§2.4 Method for Solving Exact Equations

1. If $Mdx + Ndy = 0$ is exact (that is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$), then $\frac{\partial F}{\partial x} = M$. Integrate this last equation with respect to $x$ to get

$$F(x, y) = \int M(x, y)dx + g(y).$$  \hspace{1cm} (1)

2. To determine $g(y)$, take the partial derivative with respect to $y$ of both sides of equation (1) and use

$$\frac{\partial F}{\partial y} = N.$$

3. Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (1) gives $F(x, y)$.

4. The solution to $Mdx + Ndy = 0$ is given implicitly by

$$F(x, y) = C.$$  \hspace{1cm} (Alternatively, starting with $\frac{\partial F}{\partial y} = N$, the implicit solution can be found by first integrating with respect to $y$ to get

$$F(x, y) = \int N(x, y)dy + h(x)$$

and then proceed to find $h(x)$.)

§2.6 Substitutions and Transformations

1. **Homogeneous equation:** If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio $\frac{y}{x}$ alone, the we say the equation is homogeneous. Use substitution $v = \frac{y}{x}$ to solve it.
2. **Equations of the form** \( \frac{dy}{dx} = G(ax + by) \): Use substitution \( v = ax + by \) to solve it.

3. **Bernoulli equation**: A first-order equation that can be written in the form

\[
\frac{dy}{dx} + P(x)y = Q(x)y^n,
\]

where \( P(x) \) and \( Q(x) \) are continuous on an interval \((a, b)\) and \( n \) is a real number, is called a *Bernoulli equation*. Use substitution \( v = y^{1-n} \) to solve it.

4. **Equations with linear coefficients**:

\[
(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0,
\]  

(2)

where \( a_i, b_i, c_i, i = 1, 2 \) are constants. If \( a_1 b_2 \neq a_2 b_1 \), then we seek a transformation of axes of the form

\[
x = u + h, \quad \text{and} \quad y = v + k,
\]

where \( h \) and \( k \) satisfy

\[
a_1 h + b_1 k + c_1 = 0, \quad a_2 h + b_2 k + c_2 = 0.
\]

Remark: If \( b_1 = a_2 \), then equation (2) is exact. Since

\[
d \left( \frac{a_1}{2} x^2 + b_1 xy + c_1 x + \frac{b_2}{2} y^2 + c_2 y \right)
\]

\[
= (a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0,
\]

the solution to equation (2) is

\[
\frac{a_1}{2} x^2 + b_1 xy + c_1 x + \frac{b_2}{2} y^2 + c_2 y = C.
\]

§4.2-4.3 Homogeneous Linear Second Order Equations

**Definition**: The *auxiliary equation* (or *characteristic equation*) associated with the homogeneous second-order constant-coefficient differential equation

\[
ay'' + by' + cy = 0
\]

(3)

is given by

\[
ar^2 + br + c = 0.
\]

(4)
1. If the auxiliary equation (4) has distinct real roots $r_1$ and $r_2$, then both $y_1(t) = e^{r_1t}$ and $y_1(t) = e^{r_2t}$ are solutions to (3) and $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$ is a general solution.

2. If the auxiliary equation (4) has a repeated (or double) root $r$, then both $y_1(t) = e^{rt}$ and $y_1(t) = te^{rt}$ are solutions to (3) and $y(t) = c_1e^{rt} + c_2te^{rt}$ is a general solution.

3. If the auxiliary equation (4) has complex conjugate roots $\alpha \pm i\beta$, then two linear independent solutions to (3) are $e^{\alpha t}\cos \beta t$ and $e^{\alpha t}\sin \beta t$, and a general solution is $y(t) = c_1e^{\alpha t}\cos \beta t + c_2e^{\alpha t}\sin \beta t$ where $c_1$ and $c_2$ are arbitrary constants.

§4.4 Nonhomogeneous Equations: The Method of Undetermined Coefficients

**Method of Undetermined Coefficients:**

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt},$$

use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt},$$

with

1. $s = 0$ if $r$ is not a root of the associated auxiliary equation;
2. $s = 1$ if $r$ is a simple root of the associated auxiliary equation;
3. $s = 2$ if $r$ is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{\alpha t}\cos \beta t,$$

or

$$ay'' + by' + cy = Ct^m e^{\alpha t}\sin \beta t,$$

use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos \beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin \beta t,$$

with

1. $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation;
2. $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.