Finite Dimensional Irreducible Representations of Elementary Unitary Lie Algebras over Quantum Tori

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December 31, 2013

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Outlines

Matrix Lie Algebras Coordinated By Associative Algebras

Quantum Tori

Finite dimensional irreducible representations of \( \mathfrak{eu}_n(\mathbb{C}_q, -) \)
Ideals of \( \mathfrak{eu}_n(\mathbb{R}, -) \)
Involuntary Ideals of \( \mathbb{C}_q \) I
Construction of irreducible representations
Finite dimensional irreducible representations of \( \mathfrak{eu}_n(\mathbb{C}_q, -) \)
Classical Lie algebras over $\mathbb{C}$

- $\mathbb{C}$: the field of complex numbers
- $\mathfrak{gl}_n(\mathbb{C})$: the Lie algebra of $n \times n$ matrices with entries in $\mathbb{C}$

### Classical Lie algebras

<table>
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<th>Lie Algebra</th>
<th>Description</th>
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<td>$A_\ell, (\ell \geq 1)$</td>
<td>$\mathfrak{sl}<em>{\ell+1}(\mathbb{C}) := { X \in \mathfrak{gl}</em>{\ell+1}(\mathbb{C})</td>
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<tr>
<td>$B_\ell, (\ell \geq 2)$</td>
<td>$\mathfrak{so}<em>{2\ell+1}(\mathbb{C}) := { X \in \mathfrak{gl}</em>{2\ell+1}(\mathbb{C})</td>
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<tr>
<td>$C_\ell, (\ell \geq 3)$</td>
<td>$\mathfrak{sp}<em>{2\ell}(\mathbb{C}) := { X \in \mathfrak{gl}</em>{2\ell}(\mathbb{C})</td>
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<td>$D_\ell, (\ell \geq 4)$</td>
<td>$\mathfrak{so}<em>{2\ell}(\mathbb{C}) := { X \in \mathfrak{gl}</em>{2\ell}(\mathbb{C})</td>
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Where \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \dagger = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \) is the symplectic involution.
Notations:

- $k$: a field of characteristic zero.
- $R$: a unital associative algebra over a field $k$.
- $M_m(R)$: the associative algebra of $n \times n$ matrices with entries in $R$.
- $\mathfrak{g}l_n(R)$: the Lie algebra of $n \times n$ matrices with entries in $R$.
- $e_{ij}(a) \in \mathfrak{g}l_n(R)$: the unit matrix with $a$ at the $(i,j)$ position and 0 elsewhere.
- $E_{ij}(a) \in M_m(R)$: the unit matrix with $a$ at the $(i,j)$ position and 0 elsewhere.
The special linear Lie algebra

**Definition**

\[ \mathfrak{sl}_n(R) := \{ X \in \mathfrak{gl}_n(R) | Tr(X) \in [R, R] \} \]

= the Lie subalgebra of \( \mathfrak{gl}_n(R) \) generated by

\[ \{ e_{ij}(a), a \in R, 1 \leq i \neq j \leq n \} \]

**Examples**

- If \( R \) is commutative, then \( \mathfrak{sl}_n(R) \cong \mathfrak{sl}_n(k) \otimes_k R \).
- If \( R = M_m(k) \), then

\[ \mathfrak{sl}_n(M_m(k)) \cong \mathfrak{sl}_{mn}(k). \]
Unitary Lie algebras I

- An anti-involution $\bar{\cdot}$ on $R$ is a $k$–linear map $\bar{\cdot} : R \to R$ satisfying
  \[ \bar{ab} = \bar{b}\bar{a}, \text{ and } \bar{a} = a \]
  for all $a, b \in R$.

- The unitary Lie algebra:
  \[ u_n(R, -) = \{ X \in \mathfrak{gl}_n(R) | X^T = -X \} \]

- The elementary unitary Lie algebra: $\mathfrak{eu}_n(R, -)$ is the Lie subalgebra of $\mathfrak{gl}_n(R)$ generated by
  \[ \{ e_{ij}(a) - e_{ji}(\bar{a}) | 1 \leq i \neq j \leq n, a \in R \} \]
Unitary Lie algebras II

There is an exact sequence of Lie algebras

\[ 0 \rightarrow \mathfrak{eu}_n(R, -) \rightarrow \mathfrak{u}_n(R, -) \rightarrow \frac{R_-}{[R, R] \cap R_-} \rightarrow 0, \]

where \( R_- = \{ a \in R | \bar{a} = -a \} \).

Examples

If \( R \) is commutative, then the identity map is an anti-involution

\[ \mathfrak{eu}_n(R, \text{id}) \cong \mathfrak{so}_n(k) \otimes_k R. \]
Examples

- If $S$ is a unital associative algebra over $k$, then $S \oplus S^{op}$ is a unital associative algebra over $k$ with the anti-involution $ex(a, b) = (b, a)$.

\[ eu_n(S \oplus S^{op}, ex) \cong sl_n(S). \]

- $eu_n(M_m(k), trs) \cong so_{mn}(k)$.

- $eu_n(M_{2m}(k), syp) \cong sp_{2mn}(k)$. 
Motivation for studying matrix Lie algebras I:
Central extensions, cyclic and dihedral homology

- The Lie algebra $\mathfrak{sl}_n(S)$ fits into the exact sequence

\[ 0 \to HC_1(S) \to \mathfrak{st}_n(S) \to \mathfrak{sl}_n(S) \to 0, \]

where $HC_1(S)$ is the 1st cyclic homology of $S$ and $\mathfrak{st}_n(S)$ is the universal central extension of $\mathfrak{sl}_n(S)$, which is called the Steinberg Lie algebra over $S$.

- The Lie algebra $\mathfrak{eu}_n(R, -)$ fits into the exact sequence

\[ 0 \to _\mathbb{H}D_1(R) \to \mathfrak{stu}_n(R, -) \to \mathfrak{eu}_n(R, -) \to 0, \]

where $\mathbb{H}D_1(R)$ is the 1st dihedral homology of $R$ and $\mathfrak{stu}_n(R, -)$ is the universal central extension of $\mathfrak{eu}_n(R, -)$, which is called the Steinberg unitary Lie algebra over $(R, -)$. (c.f. [Gao1996])
Motivation for studying matrix Lie algebras II: Lie algebras graded by finite root systems

Let $k$ be a field and $\Delta$ a finite root system with root lattice $Q$. A Lie algebra $\mathfrak{g}$ over $k$ is called $\Delta$–graded if

1. $\mathfrak{g}$ has a $Q$–grading

\[ \mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \]

such that $\mathfrak{g}_\alpha \neq 0$ iff $\alpha \in \Delta \cup \{0\}$.

2. $\mathfrak{g}$ has a Lie subalgebra:

\[ \dot{\mathfrak{g}} := \bigoplus_{\alpha \in Q} \dot{\mathfrak{g}}_\alpha \subseteq \mathfrak{g} \]

isomorphic to the finite-dimensional split-simple Lie algebra of type $\Delta$ over $k$ such that $\dot{\mathfrak{g}}_\alpha \subseteq \mathfrak{g}_\alpha$.

3. $\text{ad}_{\mathfrak{g}}h$ acts on $\mathfrak{g}_\alpha$ as the scalar $\alpha(h)$, $\forall h \in \dot{\mathfrak{g}}_0$, $\alpha \in \Delta \cup \{0\}$.

4. $\mathfrak{g}$ is generated by $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$. 
Motivation for studying matrix Lie algebras II:
Lie algebras graded by finite root systems

Theorem (Berman-Moody, 1992)
Let $\Delta$ be a root system of type $A_\ell$ with $\ell \geq 3$.
Then every $\Delta$–graded Lie algebra $g$ is centrally isogenous to $\mathfrak{sl}_{\ell+1}(R)$ for some unital associative algebra $R$ over $k$,
i.e., $g$ and $\mathfrak{sl}_{\ell+1}(R)$ have the same universal central extensions.

Remark
A realization of a root graded Lie algebra of type $D_\ell, E_6, E_7, E_8$
(resp. $B_\ell, C_\ell, G_2, F_4$) has been given in [Berman-Moody 1992]
(resp. [Benkart-Zelmanov 1996]). However, these realizations are
not in the form of matrix Lie algebras in general.
Quantum Tori

Let $q = (q_{ij}) \in M_{\nu}(\mathbb{C})$ such that

$$q_{ii} = 1 \text{ and } q_{ji} = q_{ij}^{-1}$$

for $i, j = 1, \ldots, \nu$.

The quantum torus associated to $q$ is the associative $\mathbb{C}$–algebra generated by $x_i^\pm, \cdots, x_\nu^\pm$ subject to the relations:

$$x_i^+ x_i^- = x_i^- x_i^+ = 1, \text{ and } x_i^+ x_j^+ = q_{ij}x_j^+ x_i^+,$$

for $i, j = 1, \ldots, \nu$. 
Motivation for studying quantum tori I
Extended affine Lie algebras of type $A_\ell$

An extended affine Lie algebra is a pair $(E, H)$ consisting of a Lie algebra $E$ over $k$ and a finite-dimensional abelian subalgebra $H$ satisfying:

(EA1) $E$ has an invariant, non-degenerate bilinear form.

(EA2) $E = \bigoplus_{\alpha \in H^*} E_\alpha$ such that $E_0 = H$.

$$R := \{ \alpha \in H^* : E_\alpha \neq 0 \}, \quad R^{an} = \{ \alpha \in R | (\alpha, \alpha) \neq 0 \}, \quad R^o = \{ \alpha \in R | (\alpha, \alpha) = 0 \}.$$

(EA3) If $\alpha \in R^{an}$, then $\text{ad}E_\alpha$ is locally nilpotent.

(EA4) $R^{an}$ is connected.

(EA5) $E_c := \langle E_\alpha : \alpha \in R^{an} \rangle_{\text{subalg}}$ is centrally closed.

(EA6) $\text{span}_\mathbb{Z}(R^o)$ is a free abelian group of finite rank $\nu$, which is called the nullity of $(E, H)$. 
Motivation for studying quantum tori I
Extended affine Lie algebras of type $A_\ell$

Theorem (Berman-Gao-Krylyuk, 1996)
Let $(E, H)$ be an extended affine Lie algebra such that
$R = \{ \alpha \oplus \lambda | \alpha \in \Delta \cup \{0\}, \lambda \in \mathbb{Z}^\nu \}$,
where $\Delta$ is a finite root system of type $A_\ell$.
Then its core $E_c$ a central extension of $\mathfrak{sl}_{\ell+1}(\mathbb{C}_q[x_1^\pm, \cdots, x_\nu^\pm])$ for
some quantum torus $\mathbb{C}_q[x_1^\pm, \cdots, x_\nu^\pm]$. 
Motivation for studying quantum tori II
$q$-Virasoro-like algebra

We consider the quantum tori in variables:

\[ \mathbb{C}_q := \mathbb{C}_q[x^\pm, y^\pm], \]

satisfying

\[ yx = qxy. \]

The Lie algebra $\mathfrak{gl}_1(\mathbb{C}_q)$ has a basis \{${x^i y^j | i, j \in \mathbb{Z}}$\} satisfying

\[ [x^i y^j, x^k y^l] = (q^{jk} - q^{il}) x^{i+k} y^{j+l} \]

for $i, j, k, l \in \mathbb{Z}$. This Lie algebra is called the $q$-Virasoro-like algebra.
Properties of $\mathbb{C}_q$

We only consider $\mathbb{C}_q$ in two variable, which has an anti-involution – given by
\[ \bar{x} = x, \quad \bar{y} = y^{-1}. \]

Properties of $\mathbb{C}_q$

- The center of $\mathbb{C}_q$ is
\[
Z(\mathbb{C}_q) = \begin{cases} 
\mathbb{C}, & \text{if } q \text{ is not a root of unity.} \\
\bigoplus_{m|i,m|j} \mathbb{C}x^i y^j, & \text{if } q \text{ is an } m\text{-th root of unity.}
\end{cases}
\]

- $\mathbb{C}_q = [\mathbb{C}_q, \mathbb{C}_q] \oplus Z(\mathbb{C}_q)$.

- $\mathbb{C}_q$ is simple (as an associative algebra) iff $Z(\mathbb{C}_q) = \mathbb{C}$ iff $q$ is not a root of unity.
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Finite dimensional irreducible representations of $\mathfrak{eu}_n(\mathbb{C}_q, -)$
Finite dimensional representation of Lie algebras

Facts:
Let $V$ be a finite dimensional $\mathbb{C}$–vector space and $\mathfrak{g} \subseteq \text{gl}(V)$ a Lie subalgebra.
If the inclusion $\mathfrak{g} \subseteq \text{gl}(V)$ is an irreducible representation of $V$, then $\mathfrak{g}$ is reductive with $\dim Z(\mathfrak{g}) \leq 1$.
If in addition $\mathfrak{g} \subseteq \text{sl}(V)$, then $\mathfrak{g}$ is semi-simple.
Finite dimensional irreducible representations of $\mathfrak{eu}_n(\mathbb{C}_q, -)$

Let $\rho : \mathfrak{eu}_n(\mathbb{C}_q, -) \to \mathfrak{gl}(V)$ be a finite dimensional irreducible representation of $\mathfrak{eu}_n(\mathbb{C}_q, -)$. Then $\ker \rho$ is an ideal of $\mathfrak{eu}_n(\mathbb{C}_q, -)$ and $\mathfrak{eu}_n(\mathbb{C}_q, -)/\ker \rho$ acts on $V$ irreducibly.

**Proposition**

For $n \geq 3$, the Lie algebra $\mathfrak{eu}_n(\mathbb{C}_q, -)$ is perfect.

**Conclusion:**

The Lie algebra $\mathfrak{eu}_n(\mathbb{C}_q, -)/\ker \rho$ is finite-dimensional semi-simple.

Hence, our problem is reduced to find ideals $I$ of $\mathfrak{eu}_n(\mathbb{C}_q, -)$ such that $\mathfrak{eu}_n(\mathbb{C}_q, -)/I$ is semi-simple and finite-dimensional.
Well-known Fact
If $R$ is commutative, then every ideal $\mathfrak{P}$ of $\mathfrak{e}u_n(R, -) \cong \mathfrak{e}u_n(k) \otimes_k R$ for $n \geq 5$ is of the form $\mathfrak{e}u_n(k) \otimes a$ for some ideal $a \subseteq R$.

From now on, assume $R$ is non-commutative.

Notations:
For a subset $S \subseteq R$, we denote

\[
I(S) := \left\{ sI_n | s \in S \right\},
\]
\[
H_i(S) := \left\{ e_{ii}(s) - e_{i+1,i+1}(s) | s \in S \right\},
\]
\[
\xi_{ij}(S) := \left\{ e_{ij}(s) - e_{ji}(\bar{s}) | s \in S \right\}.
\]
I!deals of $\mathfrak{e}u_n(R, -)$ II

Definition
A two-sided ideal $\alpha$ of $R$ is involutory if $\bar{r} \in \alpha$ for any $r \in \alpha$.

Proposition (Zheng-Gao-Chang 2011)
For every involutory ideal $\alpha$ of $R$, we can create three ideals of $\mathfrak{e}u_n(R, -)$ for $n \geq 5$:

$\overline{I}(\alpha) := I(\hat{\alpha} \cap [R, R] \cap R_{-}) \oplus \bigoplus_{i=1}^{n-1} H_i(\alpha \cap R_{-}) \oplus \bigoplus_{i<j} \xi_{ij}(\alpha)$,

$\mathcal{I}(\alpha) := I(\alpha \cap [R, R] \cap R_{-}) \oplus \bigoplus_{i=1}^{n-1} H_i(\alpha \cap R_{-}) \oplus \bigoplus_{i<j} \xi_{ij}(\alpha)$,

$I(\alpha) := I([R, \alpha] \cap R_{-}) \oplus \bigoplus_{i=1}^{n-1} H_i(\alpha \cap R_{-}) \oplus \bigoplus_{i<j} \xi_{ij}(\alpha)$,

where $\hat{\alpha} = \{ r \in R | [r, R] \subseteq \alpha \}$. In addition,

$\overline{I}(\alpha) \subseteq \mathcal{I}(\alpha) \subseteq I(\alpha)$. 
Proposition (Zheng-Gao-Chang 2011)

Let \( \mathfrak{P} \) be an ideal of \( \mathfrak{e}u_n(R, -) \), \( n \geq 5 \). Then

- \( \mathfrak{a} := \{ r \in R | e_{ij}(r) - e_{ji}(\bar{r}) \} \), independent of \( (i, j) \) with \( i \neq j \), is an involutory ideal of \( (R, -) \).
- \( \mathfrak{P} \) has a decomposition as a direct sum of \( k \)-vector spaces:
  \[
  \mathfrak{P} = (\mathfrak{P} \cap I([R, R] \cap R_-)) \oplus \bigoplus_{i=1}^{n-1} H_i(\mathfrak{a} \cap R_-) \oplus \bigoplus_{i<j} \xi_{ij}(\mathfrak{a}).
  \]
- \( \mathfrak{I}(\mathfrak{a}) \subseteq \mathfrak{P} \subseteq \overline{\mathfrak{I}}(\mathfrak{a}) \).
Proposition
Let $a$ be an involutory ideal of $R$ and $n \geq 5$. Then

$$\mathfrak{e}u_n(R, -)/\mathfrak{I}(a) \cong \mathfrak{e}u_n(R/a, -).$$

Corollary
If $q$ is not a root of unity,
$\mathbb{C}q$ is a simple associative algebra,
and hence $\mathfrak{e}u_n(\mathbb{C}q, -), n \geq 5$ is a simple Lie algebra of infinite dimension.
Therefore, $\mathfrak{e}u_n(\mathbb{C}q, -)$ has no non-trivial finite dimensional irreducible representation.
Involutory Ideals of $\mathbb{C}_q$

We assume $q$ is a primitive $m$-th root of unity.

**Lemma**

For $\alpha, \beta \in \mathbb{C}^*$, the two-sided ideal

$$J(\alpha, \beta) := \begin{cases} 
\langle x^m - \alpha, y^m - \beta \rangle, & \text{if } \beta = \pm 1, \\
\langle x^m - \alpha, (y^m - \beta)(y^m - \beta^{-1}) \rangle, & \text{if } \beta \neq \pm 1,
\end{cases}$$

is an involutory ideal of $(\mathbb{C}_q, -)$.

Moreover,

$$(\mathbb{C}_q/J(\alpha, \beta), -) \cong \begin{cases} 
(M_m(\mathbb{C}), \text{trs}), & \text{if } \beta = \pm 1, \\
(M_m(\mathbb{C}) \oplus M_m(\mathbb{C})^{\text{op}}, \text{ex}), & \text{if } \beta \neq \pm 1.
\end{cases}$$
Sketch of the proof I

For $\beta = \pm 1$, consider the homomorphism $\mathbb{C}_q \to M_m(\mathbb{C})$:

\[ x \mapsto \alpha^{\frac{1}{m}} \begin{pmatrix} 1 & q & \cdots & q^{m-1} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix} \]

\[ y \mapsto \beta^{\frac{1}{m}} \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ \cdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 1 & 0 \end{pmatrix} \]
Sketch of the proof II

For $\beta \neq \pm 1$, consider the homomorphism
\[ \mathbb{C}_q \to (M_m(\mathbb{C}) \oplus M_m(\mathbb{C})^{\text{op}}) : \]

\[
x \mapsto \begin{pmatrix} \frac{1}{\alpha^m} \begin{pmatrix} 1 \\ q \\ \vdots \\ q^{m-1} \end{pmatrix}, \alpha^m \begin{pmatrix} 1 \\ q \\ \vdots \\ q^{m-1} \end{pmatrix} \\ \frac{1}{\beta^m} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & & \cdots & 0 \end{pmatrix}, \beta^m \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix} \end{pmatrix},
\]

\[
y \mapsto \begin{pmatrix} \frac{1}{\alpha^m} \begin{pmatrix} 1 \\ q \\ \vdots \\ q^{m-1} \end{pmatrix}, \alpha^m \begin{pmatrix} 1 \\ q \\ \vdots \\ q^{m-1} \end{pmatrix} \\ \frac{1}{\beta^m} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix}, \beta^m \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix} \end{pmatrix}.
\]
Involutory Ideals of $\mathbb{C}_q$ II

Lemma

Let

- $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ be distinct,
- $\beta_1, \ldots, \beta_s \in \mathbb{C}\setminus\{0, 1\}$ with $\beta_1, \ldots, \beta_s, \beta_1^{-1}, \ldots, \beta_s^{-1}$ distinct,
- $\epsilon_1, \epsilon_2 \in \{0, 1\}$.

We define

$$f(x^m) := (x^m - \alpha_1) \cdots (x^m - \alpha_r),$$
$$g(y^m) := (y^m - 1)^{\epsilon_1}(y^m + 1)^{\epsilon_2}(y^m - \beta_1)(y^m - \beta_1^{-1}) \cdots (y^m - \beta_s)(y^m - \beta_s^{-1}).$$

Then $(\mathbb{C}_q/\langle f(x^m), g(y^m) \rangle, -)$ is isomorphic to

$$(M_m(\mathbb{C}), \text{trs}) \oplus r(\epsilon_1 + \epsilon_2) \oplus (M_m(\mathbb{C}) \oplus M_m(\mathbb{C})^\text{op}, \text{ex}) \oplus rs$$
Construction of irreducible representations

- Let $k, l$ be two non-negative integer such that $k + l > 0$.
- Let $V$ be a finite dimensional irreducible representation of $\mathfrak{eu}_{mn}(\mathbb{C}) \oplus k \oplus \mathfrak{sl}_{mn}(\mathbb{C}) \oplus l$.
- Let $f(x^m), g(y^m)$ as in the previous lemma such that $r(\epsilon_1 + \epsilon_2) \geq k$ and $rs \geq l$.

Then

$$
\mathfrak{eu}_n(\mathbb{C}_q, -) \rightarrow \mathfrak{eu}_n\left(\frac{\mathbb{C}_q}{\langle f(x^m), g(y^m) \rangle}, -\right) \rightarrow \mathfrak{eu}_{mn}(\mathbb{C}) \oplus k \oplus \mathfrak{sl}_{mn}(\mathbb{C}) \oplus l \rightarrow \mathfrak{gl}(V)
$$

$$
\mathfrak{eu}_{mn}(\mathbb{C}) \oplus r(\epsilon_1 + \epsilon_2) \oplus \mathfrak{sl}_{mn}(\mathbb{C}) \oplus rs
$$

is a finite dimensional irreducible representation of $\mathfrak{eu}_n(\mathbb{C}_q, -)$. 
Finite dimensional irreducible representations of $\mathfrak{eu}_n(C_q, -)$

**Theorem (Chang-Gao-Zheng 2013)**

Let $\rho : \mathfrak{eu}_n(C_q, -) \to \mathfrak{gl}(V)$ be a finite dimensional irreducible representation of $\mathfrak{eu}_n(C_q, -)$. Then there are non-negative integer $k, l$ with $(k, l) \neq (0, 0)$ such that $\rho$ factors through

$$
\begin{array}{c}
\mathfrak{eu}_n(C_q, -) \\
\rho \\
\text{eu}_{mn}(C) \oplus k \oplus \mathfrak{sl}_{mn}(C) \oplus l \\
\bar{\rho} \\
\mathfrak{gl}(V)
\end{array}
$$

where $\bar{\rho}$ is a finite dimensional irreducible representation of $\mathfrak{eu}_{mn}(C) \oplus k \oplus \mathfrak{sl}_{mn}(C) \oplus l$. 
Sketch of the proof I

\[ \rho \text{ factors through} \]

\[ \mathfrak{eu}_n(\mathbb{C}_q, -) \xrightarrow{\rho} \mathfrak{gl}(V) \]

\[ \mathfrak{eu}_n(\mathbb{C}_q, -) / \ker \rho \]

and \( \mathfrak{eu}_n(\mathbb{C}_q, -) / \ker \rho \) is finite-dimensional and semi-simple.
Sketch of the proof II

There are

\[ f(x^m) := (x^m - \alpha_1)^{p_1} \cdots (x^m - \alpha_r)^{p_r} \in \mathbb{C}_q, \]
\[ g(y^m) := (y^m - 1)^{q_0} (y^m + 1)^{q'_0} (y^m - \beta_1)^{q_1} (y^m - \beta_1^{-1})^{q_1} \cdots (y^m - \beta_s)^{q_s} (y^m - \beta_s^{-1})^{q_s} \in \mathbb{C}_q, \]

such that

\[ \ker \rho \supseteq \mathcal{I}(a) = \mathcal{I}(a) \]

for \( a = \langle f(x^m), g(y^m) \rangle. \)
Sketch of the proof III

Let

\[ \tilde{f}(x^m) := (x^m - \alpha_1) \cdots (x^m - \alpha_r) \in \mathbb{C}_q, \]
\[ \tilde{g}(y^m) := (y^m - 1)^{\epsilon_1} (y^m + 1)^{\epsilon_2} (y^m - \beta_1)(y^m - \beta_1^{-1}) \cdots (y^m - \beta_s)(y^m - \beta_s^{-1}) \in \mathbb{C}_q, \]

and

\[ \tilde{a} = \langle \tilde{f}(x^m), \tilde{g}(y^m) \rangle. \]

Then

- \( \mathcal{I}(\tilde{a})/\mathcal{I}(\tilde{a}) \) is a nilpotent ideal of \( \mathfrak{e}u_n(\mathbb{C}_q, -)/\mathcal{I}(\mathfrak{a}) \cong \mathfrak{e}u_n(\mathbb{C}_q/\tilde{a}, -) \).
- \( \frac{\mathfrak{e}u_n(\mathbb{C}_q, -)/\mathcal{I}(\mathfrak{a})}{\mathcal{I}(\tilde{a})/\mathcal{I}(\tilde{a})} \cong \mathfrak{e}u_n(\mathbb{C}_q/\tilde{a}, -) \) is semi-simple.

Hence, we conclude that \( \mathcal{I}(\tilde{a})/\mathcal{I}(\tilde{a}) \) is the radical of \( \mathfrak{e}u_n(\mathbb{C}_q, -)/\mathcal{I}(\mathfrak{a}) \).
Sketch of the proof IV

Since

$$\text{eu}_n(\mathbb{C}_q, -)/\ker \rho \cong \frac{\text{eu}_n(\mathbb{C}_q, -)/\mathcal{I}(a)}{\ker \rho/\mathcal{I}(a)}$$

is semisimple, we conclude

$$\ker \rho \supseteq \mathcal{I}(\tilde{a}).$$

Hence, \(\text{eu}_n(\mathbb{C}_q, -)/\ker \rho\) is a quotient of

$$\text{eu}_n(\mathbb{C}_q, -)/\mathcal{I}(\tilde{a}) \cong \text{eu}_{mn}(\mathbb{C}) \oplus r(\epsilon_1 + \epsilon_2) \oplus \mathfrak{sl}_{mn}(\mathbb{C}) \oplus rs.$$
Final Comments

Why do we consider only $n \geq 5$?

- $\mathfrak{eu}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.
- $\mathfrak{eu}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$.
- $\mathfrak{eu}_2(\mathbb{C})$ is a one-dimensional abelian Lie algebra.
Reference


Thank You!