Assignment #6 Solutions

§9.1 Problem 1a. The matrix is $A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{bmatrix}$. The determinant of $A - \lambda I$ is

$$(2-\lambda) * [(3-\lambda)(-3-\lambda) + 8] = (2-\lambda) * [-1 + \lambda^2]$$

which implies the eigenvalues are $\lambda = -1, 1, 2$. Since the eigenvalues are distinct there is a set of linearly independent eigenvectors by Theorem 9.3.

§9.1 Problem 2a. If we look only at 1g, we find the eigenvalues are $1, 1+\sqrt{2}, 1-\sqrt{2}$. Since the last eigenvalue is negative, the matrix in 1g is not positive definite.

§9.1 Problem 2b. If we look only at 1g, we have to apply the Gram-Schmidt orthogonalization process to get an orthonormal set of eigenvectors. If we check, we find $v^{(1)t}v^{(2)} = v^{(1)t}v^{(3)} = v^{(2)t}v^{(3)} = 0$ so in fact our original choice is orthogonal. We simply must normalize: $||v^{(1)}||_2 = \sqrt{2}$, $||v^{(2)}||_2 = 2$, $||v^{(3)}||_2 = 2$. Hence,

$$P = \begin{bmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

§9.1 Problem 1g. The matrix is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The determinant of $A - \lambda I$ is

$$\begin{aligned} &(1-\lambda) * [(1-\lambda)^2 - 0] - 1 * [1 * (1-\lambda) - 0 * 1] \\ &\qquad + 1 * [1 * 0 - 1 * (1-\lambda)] = \\ &(1-\lambda)^3 - (1-\lambda) - (1-\lambda) = (1-\lambda) * [(1-\lambda)^2 - 2] \\ &= (1-\lambda) * [(1-\lambda) - \sqrt{2}] * [(1-\lambda) + \sqrt{2}] \end{aligned}$$

which implies $\lambda = 1, 1-\sqrt{2}, 1+\sqrt{2}$. Solving $(A - I)y = 0$, $(A - (1-\sqrt{2})I)y = 0$, $(A - (1+\sqrt{2})I)y = 0$ will give the eigenvectors. From

$$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we find $v^{(1)} = (0, 1, -1)$ is an eigenvector for $\lambda = 1$. From

$$\begin{aligned} A - (1-\sqrt{2})I &= \begin{bmatrix} \sqrt{2} & 1 & 1 \\ 1 & \sqrt{2} & 0 \\ 1 & 0 & \sqrt{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & -1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

we find $v^{(2)} = (-\sqrt{2}, 1, 1)$ is an eigenvector for $\lambda = 1$. From

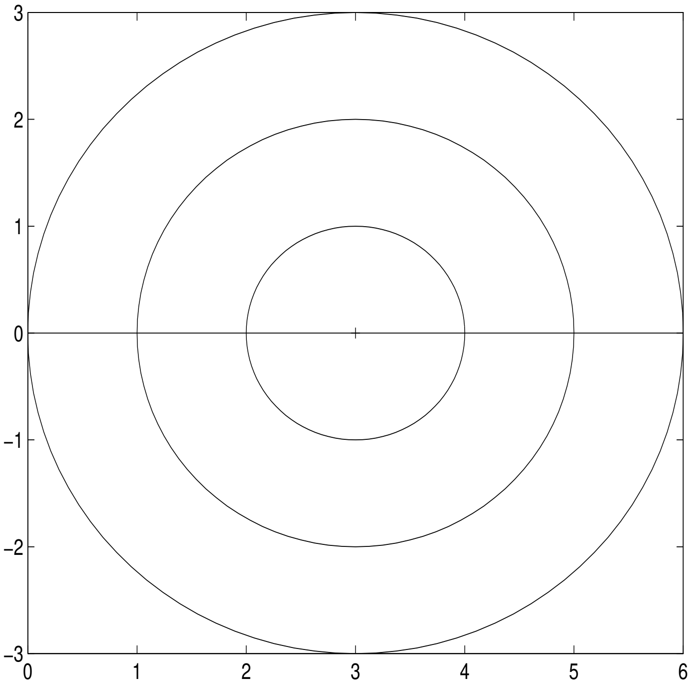
$$A - (1+\sqrt{2})I = \begin{bmatrix} -\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & 0 \\ 1 & 0 & -\sqrt{2} \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -\sqrt{2} & 0 \\ 0 & -1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\sqrt{2} \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we find $v^{(3)} = (\sqrt{2}, 1, 1)$ is an eigenvector for $\lambda = 1$. It is easy to check that these vectors are independent.

§9.1 Problem 3c. The matrix is $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. Hence,

$$\begin{aligned} R_1 &= \{z \in \mathbb{C} : |z-3| \le 3\} \\ R_2 &= \{z \in \mathbb{C} : |z-3| \le 2\} \\ R_3 &= \{z \in \mathbb{C} : |z-3| \le 1\} \end{aligned}$$

These are "nested" disks, the largest of which intersects the real axis in the interval $J := \{x : 0 \le x \le 6\}$. Since the matrix is symmetric, the eigenvalues are real and lie in the interval J.

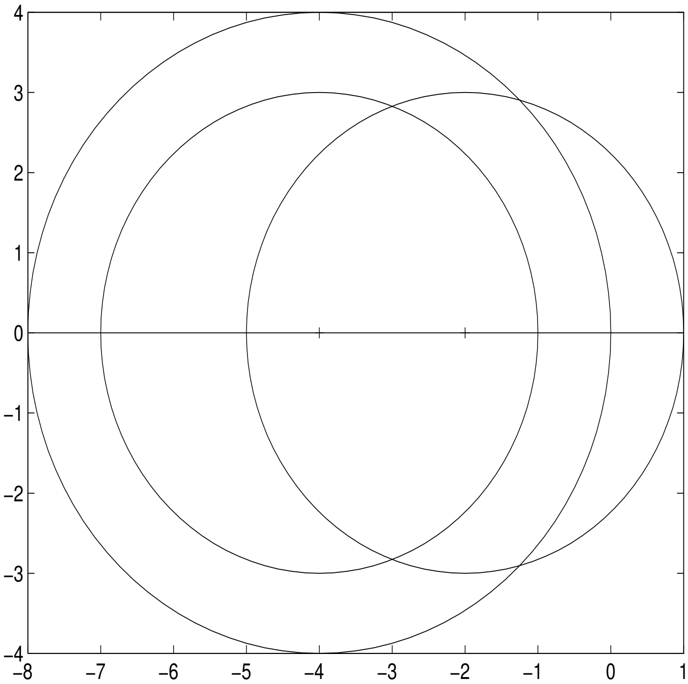


Plot for §9.1 Problem 3c.

§9.1 Problem 3e. The matrix is $A = \begin{bmatrix} -4 & 0 & 1 & 3 \\ 0 & -4 & 2 & 1 \\ 1 & 2 & -2 & 0 \\ 3 & 1 & 0 & -4 \end{bmatrix}$. Hence,

$$\begin{aligned} R_1 &= \{z \in \mathbb{C} : |z+4| \le 4\} \\ R_2 &= \{z \in \mathbb{C} : |z+4| \le 3\} \\ R_3 &= \{z \in \mathbb{C} : |z+2| \le 3\} \\ R_4 &= \{z \in \mathbb{C} : |z+4| \le 4\} \end{aligned}$$

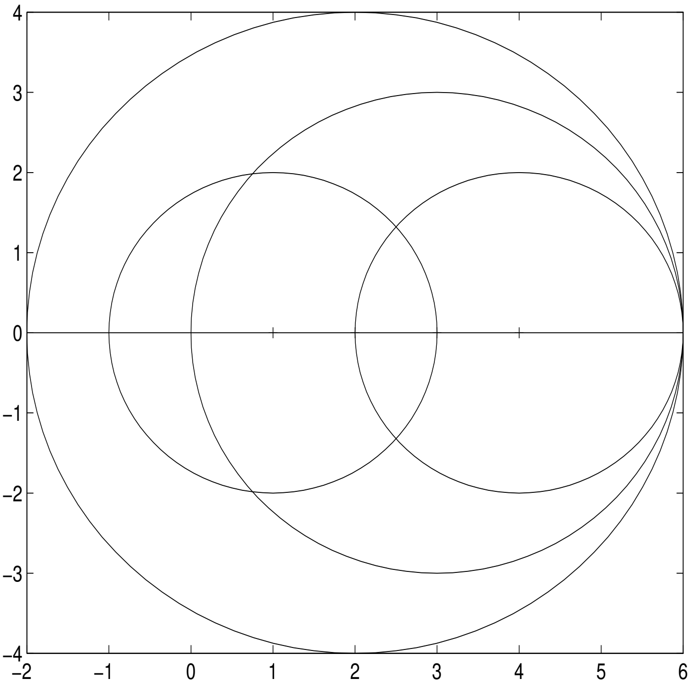
The disks intersect the real axis in the interval $J = \{x : -8 \le x \le 1\}$. Since the matrix is symmetric, the eigenvalues are real and lie in the interval J.



Plot for §9.1 Problem 3f.

§9.1 Problem 3f. The matrix is $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 1 & 0 & 1 & 4 \end{bmatrix}$. Hence,

$$\begin{aligned} R_1 &= \{z \in \mathbb{C} : |z-1| \le 2\} \\ R_2 &= \{z \in \mathbb{C} : |z-2| \le 4\} \\ R_3 &= \{z \in \mathbb{C} : |z-3| \le 3\} \\ R_4 &= \{z \in \mathbb{C} : |z-4| \le 2\} \end{aligned}$$



Plot for §9.1 Problem 3f.

The matrix is NOT symmetric, so we don't know that the eigenvalues are real in this case. Hence, we can only say that they lie in the region determined by the disks R_1, R_2, R_3, R_4.

§**9.1 Problem 8.** We prove this by induction on the number of distinct eigenvalues. Suppose $\lambda_1 \neq \lambda_2$ are distinct eigenvalues with eigenvectors $v^{(1)}$ and $v^{(2)}$ respectively. Then

$$Av^{(1)} = \lambda_1 v^{(1)}, \quad \text{and} \quad Av^{(2)} = \lambda_2 v^{(2)}.$$

If

$$\alpha_1 v^{(1)} + \alpha_2 v^{(2)} = 0, \tag{8.1}$$

then applying the matirx A, we get

$$\begin{aligned} 0 &= \alpha_1 A v^{(1)} + \alpha_2 A v^{(2)} \qquad \text{or} \\ 0 &= \alpha_1 \lambda_1 v^{(1)} + \alpha_2 \lambda_2 v^{(2)} \end{aligned} \tag{8.2}$$

Since λ_1 and λ_2 cannot both be zero, we may assume that $\lambda_1 \neq 0$. Then dividing (8.2) by λ_1, we get

$$0 = \alpha_1 v^{(1)} + \alpha_2 (\lambda_2/\lambda_1) v^{(2)} \tag{8.3}$$

Subtracting (8.3) from (8.1) gives

$$0 = \alpha_2 (1 - \lambda_2/\lambda_1) v^{(2)}.$$

Since $1 - \lambda_2/\lambda_1 \neq 0$, this implies $\alpha_2 = 0$ and by (8.1), $\alpha_1 = 0$. This shows that $\{v^{(1)}, v^{(2)}\}$ is independent.

Suppose for the induction hypothesis that the eigenvectors for any $k-1$ distinct eigenvalues of A are linearly independent. Let $\lambda_1, \ldots, \lambda_k$ be k distinct eigenvalues for A with respective eigenvectors $v^{(1)}, \ldots, v^{(k)}$. Suppose now that

$$\alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \ldots + \alpha_k v^{(k)} = 0. \tag{8.4}$$

We must show that $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. Applying A to both sides of (8.4) and using the fact that $Av^{(j)} = \lambda_j v^{(j)}$, we get

$$\begin{gathered} \alpha_1 A v^{(1)} + \alpha_2 A v^{(2)} + \ldots + \alpha_k A v^{(k)} = 0 \qquad \text{or} \\ \alpha_1 \lambda_1 v^{(1)} + \alpha_2 \lambda_2 v^{(2)} + \ldots + \alpha_k \lambda_k v^{(k)} = 0. \end{gathered} \tag{8.5}$$

Since at most one λ can equal 0, we may assume that $\lambda_1 \neq 0$. Then we divide (8.5) by λ_1 to obtain

$$\alpha_1 v^{(1)} + \alpha_2 (\lambda_2/\lambda_1) v^{(2)} + \ldots + \alpha_k (\lambda_k/\lambda_1) v^{(k)} = 0. \tag{8.6}$$

Then subtracting (8.6) from (8.4), we get

$$\alpha_2 (1 - \lambda_2/\lambda_1) v^{(2)} + \ldots + \alpha_k (1 - \lambda_k/\lambda_1) v^{(k)} = 0$$

which because $\lambda_j/\lambda_1 \neq 1$ implies that $\alpha_2 = \ldots = \alpha_k = 0$ by the induction hypothesis. Hence, also $\alpha_1 = 0$ by (8.4) and the result is established.