omega}$ and depends only on the distribution of initial exposures and sizes, but not on the banks' default probabilities.*
   - *(c) Risky banks with initial exposure below α will have post-trade exposures strictly smaller than $\bar{\Omega}$.*

2. Risky banks with initial exposure above α trade as follows:

(a) They do not trade between each other.

(b) They do not sell protection.

(c) Their purchases depend only on their initial exposure, but not on their default probabilities.

3. Risky banks with initial exposure below α trade as follows:

(a) Any two risky banks i and n with initial exposure below α do not trade between each other if their exposures $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ after trading with other banks but before trading between themselves satisfy $\tilde{\Omega}_i \leq \tilde{\Omega}_n$ and

$$\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y\big(\sum_{\ell \neq n: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{1 - p_i}. \tag{25}$$

(b) They do not purchase protection from safe banks or risky banks with initial exposures above α.

(c) If all banks have the same size, then they sell the same amount of protection to each safe bank and risky bank with initial exposure above α.

Proof. To prove the first part, we define $\bar{k}_1$ by

$$\bar{k}_1 = \inf\big\{k > 0 : \Omega_i = \Omega_j \text{ for all } i, j \text{ with } p_i = p_j = 0\big\}. \tag{26}$$

We can prove that $0 < \bar{k}_1 < \infty$ and that the infimum in (26) is attained along the same lines as on page 2273 of Atkeson et al. (2015), restricting their arguments to the safe banks. We choose $\bar{k}$ as the smallest number $k \geq \bar{k}_1$ such that

$$\Omega_i \leq \Omega_j \tag{27}$$

for all i, j with $p_i > 0$ and $p_j = 0$. We next show that such a finite $\bar{k}$ exists. If (27) holds for $k = \bar{k}_1$, we set $\bar{k} = \bar{k}_1$. Moreover, (27) always holds for k big enough. To see this, let i be such that $p_i > 0$ and, working towards a contradiction, assume that

$$\Omega_i > \Omega_j \tag{28}$$

for some j with $p_j = 0$. From (C1a) and (C1) in the proof of Proposition IV.5 with $p_j = 0$, it follows that $\gamma_{i,j} \leq 0$ and $\gamma_{i,n} \leq \gamma_{j,n}$ for all n; hence,

$$\begin{aligned}\Gamma^j_{y_i}(\gamma_j s) &= \Xi'(\Omega_j)\eta f_y\left(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j\right) = \Xi'(\Omega_j)\eta \\ &< \Xi'(\Omega_i)\eta = \Xi'(\Omega_i)\eta f_y(\gamma_{i,j} s_j, p_j) = \Gamma^i_{y_j}(\gamma_i s)\end{aligned}$$

using that $f_y(y,p_j)=1$ because $p_j=0$, Ξ is strictly increasing and strictly convex, and $\Omega_i>\Omega_j$. Then $\gamma_{i,j}=-k$ follows from $\Gamma^j_{y_i}(\gamma_j s)<\Gamma^i_{y_j}(\gamma_i s)$ by (4), and thus

$$\begin{aligned}\Omega_j &= \omega_j + f\Bigg(\sum_{n:\gamma_{j,n}\geq 0}\gamma_{j,n}s_n, p_j\Bigg) + \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n}s_n, p_n)\\ &\geq ks_i + \omega_j + f\Bigg(\sum_{n:\gamma_{i,n}\geq 0}\gamma_{i,n}s_n, p_i\Bigg) + \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n}s_n, p_n)\\ &= ks_i+\omega_j-\omega_i+\Omega_i.\end{aligned}$$

However, for $k\geq(\omega_i-\omega_j)/s_i$, this gives $\Omega_j\geq\Omega_i$ in contradiction to (28). Hence, we have that (27) holds for k big enough. By a compactness argument similar to page 2273 of Atkeson et al. (2015), we deduce that (27) holds for $k=\bar{k}$. By definition of $\bar{k}$, for $k<\bar{k}$, there exist i and j with $p_j=0$ such that $\Omega_i>\Omega_j$.

We now consider $k\geq\bar{k}$ and

$$\beta(p,s)=\max_{i:p_i=p,s_i=s}\Omega_i,\quad \bar{i}(p,s)=\begin{cases}\operatorname{arg\,max}_{i:p_i=p,s_i=s}\Omega_i & \text{if } \beta(p,s)=\Omega_j \text{ for } j \text{ with } p_j=0,\\ \emptyset & \text{otherwise.}\end{cases}$$
$$\overline{\delta}(p,s)=\min_{i\in\bar{i}(p,s)}\omega_i,\quad \underline{\delta}(p,s)=\max_{\{i:p_i=p,s_i=s\}\setminus\bar{i}(p,s)}\omega_i$$

for $p\in\{p_1,\dots,p_M\}$ and $s\in\{s_1,\dots,s_M\}$ where the minimum (and maximum) over an empty set equals $+\infty$ and $-\infty$ by the usual convention. Several p_j and s_j for different j can take the same values, and thus $\bar{i}(p,s)$ can be a set with several entries because the maximum does not need to be attained at a unique i. We can choose a function $\bar{\alpha}:(0,1]\times[0,1]\to[0,\infty)$ for all s such that $\underline{\delta}(p,s)<\bar{\alpha}(p,s)\leq\overline{\delta}(p,s)$ for all $p\in\{p_1,\dots,p_M\}$ and $s\in\{s_1,\dots,s_M\}$. Note that $\bar{\alpha}(p,s)$ may depend here on both arguments p and s, but in the next paragraph, we will show that $\bar{\alpha}$ can be chosen independently of p. From $\bar{\alpha}(p,s)\leq\overline{\delta}(p,s)$, it follows that $A(\bar{\alpha})$ defined by

$$A(\bar{\alpha})=\{i:\omega_i\geq\bar{\alpha}(p_i,s_i)\text{ or } p_i=0\}$$

contains all indices i with $\Omega_i=\Omega_j$ for j with $p_j=0$. To show that $A(\bar{\alpha})$ contains only such indices i, assume that there exists $i\in A(\bar{\alpha})$ with $\Omega_i<\Omega_j$ for j with $p_j=0$. This implies

$$\omega_i\geq\bar{\alpha}(p_i,s_i)>\underline{\delta}(p_i,s_i);$$

hence, $\omega_i>\omega_\ell$ for all ω_ℓ with $\Omega_\ell<\Omega_j$, which contradicts $\Omega_i<\Omega_j$. Therefore, all banks $i\in A(\bar{\alpha})$ have the same post-trade exposure Ω_i while banks $i\notin A(\bar{\alpha})$ have a strictly smaller post-trade exposure. Thus, we can set $\bar{\Omega}=\Omega_i$ for some $i\in A(\bar{\alpha})$.

We next show that $\bar{\alpha}$ can be chosen independently of p, consider $k\geq\bar{k}$ and i with $p_i>0$ and $\Omega_i=\Omega_j$ for j with $p_j=0$. Because $k\geq\bar{k}$, it follows from (27) that $\Omega_i\geq\Omega_\ell$ for all ℓ. In the case $\Omega_i>\Omega_\ell$, we obtain $\gamma_{i,\ell}\leq 0$ by (C1a). In the case $\Omega_i=\Omega_\ell$, we argue similarly to the proof of (C1a) to show $\gamma_{i,\ell}\leq 0$. Indeed, to derive a contradiction, we assume that $\gamma_{i,\ell}>0$ and $\Omega_i=\Omega_\ell$, which implies

$$\Gamma^i_{y_\ell}(\gamma_i s)=\Xi'(\Omega_i)\eta f_y\Bigg(\sum_{n:\gamma_{i,n}\geq 0}\gamma_{i,n}s_n, p_i\Bigg)$$

$$
\begin{aligned}
&= \Xi'(\Omega_\ell)\eta f_y\left(\sum_{n:\gamma_{i,n}\geq 0}\gamma_{i,n}s_n, p_i\right)\\
&> \Xi'(\Omega_\ell)\eta f_y(\gamma_{\ell,i}s_i, p_i)\\
&= \Gamma^\ell_{y_i}(\gamma_\ell s)
\end{aligned}
$$

by strict convexity of $f(., p_i)$ from Lemma IV.1, using that $p_i > 0$. However, this implies $\gamma_{i,\ell} = -k$ by (4) in contradiction to the assumption $\gamma_{i,\ell} > 0$. Hence, we have $\gamma_{i,\ell} \leq 0$, and the trading choices of bank i do not depend on p_i. Using Lemma IV.1, we then deduce that, for all ℓ, $\Gamma^\ell(\gamma_\ell s)$ does not depend on p_i if $\gamma_{i,\ell} \leq 0$, and thus the objective function $\sum_{\ell=1}^M s_\ell \Gamma^\ell(\gamma_\ell s)$ in (15) does not depend on p_i in the optimum. Therefore, $\bar{\alpha}$ can be chosen independently of p. From $\gamma_{i,\ell} \leq 0$ for all ℓ, we also deduce that $\bar{\alpha} \geq \bar{\Omega}$. This concludes the proof of the first part of the theorem.

To prove part 2a of the theorem, we consider two risky banks i and n with $\omega_i \geq \alpha$ and $\omega_n \geq \alpha$. From part 1b of the theorem, we know that the banks' post-trade exposures are $\Omega_i = \Omega_n = \bar{\Omega}$. Working towards a contradiction, we assume that $\gamma_{i,n} > 0$ so that bank i sells protection to bank n. We then have $\Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n)$ by (4), which implies

$$
\Xi'(\Omega_i)\eta f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i}s_n, p_i).
$$

This equality cannot hold because $\Xi'(\Omega_i) = \Xi'(\Omega_n) = \Xi'(\bar{\Omega})$ and

$$
f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right) > f_y(\gamma_{n,i}s_n, p_j).
$$

Therefore, we deduce that $\gamma_{i,n} > 0$ leads to a contradiction and so does $\gamma_{i,n} < 0$ by symmetry. Hence, we must have $\gamma_{i,n} = 0$, proving part 2a. Moreover, any bank i with initial exposure $\omega_i \geq \alpha$ will have a post-trade exposure $\bar{\Omega}$, which is strictly greater than the post-trade exposure of any risky bank n that has initial exposure $\omega_n < \alpha$. Therefore, bank i does not sell to bank n by (C1a). Similarly to part 2a, we can show that bank i does not sell to any safe bank, either, establishing part 2b. Consequently, the post-trade exposure of banks with initial exposure greater than or equal to α does not depend on the banks' own default probabilities, and neither does their purchased quantities. This shows part 2c.

For part 3, we consider two banks i and n that satisfy (25) and have exposures $\tilde{\Omega}_i \leq \tilde{\Omega}_n$ after trading with other banks, but before trading between themselves. We deduce $\gamma_{i,n} \geq 0$ from (C1a). Working towards a contradiction, we assume that $\gamma_{i,n} > 0$. We then have $\Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n)$ by (4), which implies

$$
\Xi'(\Omega_i)\eta f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i}s_n, p_i)
$$

so that

$$
\Xi'(\Omega_i)\eta f_y\left(\sum_{\ell\neq n:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right) < \Xi'(\Omega_n)\eta f_y(0, p_i) = \Xi'(\Omega_n)\eta(1-p_i),
$$

which is a contradiction to

$$\frac{\Xi'(\Omega_n)}{\Xi'(\Omega_i)} \leq \frac{f_y\big(\sum_{\ell\neq n:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\big)}{1-p_i}.$$

which is implied by (25), $\gamma_{i,n} > 0$ and the convexity of Ξ by Lemma IV.1. This shows that $\gamma_{i,n} = 0$ so that banks i and n do not trade with each other, proving part 3a. Moreover, any bank i with initial exposure $\omega_i < \alpha$ will have a post-trade exposure smaller than $\bar{\Omega}$. Hence, its post-trade exposure is strictly smaller than that of any bank n that has initial exposure $\omega_n \geq \alpha$. Therefore, bank i does not buy from bank n by (C1a), proving part 3a. Finally, we note that safe banks and risky banks with initial exposures above α have all the same post-trade exposure $\bar{\Omega}$. Therefore, when traders of these banks meet traders of a risky bank i with initial exposure $\omega_i < \alpha$, they all have the same incentive compatibility condition, namely, either they do not trade or

$$\Xi'(\Omega_i)\eta f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\right) = \Xi'(\bar{\Omega})\eta f_y(\gamma_{n,i}s_n, p_j),$$

where n refers to any of the safe banks or risky banks with initial exposures above α. If all sizes s_n are equal, each bank n satisfies the same condition, thus each of them buys the same amount from bank i, as stated in part 3c. □

Proof of Corollary IV.8. Part 2b of Theorem A.4 implies that risky banks with initial exposures above α only purchase protection, hence they are not intermediaries.

Next we consider a bank n with initial exposure below α and that satisfies $\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y(\Gamma_i, p_i)}{1-p_i}$ for any risky bank i with $\tilde{\Omega}_i < \tilde{\Omega}_n$, where $\Gamma_i = \sum_{\ell\neq n:s_i\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}$ is the sum of contracts sold by bank i, and $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ are the exposures of banks i and n, respectively, after trading with other banks but before trading between themselves. From part 3a of Theorem A.4, we obtain $\gamma_{i,n} = 0$ so that bank i and n do not trade between themselves. Moreover, part 3b of Theorem A.4 implies that bank n does not purchase protection from safe banks or risky banks with initial exposures above α. Therefore, bank n only sells protection, hence it is not an intermediary. □

A.D Results of Section V and their Proofs

Lemma A.5 (Lemma V.1). *For given $s_1, \ldots, s_M$, the value of $x_i(p_1, \ldots, p_M)$ is uniquely determined.*

Proof. For general s_i, (9) becomes

$$x_i(p_1, \ldots, p_M) = \omega_i + \sum_{n\neq i}\gamma_{i,n}s_n R_{i,n} - \Gamma^i(\gamma_i s).$$

Using the definition (5) of $R_{i,n}$ and (4), we can write

$$x_i(p_1, \ldots, p_M) = \omega_i - \Gamma^i(\gamma_i s) + \sum_{n:\gamma_{i,n}>0}\gamma_{i,n}s_n\big(\nu\Gamma^n_{y_i}(\gamma_n s) + (1-\nu)\Gamma^i_{y_n}(\gamma_i s)\big)$$

$$
\begin{aligned}
&+\sum_{n:\gamma_{i,n}<0}\gamma_{i,n}s_n\big(\nu\Gamma^i_{y_n}(\gamma_i s)+(1-\nu)\Gamma^n_{y_i}(\gamma_n s)\big)\\
&=\omega_i-\Gamma^i(\gamma_i s)+\nu\sum_{n:\gamma_{i,n}>0}\gamma_{i,n}s_n\big(\Gamma^n_{y_i}(\gamma_n s)-\Gamma^i_{y_n}(\gamma_i s)\big)\\
&\quad+(1-\nu)\sum_{n:\gamma_{i,n}<0}\gamma_{i,n}s_n\big(\Gamma^n_{y_i}(\gamma_n s)-\Gamma^i_{y_n}(\gamma_i s)\big)+\sum_{n\neq i}\gamma_{i,n}s_n\Gamma^i_{y_n}(\gamma_i s)\\
&=\omega_i-\Gamma^i(\gamma_i s)+\nu k\sum_{n:\gamma_{i,n}>0}s_n\big(\Gamma^n_{y_i}(\gamma_n s)-\Gamma^i_{y_n}(\gamma_i s)\big)\\
&\quad-(1-\nu)k\sum_{n:\gamma_{i,n}<0}s_n\big(\Gamma^n_{y_i}(\gamma_n s)-\Gamma^i_{y_n}(\gamma_i s)\big)+\sum_{\substack{p_n>0,\,\gamma_{i,n}<0,\text{ or}\\ p_i>0,\,\gamma_{i,n}>0}}\gamma_{i,n}s_n\Gamma^i_{y_n}(\gamma_i s)\\
&\quad+\overline{\Gamma}^i(\gamma_i s)\sum_{\substack{p_n=0,\,\gamma_{i,n}<0,\text{ or}\\ p_i=0,\,\gamma_{i,n}>0}}\gamma_{i,n}s_n,
\end{aligned}
$$

where

$$
\overline{\Gamma}^i(y):=\Gamma^i_{y_n}(y)=\frac{q\mathrm{e}^{\eta\omega_i+\eta f(\sum_{n:y_n\geq 0}y_n,p_i)+\eta\sum_{n:y_n<0}f(y_n,p_n)}}{1-q+q\mathrm{e}^{\eta\omega_i+\eta f(\sum_{n:y_n\geq 0}y_n,p_i)+\eta\sum_{n:y_n<0}f(y_n,p_n)}}
$$

does not depend on the specific n for all n with $p_n=0$ and $\gamma_{i,n}<0$, or $p_i=0$ and $\gamma_{i,n}>0$. This means that $\Gamma^i_{y_n}$ is the same for all banks n that are (I) default-free protection sellers to i, or (II) protection buyers from i, and i is default-free. All these pairwise transactions do not bear any counterparty risk. Uniqueness of $x_i(p_1,\dots,p_M)$ now follows from Theorem A.2. □

Proof of Proposition V.3. We first note that the mapping $p_i\mapsto x_i(p_1,\dots,p_M)$ is continuous. This follows from the Envelope theorem using that Γ^i and its partial derivatives are differentiable. For $p_{-i}=(p_j)_{j\neq i}$, we define set-valued functions

$$
r_i(p_{-i})=\underset{p_i\in[0,\bar{p}_i]}{\arg\max}\big(x_i(p_1,\dots,p_M)-C_i(p_i)\big),\quad r(p)=\big(r_1(p_{-1}),\dots,r_M(p_{-M})\big)
$$

so that r is a mapping from $[0,\bar{p}_1]\times\cdots\times[0,\bar{p}_m]$ onto its power set. It has the following properties:

- $[0,\bar{p}_1]\times\cdots\times[0,\bar{p}_m]$ is compact, convex, and nonempty.
- For each p, $r(p)$ is nonempty because a continuous function over a compact set has always a maximizer.
- $r(p)$ is convex by assumption.
- It follows from Berge's maximum theorem that $r(p)$ has a closed graph.

Thanks to these properties, Kakutani's fixed point theorem implies that there exists a fixed point of the mapping r, which means that there exists an equilibrium. □

Proof of Proposition V.4. Because the function

$$
\sum_{i=1}^{M}s_i\Gamma^i(\gamma_i s,p)+\sum_{i=1}^{M}s_iC_i(p_i)
$$

is continuous over the compact set $[0,\bar{p}_1]\times\cdots\times[0,\bar{p}_M]$, it has a maximum, which shows the statement of the proposition, using that the social planner's optimization problem over $(\gamma_{i,n})_{i,n=1,\dots,M}$ conditional on the choice of the default probabilities has a solution by Theorems IV.3 and IV.4. □

Theorem A.6 (Theorem V.5)**.** *The externality imposed by bank i on the system equals*

$$MSV_i(p)-MPV_i(p)=\frac{\partial S_i}{\partial p_i}(p)+k(1-\nu)\frac{\partial T_i}{\partial p_i}(p), \tag{29}$$

where

$$S_i(p):=\sum_{n\neq i}s_n\left(\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n s,p)+\frac{1}{s_i}\Gamma^n(\gamma_n s,p)\right),\quad T_i(p):=\sum_{n\neq i}s_n\big(\Gamma^i_{y_n}(\gamma_i s,p)-\Gamma^n_{y_i}(\gamma_n s,p)\big). \tag{30}$$

For small enough p_i and large enough q, we have $\frac{\partial S_i}{\partial p_i}(p)<0$ and $\frac{\partial T_i}{\partial p_i}(p)\geq 0$ with strict inequality when the trade size limit is binding for at least one bilateral trading relationship.

Proof. 1. part: proof of (29)*.*
For arbitrary bank sizes, marginal private and social values for bank i are given by

$$\begin{aligned}MSV_i(p)&=\sum_{n=1}^{M}s_n\frac{\partial\Gamma^n}{\partial p_i}(\gamma_n s,p)+s_iC_i'(p_i),\\ MPV_i(p)&=s_i\frac{\partial\Gamma^i}{\partial p_i}(\gamma_i s,p)-s_i\sum_{n\neq i}\gamma_{i,n}s_n\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s,\gamma_n s,p)+s_iC_i'(p_i)\end{aligned}$$

so that its difference is

$$MSV_i(p)-MPV_i(p)=\sum_{n\neq i}s_n\left(s_i\gamma_{i,n}\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s,\gamma_n s,p)+\frac{\partial\Gamma^n}{\partial p_i}(\gamma_n s,p)\right).$$

If $\gamma_{i,n}\leq 0$, then we obtain from (5) that

$$R_{i,n}(\gamma_i s,\gamma_n s,p)=\nu\Gamma^i_{y_n}(\gamma_i s,p)+(1-\nu)\Gamma^n_{y_i}(\gamma_n s,p).$$

We then have that $\frac{\partial\Gamma^n}{\partial p_i}(\gamma_n s,p)=0$ and $\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s,\gamma_n s,p)=0$ because $\Gamma^n(\gamma_n s,p)$, $\Gamma^i_{y_n}(\gamma_i s,p)$, and $R_{i,n}(\gamma_i s,\gamma_n s,p)$ do not depend on p_i for $\gamma_{i,n}\leq 0$; if traders of bank i are buying CDSs from bank n, the default probability of bank i does not affect the terms of trade between traders of banks i and n. For $\gamma_{i,n}>0$, we find

$$R_{i,n}(\gamma_i s,\gamma_n s,p)=\nu\Gamma^n_{y_i}(\gamma_n s,p)+(1-\nu)\Gamma^i_{y_n}(\gamma_i s,p)$$

by (4) and (5) so that

$$MSV_i(p)-MPV_i(p)=\sum_{n:\gamma_{i,n}>0}s_n\left(s_i\gamma_{i,n}\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s,\gamma_n s,p)+\frac{\partial\Gamma^n}{\partial p_i}(\gamma_n s,p)\right)$$

$$
\begin{aligned}
&= \sum_{n:\gamma_{i,n}>0} s_n \Big(s_i\gamma_{i,n} \frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \Big) \\
&\quad + \sum_{n:\gamma_{i,n}>0} s_n s_i \gamma_{i,n}(1-\nu) \Big(\frac{\partial \Gamma^i_{y_n}}{\partial p_i}(\gamma_i s, p) - \frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) \Big) \\
&= \frac{\partial}{\partial p_i} \sum_{n\neq i} s_n \big(s_i\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n s, p) + \Gamma^n(\gamma_n s, p) \big) \\
&\quad + \frac{\partial}{\partial p_i} \sum_{n\neq i} s_n s_i k(1-\nu) \big(\Gamma^i_{y_n}(\gamma_i s, p) - \Gamma^n_{y_i}(\gamma_n s, p) \big),
\end{aligned}
$$

using for the last equality that $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$, $\frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) = 0$ and $\frac{\partial \Gamma^i_{y_n}}{\partial p_i}(\gamma_i s, p) = 0$ for $\gamma_{i,n} \leq 0$ and $\Gamma^i_{y_n}(\gamma_i s, p) = \Gamma^n_{y_i}(\gamma_n s, p)$ for $\gamma_{i,n} \in (-k, k)$. Combining this with (30), we conclude the proof of (29).

2. part: $\frac{\partial S_i}{\partial p_i}(p) < 0$ for small enough p_i and large enough q.
We recall from Lemma IV.1 that

$$
\begin{aligned}
\Gamma^n(y,p) &= \frac{1}{\eta} \log \Big(1 - q + q e^{\eta\omega_n + \eta f(\sum_{\ell:y_\ell\geq 0} y_\ell, p_n) + \eta \sum_{\ell:y_\ell<0} f(y_\ell, p_\ell)} \Big) \\
&= \frac{1}{\eta} \log \Big(1 - q + a(1-p_i)e^{\eta y_i} + a p_i e^{\eta r y_i} \Big)
\end{aligned}
$$

for $y_i < 0$, where we use the abbreviation $a = q e^{\eta\omega_n + \eta f(\sum_{\ell:y_\ell\geq 0} y_\ell, p_n) + \eta \sum_{\ell\neq i:y_\ell<0} f(y_\ell, p_\ell)}$ in this proof. We compute

$$
\Gamma^n_{p_i}(y,p) = \frac{1}{\eta} \frac{-a e^{\eta y_i} + a}{1 - q + a(1-p_i)e^{\eta y_i} + a p_i} = \frac{1}{\eta} \frac{1 - e^{\eta y_i}}{b + (1-p_i)e^{\eta y_i} + p_i},
$$

using the abbreviation $b = (1-q)/a$. Next, we find

$$
\begin{aligned}
\Gamma^n_{y_i,p_i}(y,p) &= \frac{(b + (1-p_i)e^{\eta y_i} + p_i)(-e^{\eta y_i}) - (1 - e^{\eta y_i})(1-p_i)e^{\eta y_i}}{(b + (1-p_i)e^{\eta y_i} + p_i)^2} \\
&= \frac{b(-e^{\eta y_i}) - p_i e^{\eta y_i} - (1-p_i)e^{2\eta y_i} - (1-p_i)e^{\eta y_i} + (1-p_i)e^{2\eta y_i}}{(b + (1-p_i)e^{\eta y_i} + p_i)^2} \\
&= \frac{b(-e^{\eta y_i}) - e^{\eta y_i}}{(b + (1-p_i)e^{\eta y_i} + p_i)^2}. \qquad (31)
\end{aligned}
$$

For $p_i = 0$, $q = 1$ and $\gamma_{i,n} > 0$, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial p_i} \big(s_i\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n s, p) + \Gamma^n(\gamma_n s, p) \big) \Big|_{p_i=0,q=1} = \big(s_i\gamma_{i,n}\Gamma^n_{y_i,p_i}(\gamma_n s, p) + \Gamma^n_{p_i}(\gamma_n s, p) \big) \big|_{p_i=0,q=1} \\
&= \Big(s_i\gamma_{i,n} \frac{b(-e^{-\eta s_i\gamma_{i,n}}) - e^{-\eta s_i\gamma_{i,n}}}{(b + (1-p_i)e^{-\eta s_i\gamma_{i,n}} + p_i)^2} + \frac{1}{\eta} \frac{1 - e^{-\eta s_i\gamma_{i,n}}}{b + (1-p_i)e^{-\eta s_i\gamma_{i,n}} + p_i} \Big) \Big|_{p_i=0,q=1} \\
&= \frac{-s_i\gamma_{i,n}}{e^{-\eta s_i\gamma_{i,n}}} + \frac{1 - e^{-\eta s_i\gamma_{i,n}}}{\eta e^{-\eta s_i\gamma_{i,n}}} < \frac{-s_i\gamma_{i,n}}{e^{-\eta s_i\gamma_{i,n}}} + \frac{\eta s_i\gamma_{i,n}}{\eta e^{-\eta s_i\gamma_{i,n}}} = 0.
\end{aligned}
$$

Using that $\gamma_{i,n}\Gamma^n_{y_i,p_i}(\gamma_n s,p) = 0$ and $\Gamma^n_{p_i}(\gamma_n s,p) = 0$, we deduce $\frac{\partial S_i}{\partial p_i}(p)\big|_{p_i=0,q=1} < 0$ by the definition (30) of $S_i(p)$, which implies $\frac{\partial S_i}{\partial p_i}(p) < 0$ for small enough p_i and large enough q by continuity.

3. part: $\frac{\partial T_i}{\partial p_i}(p) \geq 0$ *for small enough* p_i *and large enough* q *with strict inequality when the trade size limit is binding for at least one bilateral trading relationship.*
We compare $\Gamma^n_{y_i,p_i}(\gamma_n,p)$ and $\Gamma^i_{y_n,p_i}(\gamma_i,p)$. We first note that $\Gamma^n_{y_i,p_i}(\gamma_n,p) = 0$ and $\Gamma^i_{y_n,p_i}(\gamma_i,p) = 0$ for $\gamma_{n,i} = -\gamma_{i,n} \geq 0$. For $p_i = 0$ and $q = 1$, we obtain from (31) that

$$\Gamma^n_{y_i,p_i}(y,p)\big|_{p_i=0,q=1} = \frac{-b\mathrm{e}^{\eta y_i} - \mathrm{e}^{\eta y_i}}{(b + (1-p_i)\mathrm{e}^{\eta y_i} + p_i)^2}\Bigg|_{p_i=0,q=1} = -\mathrm{e}^{-\eta y_i}$$

for $y_i < 0$. A calculation similar to (31) gives

$$\Gamma^i_{y_n,p_i}(y,p)\big|_{p_i=0,q=1} = \frac{-\tilde{b}\mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell} - \mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell}}{(\tilde{b} + (1-p_i)\mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell} + p_i)^2}\Bigg|_{p_i=0,q=1} = -\mathrm{e}^{-\eta\sum_{\ell:y_\ell\geq 0} y_\ell}$$

for $y_n > 0$, where $\tilde{b} = (1-q)/\big(q\mathrm{e}^{\eta\omega_i + \eta\sum_{\ell:y_\ell<0} f(y_\ell,p_\ell)}\big)$. Therefore, for $\gamma_{n,i} = -\gamma_{i,n} < 0$, we obtain

$$\Gamma^n_{y_i,p_i}(\gamma_n s,p)\big|_{p_i=0,q=1} < \Gamma^i_{y_n,p_i}(\gamma_i s,p)\big|_{p_i=0,q=1}$$

and thus $\frac{\partial T_i}{\partial p_i}(p) > 0$ in this case when k is binding for some bilateral trades. □

B Description of Data and Plot Generation Procedure

In this section, we test the empirical predictions of our model conditional on banks' chosen default probabilities. We use different data sources for the bilateral exposures in the CDS market, the initial exposures of banks, and their default probabilities.

CDS volume. CDS data come from the confidential Trade Information Warehouse of the DTCC. We use position data from December 31, 2011. This data set allows for a post-crisis analysis in which a large part of CDS trades were not yet centrally cleared.[8] We eliminate from our data set the following transactions:

- All swaps with governments, states, or sovereigns as reference entities. We eliminate these transactions because we expect the default probabilities of corporate reference entities to have stronger dependence on the risk stemming from banks' exposures than on that of sovereign entities.
- All swaps with reference entities that are considered systemically important financial institutions. By doing so, we avoid problems related to specific wrong-way risk, where the seller of the transaction also happens to be the reference entity.
- All transactions done by nonbanking institutions. For nonbanking institutions, there is no consistent way to measure initial exposures, which are needed in our analysis. While

[8]Distortion on the CDS market due to the "London Whale" (large unauthorized trading activities in JPMorgan's Chief Investment Office) occurred only after December 31, 2011 and, thus, does not affect our analysis.

we consider only banks, we adjust their initial exposures by including CDS trades done with nonbanks. This procedure is consistent with our model and means that initial exposures of banks are determined after they have traded with nonbanks.

- The transactions done by two small private banks for which there were no data available on their initial exposures. Because these two banks are small players, the conclusions of our analysis are not affected by their exclusion.

Other than these four restrictions, we do not make any further adjustments. In particular, our data set also includes settlement locations outside of the United States, which allows for a more complete coverage of CDS trades and, importantly, guarantees symmetry in the inclusion of CDS trades (the transactions of both buyers and sellers are accounted for). The resulting set consists of CDS data for 81 banks.

Initial exposure. For each of these 81 banks, we compute its initial exposure by using 2011 data from the Federal Financial Institutions Examination Council (FFIEC) form 031 ("call report"), as in Begenau, Piazzesi, and Schneider (2015). We compute the initial exposure of each bank as the discounted valuation of its securities and loan portfolio, including CDSs traded with nonbanks as explained above. For large banks that book their assets mainly in holding companies, we use securities and loan portfolios at the bank holding company level. We group the securities and loans into three categories and use a specific discount factor for each group: less than one year (using the six-month U.S. Treasury rate to discount), one to five years (using the two-year U.S. Treasury rate to discount), and more than five years (using the seven-year U.S. Treasury rate to discount). Given the low interest rate environment in 2011, the precise choice of the discounting date and rate does not have a significant effect on our results. For foreign banks that do not report to the FFIEC, we analyze individual annual reports from 2011 to find the maturity profile of their securities and loans. Most of these annual reports are dated December 31, 2011, making them consistent with the domestic bank data. Some of them were released in March, June, or October of 2011, in line with the respective country's regulatory guidelines.

Default probabilities. The banks' default probabilities are calculated using CDS spread data from IHS Markit Ltd. (2018) via Wharton Research Data Services (WRDS). Because the default probabilities that are relevant for the analysis are those around the time of the transaction, we fix January 3, 2011 as the proxy date for CDS transactions and use the spread on this date to infer the default probability. We use the average five-year spread for Senior Unsecured Debt (Corporate/Financial) and Foreign Currency Sovereign Debt (Government) (SNRFOR). We compute the default probabilities from the CDS spreads applying standard techniques (credit triangle relation). For 19 among the 81 banks, CDS spread data were not available. For each of these banks, we instead use Moody's credit rating as of January 2011 for its Senior Unsecured Debt, and relate the ratings to default probabilities by using corporate default rates over the 1982–2010 period from Moody's.

Intermediation volume. For each bank i, we compute the intermediation volume as $I_i = \min\{G_i^+, G_i^-\}$, where $G_i^+ = \sum_{n \neq i} \max\{\gamma_{i,n}, 0\}$ and $G_i^- = \sum_{n \neq i} \max\{-\gamma_{i,n}, 0\}$, following the definition in Section IV.C.

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