Automorphisms and Twisted Forms of Differential Conformal Superalgebras

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Outline

Affine Kac-Moody algebras

Lie algebras and Lie conformal algebras

Differential conformal algebras and their forms

The $N = 1, 2, 3$ conformal superalgebras

The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
Generalized Cartan matrices

A matrix $A = (a_{ij}) \in M_n(\mathbb{Z})$ is called a Generalized Cartan Matrix if

- $a_{ii} = 2$,
- $a_{ij} \leq 0$ if $i \neq j$,
- $a_{ij} = 0$ if and only if $a_{ji} = 0$. 
Classification of generalized Cartan matrices

A is decomposable if there is $\sigma \in S_n$ such that

$$(a_{\sigma(i),\sigma(j)}) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$ 

Otherwise, we say $A$ is indecomposable.
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Facts:
Let $A$ be an indecomposable GCM. Then exactly one of the following statements holds:

- There is $v \in \mathbb{R}^n_{>0}$ such that $Av > 0$. (Finite Type)
- There is $v \in \mathbb{R}^n_{>0}$ such that $Av = 0$. (Affine Type)
- There is $v \in \mathbb{R}^n_{>0}$ such that $Av < 0$. (Indefinite Type)
Definition of KM algebra

Let \( A \) be a generalized Cartan matrix. Then we define \( \mathcal{L}(A) \) to be the Lie algebra generated by \( \{ h_i, e_i, f_i \mid i = 1, \ldots, n \} \) subject to the relations:

\[
\begin{align*}
[h_i, h_j] &= 0, \\
[h_i, e_j] &= a_{ij} e_j, \\
[h_i, f_j] &= -a_{ij} f_j, \\
\text{ad}(e_i)^{1-a_{ij}}(e_j) &= 0, \\
\text{ad}(f_i)^{1-a_{ij}}(f_j) &= 0,
\end{align*}
\]

for \( i, j = 1, \ldots, n \).
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\[
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad \text{ad}(e_i)^{1-a_{ij}}(e_j) = 0, \quad \text{ad}(f_i)^{1-a_{ij}}(f_j) = 0,
\]

for $i, j = 1, \ldots, n$.

Facts:

- the set of equivalent classes of GCM of finite type
- the isomorphism classes of finite dimensional simple complex Lie algebras
Twisted loop realization of affine KM algebras

Let \( g \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) and \( \sigma \) an automorphism of \( g \) of order \( m \). Then

\[
g = \bigoplus_{\ell=1}^{m} g_{\ell},
\]

where \( g_{\ell} = \{ x \in g \mid \sigma(x) = \zeta_{m}^\ell x \} \), \( \zeta_{m} = e^{\frac{2\pi i}{m}} \).
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One can construct a Lie algebra

\[
L(g, \sigma) = \bigoplus_{\ell \in \mathbb{Z}} g_{\ell} \otimes \mathbb{C} t^{\frac{\ell}{m}}.
\]
Twisted loop realization of affine KM algebras

- Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) and \( \sigma \) an automorphism of \( \mathfrak{g} \) of order \( m \). Then

\[
\mathfrak{g} = \bigoplus_{\ell=1}^{m} \mathfrak{g}_\ell,
\]

where \( \mathfrak{g}_\ell = \{ x \in \mathfrak{g} | \sigma(x) = \zeta_{m}^{\ell} x \} \), \( \zeta_{m} = e^{\frac{2\pi i}{m}} \).

- One can construct a Lie algebra

\[
L(\mathfrak{g}, \sigma) = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_\ell \otimes \mathbb{C} t^{\ell/m}.
\]

- Fact: For an affine GCM \( A \), the Lie algebra \( \mathcal{L}(A) \) is isomorphic to a Lie algebra of the form

\[
L(\mathfrak{g}, \sigma) \oplus \mathbb{C}c.
\]
Affine algebras as twisted forms

Key observations by A. Pianzola and his collaborators:
Affine algebras as twisted forms

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- $L(g, \sigma)$ is a Lie algebra over $D := \mathbb{C}[t^{\pm 1}]$. 
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- $D \hookrightarrow D_m := \mathbb{C}[t^{\pm \frac{1}{m}}]$ is a finite Galois ring extension.
- $L(\mathfrak{g}, \sigma) \otimes_D D_m \cong (\mathfrak{g} \otimes_{\mathbb{C}} D) \otimes_D D_m \cong \mathfrak{g} \otimes_{\mathbb{C}} D_m$, i.e., $L(\mathfrak{g}, \sigma)$ is a $D_m/D$–twisted form of $\mathfrak{g} \otimes_{\mathbb{C}} D = L(\mathfrak{g}, \text{id})$. 

Well known results on twisted forms:

The isomorphism classes of $D_m/D$–twisted form of $\mathfrak{g} \otimes_{\mathbb{C}} D$ bijectively correspond to elements of $H^1(D_m/D, \text{Aut}(\mathfrak{g} \otimes_{\mathbb{C}} D))$. 

$N = 1, 2, 3$

small $N = 4$

large $N = 4$
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Definition of $H^1$

Let

- $R \rightarrow S$ a faithfully flat ring extension, and
- $G$ a functor from the category of commutative rings to the category of groups.

Consider the natural maps

$$d^i : G(S) \rightarrow G(S \otimes_R S),$$

for $i = 1, 2$, and

$$d^{ij} : G(S \otimes_R S) \rightarrow G(S \otimes_R S \otimes_R S),$$

for $(i, j) = (1, 2), (1, 3), (2, 3)$. 

Definition of $H^1$

A 1-cocycle is $\mathcal{z} \in \mathbf{G}(S \otimes_R S)$ such that

$$d^{13}(\mathcal{z}) = d^{23}(\mathcal{z})d^{12}(\mathcal{z}).$$

Two 1-cocycles $\mathcal{z}$ and $\mathcal{z}'$ are equivalent if there is $a \in \mathbf{G}(S)$ such that

$$\mathcal{z}' = d^2(a)\mathcal{z}(d^1(a))^{-1}.$$

$H^1(S/R, \mathbf{G})$ is the set

$$\left\{ \text{1-cocycles} \right\}_{\text{equivalence}}.$$
Researches motivated by the above viewpoint

- $H^1_{\text{ét}}(D, G)$ for a reductive group scheme $G$ over $D$.
  c.f. [P 2004], [GP 2007-2008], [CGP2012], [GP2013].
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- Application of descent theory to Lie theory:
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  - finite dimensional irreducible representation:
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- $H^1_{\text{ét}}(D, G)$ for a reductive group scheme $G$ over $D$. c.f. [P 2004], [GP 2007-2008], [CGP2012], [GP2013].
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    c.f. [P 2004], [CGP 2011], [CEGP 2012].
  - finite dimensional irreducible representation:
    c.f. [L 2010], [LP 2013].
  - invariant bilinear form:
    c.f. [NPPS 2013].
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Motivating Example:
The Centreless Virasoro Algebra

\( \mathbb{k} \): an algebraically closed field of characteristic zero
\( \mathfrak{v} \): the centreless Virasoro algebra

- \( \mathfrak{v} \) has a basis \( \{ L_n \mid n \in \mathbb{Z} \} \) satisfying \( [L_m, L_n] = (m - n)L_{m+n} \).
Motivating Example:
The Centreless Virasoro Algebra

$k$: an algebraically closed field of characteristic zero
$v$: the centreless Virasoro algebra

$\triangleright$ $v$ has a basis $\{L_n| n \in \mathbb{Z}\}$ satisfying $[L_m, L_n] = (m - n)L_{m+n}$.

Consider the formal series $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.
The Lie bracket on $v$ yields the OPE

$$[L(z), L(w)] : = \sum_{m,n \in \mathbb{Z}} [L_m, L_n] z^{-m-2} w^{-n-2}$$
$$= (\partial_w L(w)) \delta(z - w) + 2L(w) \partial_w \delta(z - w),$$

where $\delta(z - w) = \sum_{n \in \mathbb{Z}} w^n z^{-n-1}$. 
Lie Algebra $\Rightarrow$ Conformal Algebra

Let $\mathcal{V} := \text{span}_k \{ \partial^\ell_z L(z) | \ell \geq 0 \}$.

- For $a(z), b(z) \in \mathcal{V}$, we have

$$[a(z), b(w)] = \sum_{\ell \geq 0} c_\ell(w) \cdot \partial_w^{(\ell)} \delta(z - w),$$

where $c_\ell(z) \in \mathcal{V}$. 

Summarizing:

$\mathcal{V}$ is a $k[\partial]$-module generated by $L(z)$, equipped with a $\lambda$-bracket on $\mathcal{V}$ given by

$$[L(z) \lambda L(z)] := (\partial z + 2 \lambda) L(z).$$
Let \( \mathcal{V} := \text{span}_k \{ \partial_z^\ell L(z) | \ell \geq 0 \} \).

- For \( a(z), b(z) \in \mathcal{V} \), we have

\[
[a(z), b(w)] = \sum_{\ell \geq 0} c_\ell(w) \cdot \partial_w^{(\ell)} \delta(z - w),
\]

where \( c_\ell(z) \in \mathcal{V} \).

- We can define a product on \( \mathcal{V} \) for each \( \ell \geq 0 \) by

\[
a(z)(\ell)b(z) := c_\ell(z).
\]
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- We can define a product on $\mathcal{V}$ for each $\ell \geq 0$ by

  $$a(z)_{(\ell)} b(z) := c_{\ell}(z).$$

Notation: $\lambda$–bracket

$$[a(z)_{\lambda} b(z)] = \sum_{\ell \geq 0} \lambda^{(\ell)} (a(z)_{(\ell)} b(z)).$$
Lie Algebra $\Rightarrow$ Conformal Algebra

Let $\mathcal{V} := \text{span}_k \{ \partial^\ell_z L(z) | \ell \geq 0 \}$.

- For $a(z), b(z) \in \mathcal{V}$, we have
  \[ [a(z), b(w)] = \sum_{\ell \geq 0} c_\ell(w) \cdot \partial^{(\ell)}_w \delta(z - w), \]
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\[ [a(z)\lambda b(z)] = \sum_{\ell \geq 0} \lambda^{(\ell)} (a(z)(\ell)b(z)). \]

Summarizing: $\mathcal{V}$ is a $k[\partial]$–module generated by $L(z)$, equipped with a $\lambda$–bracket on $\mathcal{V}$ given by

\[ [L(z)\lambda L(z)] := (\partial_z + 2\lambda)L(z). \]
Lie conformal algebras

axiomatic definition

Due to V. G. Kac.

A Lie conformal algebra over \( k \) is a \( k[\partial] \)-module \( \mathcal{A} \) equipped with a \( \lambda \)-bracket

\[
[\lambda] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[\lambda],
\]

satisfying

(C1) \( [(\partial a)\lambda b] = -\lambda[a\lambda b] \),

(C2) \( [b\lambda a] = -[a-\partial-\lambda b] \),

(C3) \( [a\lambda[b\mu c]] = [[a\lambda b]\lambda+\mu c] + [b\mu[a\lambda c]] \),

for all \( a, b, c \in \mathcal{A} \).
Lie conformal algebra $\Rightarrow$ Lie algebra

affinization

$\mathcal{A}$: a Lie conformal algebra over $\mathbb{k}$.

Define a conformal algebra structure on $\mathcal{A} \otimes_{\mathbb{k}} \mathbb{k}[t^{\pm 1}]$ by

$$\hat{\partial}(a \otimes r) = \partial(a) \otimes r + a \otimes \frac{d}{dt}(r)$$

and

$$(a \otimes r)(\ell)(b \otimes s) := \sum_{j \geq 0} (a(\ell+j)b) \otimes (\frac{d}{dt})^{(j)}(r)s,$$

for $a, b \in \mathcal{A}$ and $r, s \in \mathbb{k}[t^{\pm 1}]$.

Terminology: the (untwisted) loop conformal algebra based on $\mathcal{A}$.

Notation: $\mathcal{A} \otimes_{\mathbb{k}} \mathcal{D}$, where $\mathcal{D} = (\mathbb{k}[t^{\pm 1}], \frac{d}{dt})$. 

Lie conformal algebra $\Rightarrow$ Lie algebra

The conformal algebra $\mathcal{A} \otimes_k \mathcal{D}$ determines a Lie algebra

$$\text{Alg}(\mathcal{A}) := (\mathcal{A} \otimes_k \mathcal{D})/\widehat{\partial}(\mathcal{A} \otimes_k \mathcal{D}),$$

with Lie bracket induced by the 0–th product on $\mathcal{A} \otimes_k \mathcal{D}$. 
Twisted loop Lie conformal algebras

Given

- $\mathcal{A}$: a Lie conformal algebra over $\mathbb{k}$
- $\sigma: \mathcal{A} \to \mathcal{A}$ an automorphism of $\mathcal{A}$ of order $m$

We know that

- $\mathcal{A} \otimes_\mathbb{k} \mathcal{D}_m$ is a Lie conformal algebra over $\mathbb{k}$, where $\mathcal{D}_m = (\mathbb{k}[t^{\pm 1/m}], \frac{d}{dt})$.

\[
\mathcal{A} = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell,
\]

where $\mathcal{A}_\ell = \{ a \in \mathcal{A} | \sigma(a) = \zeta_\ell^m a \}$ and $\zeta_m = e^{\frac{2\pi i}{m}}$.
Twisted loop Lie conformal algebras

Facts:

- The $k$–subspace

$$\mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell \otimes t^\ell k[t^{\pm 1}] \subseteq \mathcal{A} \otimes_k \mathcal{D}_m,$$

is a Lie conformal subalgebra of $\mathcal{A} \otimes_k \mathcal{D}_m$.

(the twisted loop conformal algebra based on $\mathcal{A}$ w.r.t $\sigma$)
Twisted loop Lie conformal algebras

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- The \( k \)-subspace

\[
\mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell \otimes t^\ell_m k[t^{\pm 1}] \subseteq \mathcal{A} \otimes_k \mathcal{D}_m,
\]

is a Lie conformal subalgebra of \( \mathcal{A} \otimes_k \mathcal{D}_m \).

(\text{the twisted loop conformal algebra based on } \mathcal{A} \text{ w.r.t } \sigma)

- \( \mathcal{L}(\mathcal{A}, \sigma) = (\mathcal{A} \otimes \mathcal{D}_m)^\Gamma \),

where \( \Gamma \) is a finite cyclic group of automorphisms of \( \mathcal{A} \otimes \mathcal{D}_m \) generated by

\[
\sigma \otimes \psi : \mathcal{A} \otimes_k \mathcal{D}_m \to \mathcal{A} \otimes_k \mathcal{D}_m,
\]

\[
a \otimes t^\frac{n}{m} \mapsto \sigma(a) \otimes \zeta_m^n t^\frac{n}{m}.
\]

- In particular, \( \mathcal{L}(\mathcal{A}, \text{id}) = \mathcal{A} \otimes_k \mathcal{D} \).
The associated Lie algebras

- The conformal algebra $\mathcal{L}(\mathcal{A}, \sigma)$ determines a Lie algebra

$$\text{Alg}(\mathcal{A}, \sigma) := \mathcal{L}(\mathcal{A}, \sigma) / \hat{\partial} \mathcal{L}(\mathcal{A}, \sigma)$$

with Lie bracket induced by the 0-th product on $\mathcal{L}(\mathcal{A}, \sigma)$. 

Central extensions of Lie superalgebras of this form are indeed the twisted superconformal Lie algebras which appear in physics literature.
The associated Lie algebras

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- Central extensions of Lie superalgebras of this form are indeed the twisted superconformal Lie algebras which appear in physics literature.
Lie algebras ⇐ Lie conformal algebras

Summary:

\[ \mathcal{L}(\mathcal{A}, \text{id}) \quad \text{Alg}(\mathcal{A}, \text{id}) \]

\[ g \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \otimes_k \mathcal{D} \longrightarrow \text{Alg}(\mathcal{A}) \longrightarrow g \]

\[ \mathcal{L}(\mathcal{A}, \sigma) \longrightarrow \text{Alg}(\mathcal{A}, \sigma) \]
Question

Given a conformal algebra $\mathcal{A}$ over $\mathbb{k}$, how can we classify all twisted loop conformal algebras based on $\mathcal{A}$?
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The theory of differential conformal (super)algebras was developed in

[KLP] V. G. Kac, M. Lau, and A. Pianzola,
Differential conformal superalgebras and their forms,
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Key observation I in [KLP]

\( \mathcal{L}(\mathcal{A}, \sigma) \) is not only a conformal algebra over \( k \), but also a differential conformal algebra over \( D := (k[t^{\pm 1}], \frac{d}{dt}) \).
Key observation I in [KLP]

\(\mathcal{L}(\mathcal{A}, \sigma)\) is not only a conformal algebra over \(\mathbb{k}\), but also a differential conformal algebra over \(\mathcal{D} := (\mathbb{k}[t^{\pm 1}], \frac{d}{dt})\).

Let \(\mathcal{R} = (R, d)\) be a \(\mathbb{k}\)–differential ring.
A differential Lie conformal algebra over \(\mathcal{R}\) consists of

- an \(R\)-module \(\mathcal{A}\),
- a \(\mathbb{k}\)–linear operator \(\partial : \mathcal{A} \to \mathcal{A}\) such that
  \[
  \partial(ra) = d(r)a + r\partial(a),
  \]
- a \(\mathbb{k}\)–bilinear product \(-_{(n)}-\) for each \(n \in \mathbb{Z}_+\) satisfying
  \[
  (1) \quad a_{(n)}(rb) = r(a_{(n)}b),
  (2) \quad (ra)_{(n)}b = \sum_{j \in \mathbb{Z}_+} d^{(j)}(r)(a_{(n+j)}b).
  \]
which satisfy axioms (C1), (C2) and (C3).
Terminology: \(\mathcal{R}\)–conformal algebra.
Key observation II in [KLP]

\[ \mathcal{L}(A, \sigma) \] is a \( \mathcal{D}_m/\mathcal{D} \)-form of \( \mathcal{L}(A, \text{id}) \), i.e.,

\[ \mathcal{L}(A, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m \cong \mathcal{L}(A, \text{id}) \otimes_{\mathcal{D}} \mathcal{D}_m \cong A \otimes_{k} \mathcal{D}_m \]

as \( \mathcal{D}_m \)-conformal algebras, where \( \mathcal{D}_m := (k[t^\pm \frac{1}{m}], \frac{d}{dt}) \).
Key observation II in [KLP]

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as \( \mathcal{D}_m \)--conformal algebras, where \( \mathcal{D}_m := (\mathcal{K}[t^{\pm \frac{1}{m}}], \frac{d}{dt}) \).

**Base Change**

an \( \mathcal{R} \)--conformal algebra \( A \)
\[ \mathcal{R} = (R, d_R) \rightarrow S = (S, d_S) \}

\[ \Rightarrow \] an \( S \)--conformal algebra \( A \otimes_{\mathcal{R}} S \)

- the underlying \( S \)--module: \( A \otimes_R S \).
- \( \hat{\partial} := \partial \otimes \text{id} + \text{id} \otimes d_S \)
- the \( n \)-th product

\[ (a \otimes r)_{(n)}(b \otimes s) = \sum_{j \geq 0} (a_{(n+j)}b) \otimes d_S^{(j)}(r)s, \]

for \( a, b \in A, r, s \in S \).
Classification of twisted forms

\( \mathcal{A} \): an \( \mathcal{R} \)–conformal algebra

**Automorphism group functor:**

\[
\text{Aut}(\mathcal{A}) : S \mapsto \text{Aut}_{S\text{-conf}}(\mathcal{A} \otimes \mathcal{R} S).
\]
Classification of twisted forms

\( \mathcal{A} : \text{an } \mathcal{R}\text{–conformal algebra} \)

**Automorphism group functor:**

\[ \text{Aut}(\mathcal{A}) : S \mapsto \text{Aut}_{S\text{-conf}}(\mathcal{A} \otimes_{\mathcal{R}} S). \]

**Theorem (Kac, Lau, Pianzola, 2009)**

*Let \( \mathcal{R} \rightarrow S \) be a faithfully flat extension of \( \mathbb{k}\text{–differential rings}. \) Then*

\[ \text{the set of isomorphism classes of } S/\mathcal{R}\text{–forms of } \mathcal{A} \text{ (up to } \mathcal{R}\text{–isomorphism)} \leftrightarrow H^1(S/\mathcal{R}, \text{Aut}(\mathcal{A})). \]
Classification of twisted loop conformal algebras

Recall: $L(\mathcal{A}, \sigma)$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathcal{A} \otimes_k \mathcal{D}$.

Take

$$\hat{\mathcal{D}} := \lim_{\to} \mathcal{D}_m.$$ 

We know: $L(\mathcal{A}, \sigma)$ is a $\hat{\mathcal{D}}/\mathcal{D}$–form of $\mathcal{A} \otimes_k \mathcal{D}$. 

$N = 1, 2, 3$

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\[ \hat{\mathcal{D}} := \lim_{\rightarrow} \mathcal{D}_m. \]

We know: \( L(\mathcal{A}, \sigma) \) is a \( \hat{\mathcal{D}}/\mathcal{D} \)–form of \( \mathcal{A} \otimes_k \mathcal{D} \).

Fact:

\[ \text{the set of isomorphism classes of twisted loop conformal algebras based on } \mathcal{A} \] (up to \( \mathcal{D} \)–isomorphism)

\[ \longleftrightarrow \]

\[ H^1(\hat{\mathcal{D}}/\mathcal{D}, \text{Aut}(\mathcal{A})) \]
$H^1 \Rightarrow H^1_{ct} \Rightarrow H^1_{\text{ét}}$

**Proposition (Kac, Lau, Pianzola, 2009)**

*If $\mathcal{A}$ is a finitely generated $k[\partial]$–module, then*

$$H^1(\mathcal{D}/\mathcal{D}, \text{Aut}(\mathcal{A})) = H^1_{ct}\left(\hat{\mathbb{Z}}, \text{Aut}_{\mathcal{D}-\text{conf}}(\mathcal{A} \otimes_k \mathcal{D})\right).$$
Proposition (Kac, Lau, Pianzola, 2009)

If $\mathcal{A}$ is a finitely generated $k[\partial]$–module, then

$$H^1(\hat{D}/D, \text{Aut}(\mathcal{A})) = H^1_{ct}(\hat{\mathbb{Z}}, \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{A} \otimes_k \hat{D})).$$

Proposition (Gille, Pianzola, 2008)

Let $G$ be an affine group scheme over $D$. If $G$ is an extension of a twisted finite constant group by a reductive group, then

$$H^1_{ct}(\hat{\mathbb{Z}}, G(\hat{D})) = H^1_{\text{ét}}(D, G).$$
\[ H^1 \Rightarrow H^1_{\text{ct}} \Rightarrow H^1_{\text{ét}} \]

Proposition (Kac, Lau, Pianzola, 2009)

If \( \mathcal{A} \) is a finitely generated \( \mathbb{k}[\partial] \)-module, then

\[ H^1(\hat{D}/D, \text{Aut}(\mathcal{A})) = H^1_{\text{ct}}\left(\hat{\mathbb{Z}}, \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{A} \otimes_{\mathbb{k}} \hat{D})\right). \]

Proposition (Gille, Pianzola, 2008)

Let \( G \) be an affine group scheme over \( D \). If \( G \) is an extension of a twisted finite constant group by a reductive group, then

\[ H^1_{\text{ct}}(\hat{\mathbb{Z}}, G(\hat{D})) = H^1_{\text{ét}}(D, G). \]

Proposition (Pianzola 2005)

Let \( G \) be a reductive group scheme over \( D \). Then \( H^1_{\text{ét}}(D, G) = 1 \).
Centroid trick

**Question:**
Given two twisted loop conformal algebras \( \mathcal{L}(A, \sigma_i), i = 1, 2, \)

\[
\mathcal{L}(A, \sigma_1) \not\cong_D \mathcal{L}(A, \sigma_2) \quad \Rightarrow \quad \mathcal{L}(A, \sigma_1) \not\cong_k \mathcal{L}(A, \sigma_2)
\]
Centroid trick

Question: Given two twisted loop conformal algebras $\mathcal{L}(\mathcal{A}, \sigma_i), i = 1, 2,$

$$\mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_D \mathcal{L}(\mathcal{A}, \sigma_2) \Rightarrow \mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_k \mathcal{L}(\mathcal{A}, \sigma_2)$$

Centroid: the centroid of an $\mathcal{R}$–conformal algebra $\mathcal{B}$ is

$$\text{Ctd}_\mathcal{R}(\mathcal{B}) = \{ \chi \in \text{End}_{\mathcal{R}-\text{mod}}(\mathcal{B}) | \chi(a_{(n)}b) = a_{(n)}\chi(b), \forall a, b \in \mathcal{B}, n \in \mathbb{Z}_+ \},$$
**Centroid trick**

**Question:**
Given two twisted loop conformal algebras $\mathcal{L}(\mathcal{A}, \sigma_i), i = 1, 2,$

$$\mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_D \mathcal{L}(\mathcal{A}, \sigma_2) \implies \mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_k \mathcal{L}(\mathcal{A}, \sigma_2)$$

**Centroid:** the centroid of an $\mathcal{R}$–conformal algebra $\mathcal{B}$ is

$$\text{Ctd}_\mathcal{R}(\mathcal{B}) = \{ \chi \in \text{End}_{\mathcal{R}-\text{mod}}(\mathcal{B}) | \chi(a(n)b) = a(n)\chi(b), \forall a, b \in \mathcal{B}, n \in \mathbb{Z}_+ \}$$

**Proposition (Kac, Lau, Pianzola, 2009)**

*If the canonical maps $D \to \text{Ctd}_k(\mathcal{L}(\mathcal{A}, \sigma_i)), i = 1, 2$ are both $k$–algebra isomorphisms, then*

$$\mathcal{L}(\mathcal{A}, \sigma_1) \cong_D \mathcal{L}(\mathcal{A}, \sigma_2) \iff \mathcal{L}(\mathcal{A}, \sigma_1) \cong_k \mathcal{L}(\mathcal{A}, \sigma_2)$$
Proposition (Chang 2013)

Let

- $\mathcal{A}$ be a $\mathbb{k}$–conformal superalgebra and
- $\sigma$ an automorphism of $\mathcal{A}$ of finite order.

Suppose $\mathcal{A}$ satisfies all of the following conditions:

1. There are $a_1, \ldots, a_{n_0} \in \mathcal{A}$ such that
   - $\mathcal{A}_0 = \mathbb{k}[\partial]a_1 \oplus \cdots \oplus \mathbb{k}[\partial]a_{n_0}$,
   - $L := a_1$ satisfies $[L_\lambda L] = (\partial + 2\lambda)L$ and $\sigma(L) = L$,
   - $[L_\lambda a_i] = (\partial + \lambda)a_i$, for $i = 2, \cdots, n_0$.

2. There are $b_1, \cdots, b_{n_1} \in \mathcal{A}_\bar{1}$ generating $\mathcal{A}_\bar{1}$ as a $\mathbb{k}[\partial]$–module such that $[L_\lambda b_i] = (\partial + \Delta'_i \lambda)b_i$ with $\Delta'_i \neq 0$ for $i = 1, \cdots, n_1$.

Then $\text{Ctd}_\mathbb{k}(\mathcal{L}(\mathcal{A}, \sigma)) = D$. 
Centroid: special case

Corollary

Let $\mathcal{A}$ be one of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras over $k$, and $\mathcal{L}(\mathcal{A}, \sigma)$ be an arbitrary twisted loop conformal superalgebra based on $\mathcal{A}$. Then the canonical map

$$k[t^\pm 1] \to \text{Ctd}_k(\mathcal{L}(\mathcal{A}, \sigma))$$

is an isomorphism.
Summary

Given a conformal superalgebra $\mathcal{A}$ over $k$, the twisted loop conformal superalgebras based on $\mathcal{A}$ can be classified using the following steps:

- Compute the automorphism group

$$\text{Aut}_{\hat{D}\text{-conf}}(\mathcal{A} \otimes_k \hat{D});$$

- Compute the non-abelian cohomology set

$$H^1_{\text{ct}} \left( \hat{\mathbb{Z}}, \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{A} \otimes_k \hat{D}) \right);$$

- Compute the centroid $\text{Ctd}_k(\mathcal{L}(\mathcal{A}, \sigma))$ for all twisted loop conformal superalgebra $\mathcal{L}(\mathcal{A}, \sigma)$. 
Outline

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The $N = 1, 2, 3$ conformal superalgebras

The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
The $N = 1, 2, 3$ Lie Conformal Superalgebras $\mathcal{H}_N$

\[ \mathcal{H}_N := \mathbb{k}[\partial] \otimes_{\mathbb{k}} \Lambda(N), \]

where $\Lambda(N)$ is the exterior superalgebra over $\mathbb{k}$ in $N$–variables $\xi_1, \ldots, \xi_N$.

For $n \in \mathbb{Z}_+$, the $n$-th product on $\mathcal{H}_N$ is given by

\[ f(0)g = \left( \frac{1}{2}|f| - 1 \right) \partial \otimes fg + \frac{1}{2}(-1)^{|f|} \sum_{i=1}^{N} (\partial_i f)(\partial_i g), \]

\[ f(1)g = \left( \frac{1}{2}(|f| + |g|) - 2 \right) fg, \]

\[ f(n)g = 0, \quad n \geq 2, \]

where $f$ and $g$ are monomials in $\xi_1, \ldots, \xi_N$ of degree $|f|$ and $|g|$, respectively.
Automorphism group functors: $N = 1, 2, 3$

For a $\mathbb{k}$–differential ring $\mathcal{R} = (R, d)$, we define

$$\text{GrAut}(\mathcal{H}_N)(\mathcal{R}) := \{ \phi \in \text{Aut}(\mathcal{H}_N)(\mathcal{R}) | \phi(\Lambda(N) \otimes R) \subseteq \Lambda(N) \otimes R \}.$$ 

**Fact:** $\text{GrAut}(\mathcal{H}_N)$ is a subgroup functor of $\text{Aut}(\mathcal{H}_N)$.
Automorphism group functors: $N = 1, 2, 3$

For a $\mathbb{k}$–differential ring $\mathcal{R} = (R, d)$, we define

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**Proposition (Chang, Pianzola, 2011)**

For $N = 1, 2, 3$, the following hold:

- $\text{GrAut}(\mathcal{K}_N) \cong O_N \circ f$ as functors $\mathbb{k}$-$\text{drng} \rightarrow \text{grp}$, where
  
  $$f : \mathbb{k}$-drng $\rightarrow \mathbb{k}$-rng, \quad \mathcal{R} = (R, d) \mapsto R.$$
Automorphism group functors: $N = 1, 2, 3$

For a $\mathbb{k}$–differential ring $\mathcal{R} = (R, d)$, we define

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1. $\text{GrAut}(\mathcal{K}_N) \cong O_N \circ f$ as functors $\mathbb{k}$-$\text{drng} \to \text{grp}$, where

   $$f : \mathbb{k}$-$\text{drng} \to \mathbb{k}$-$\text{rng}, \quad \mathcal{R} = (R, d) \mapsto R.$$

2. If $\mathcal{R} = (R, d)$ where $R$ is an integral domain, then

   $$\text{GrAut}(\mathcal{K}_N)(\mathcal{R}) = \text{Aut}(\mathcal{K}_N)(\mathcal{R}).$$
Sketch of Proof:

Explicit construction of automorphisms

Let $\mathcal{R} = (R, d)$ be an object in $\mathbf{k}$-$\text{drng}$.
For each $N$, we construct a group homomorphism

$$O_N(R) \to \text{GrAut}(\mathcal{H}_N)(\mathcal{R}) \subseteq \text{Aut}(\mathcal{H}_N)(\mathcal{R})$$

$$A \mapsto \phi_A.$$
Sketch of Proof:

Explicit construction of automorphisms

Let $\mathcal{R} = (R, d)$ be an object in $\mathbb{k}$-$\text{drng}$.
For each $N$, we construct a group homomorphism

$$O_N(R) \rightarrow \text{GrAut}(\mathcal{K}_N)(\mathcal{R}) \subseteq \text{Aut}(\mathcal{K}_N)(\mathcal{R})$$

$$A \mapsto \phi_A.$$

$N = 1$:

- $A = (a)$ where $a^2 = 1$.
- $\phi_A$ is given by

$$\phi_A(1) = 1, \text{ and } \phi_A(\xi_1) = \xi_1 \otimes a$$
Sketch of Proof:

Explicit construction of automorphisms

$N = 2$:

- $A = (a_{ij})_{2 \times 2}$.
- $\phi_A$ is given by

  $\phi_A(1) = 1 + \xi_1 \xi_2 \otimes r$, 
  $\phi_A(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}$, 
  $\phi_A(\xi_1 \xi_2) = \xi_1 \xi_2 \otimes \det(A)$, 
  $\phi_A(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22}$,

where

$$
\begin{pmatrix}
0 & r \\
-r & 0
\end{pmatrix} = 2d(A)A^T.
$$
Sketch of Proof:

Explicit construction of automorphisms

$N = 3$:

- For $A = (a_{ij})_{3 \times 3} \in O_3(R)$, $\phi_A$ is given by

\[
\phi_A(1) = 1 + \sum_{l=1}^{3} \epsilon_{mnl} \xi_m \xi_n \otimes r_l,
\]

\[
\phi_A(\xi_j) = \sum_{l=1}^{3} \xi_l \otimes a_{lj} + \xi_1 \xi_2 \xi_3 \otimes s_j,
\]

\[
\phi_A(\xi_i \xi_j) = \epsilon_{ijl} \sum_{l'=1}^{3} \epsilon_{mnl'} \xi_m \xi_n \otimes A_{l'1},
\]

\[
\phi_A(\xi_1 \xi_2 \xi_3) = \xi_1 \xi_2 \xi_3 \otimes \det(A),
\]

$i, j = 1, 2, 3, i \neq j$, where $A_{l'1}$ is the cofactor of $a_{l'1}$ in $A$ and

\[
\begin{pmatrix}
0 & r_3 & -r_2 \\
-r_3 & 0 & r_1 \\
r_2 & -r_1 & 0
\end{pmatrix} = 2d(A)A^T,
\]

\[
\begin{pmatrix}
0 & s_3 & -s_2 \\
-s_3 & 0 & s_1 \\
s_2 & -s_1 & 0
\end{pmatrix} = 2(\det A)A^T d(A).
\]
Twisted loop conformal superalgebras: $N = 1, 2, 3$

Theorem (Chang, Pianzola, 2011)

There are exactly two twisted loop conformal superalgebras (up to isomorphism of $k$–conformal superalgebras) based on each $\mathcal{K}_N$, $N = 1, 2, 3$, namely, $L(\mathcal{K}_N, \text{id})$ and $L(\mathcal{K}_N, \omega_N)$, where $\omega_N : \mathcal{K}_N \to \mathcal{K}_N$ is given by

\[
\begin{align*}
\omega_1 : & \quad 1 \mapsto 1, \quad \xi_1 \mapsto -\xi_1, \\
\omega_2 : & \quad 1 \mapsto 1, \quad \xi_1 \mapsto -\xi_1, \quad \xi_2 \mapsto \xi_2, \quad \xi_1\xi_2 \mapsto -\xi_1\xi_2, \\
\omega_3 : & \quad 1 \mapsto 1, \quad \xi_j \mapsto -\xi_j, j = 1, 2, 3, \quad \xi_i\xi_j \mapsto \xi_i\xi_j, i \neq j, \quad \xi_1\xi_2\xi_3 \mapsto -\xi_1\xi_2\xi_3.
\end{align*}
\]
Sketch of Proof

\[ H^1_{ct}(\hat{\mathbb{Z}}, \text{Aut}_{\mathcal{D}-\text{conf}}(\mathcal{X}_N, \mathcal{D})) \cong H^1_{ct}(\hat{\mathbb{Z}}, O_N(\mathcal{D})). \]
Sketch of Proof

- $H^1_{ct}(\hat{\mathbb{Z}}, \text{Aut}_{\mathcal{D}-\text{conf}}(\mathcal{X}_N, \hat{\mathcal{D}})) \cong H^1_{ct}(\hat{\mathbb{Z}}, O_N(\hat{D}))$.

- $H^1_{ct}(\hat{\mathbb{Z}}, O_N(\hat{D}))$ has exactly two elements.
  - There is a split exact sequence of $\hat{\mathbb{Z}}$–groups
    
    $$1 \to SO_N(\hat{D}) \to O_N(\hat{D}) \xrightarrow{\text{det}} \mathbb{Z}_2 \to 1,$$
    
    which induces an exact sequence of pointed sets
    
    $$H^1_{ct}(\hat{\mathbb{Z}}, SO_N(\hat{D})) \to H^1_{ct}(\hat{\mathbb{Z}}, O_N(\hat{D})) \xrightarrow{\psi} H^1_{ct}(\hat{\mathbb{Z}}, \mathbb{Z}_2).$$
  
  - $\psi$ is surjective and each fiber of $\psi$ contains exactly one point.

- Centroid trick.
The associated Lie superalgebras

For each of $N = 1, 2, 3$, the two non-isomorphic twisted loop conformal superalgebras $\mathcal{L}(\mathcal{K}_N, \text{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$ yield two Lie superalgebras

$$\text{Alg}(\mathcal{K}_N, \text{id}) \text{ and Alg}(\mathcal{K}_N, \omega_N).$$
The associated Lie superalgebras

For each of $N = 1, 2, 3$, the two non-isomorphic twisted loop conformal superalgebras $\mathcal{L}(\mathcal{H}_N, \text{id})$ and $\mathcal{L}(\mathcal{H}_N, \omega_N)$ yield two Lie superalgebras

$$\text{Alg}(\mathcal{H}_N, \text{id}) \text{ and } \text{Alg}(\mathcal{H}_N, \omega_N).$$

Proposition (Chang, Pianzola, 2011&2013)

For each $N = 1, 2, 3$, $\text{Alg}(\mathcal{H}_N, \text{id})$ and $\text{Alg}(\mathcal{H}_N, \omega_N)$ are not isomorphic as Lie superalgebras over $\mathbb{k}$.
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The $N = 1, 2, 3$ conformal superalgebras

The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
The small $N = 4$ conformal superalgebra \( \mathcal{W} \)

\[ \mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1, \]

where

\[ \mathcal{W}_0 = k[\partial]L \oplus k[\partial]T^1 \oplus k[\partial]T^2 \oplus k[\partial]T^3, \]

\[ \mathcal{W}_1 = k[\partial]G^1 \oplus k[\partial]G^2 \oplus k[\partial]\bar{G}^1 \oplus k[\partial]\bar{G}^2. \]

The \( \lambda \)-bracket on \( \mathcal{W} \) is given by

\[ [L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda T^i] = (\partial + \lambda)T^i, \]

\[ [L_\lambda G^p] = (\partial + \frac{3}{2}\lambda)G^p, \quad [T^i_\lambda T^j] = i\epsilon_{ijk}T^k, \]

\[ [L_\lambda \bar{G}^p] = (\partial + \frac{3}{2}\lambda)\bar{G}^p, \quad [G^p_\lambda G^q] = [\bar{G}^p_\lambda \bar{G}^q] = 0, \]

\[ [T^i_\lambda G^p] = -\frac{1}{2}\sum_{q=1}^2 \sigma^i_{pq}G^q, \quad [T^i_\lambda \bar{G}^p] = \frac{1}{2}\sum_{q=1}^2 \sigma^i_{qp}\bar{G}^q, \]

\[ [G^p_\lambda \bar{G}^q] = 2\delta_{pq}L - 2(\partial + 2\lambda)\sum_{i=1}^3 \sigma^i_{pq}T^i, \]

for \( i, j = 1, 2, 3 \) and \( p, q = 1, 2 \), where

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
Known results

Proposition (Kac, Lau, Pianzola, 2009)

\[
\text{Aut}_{\widehat{D}-\text{conf}}(\mathcal{W}_{\widehat{D}}) \cong \frac{\text{SL}_2(\widehat{D}) \times \text{SL}_2(\mathbb{k})}{\langle (-I_2, -I_2) \rangle}.
\]
Known results

Proposition (Kac, Lau, Pianzola, 2009)

$$\text{Aut}_{\hat{D}-\text{conf}}(\hat{\mathcal{W}}_\hat{D}) \cong \frac{\text{SL}_2(\hat{D}) \times \text{SL}_2(\mathbb{k})}{\langle (-I_2, -I_2) \rangle}.$$
Automorphism group functor

Fix a subspace

\[ V = \text{span}_k \{L, T^1, T^2, T^3, G^1, G^2, \overline{G}^1, \overline{G}^2\}. \]

Then \( \mathcal{W} = k[\partial] \otimes_k V. \)

For each \( \mathcal{R} = (R, d) \in k\text{-drng}, \) we define a subgroup

\[ \text{GrAut}(\mathcal{W})(\mathcal{R}) = \{ \phi \in \text{Aut}(\mathcal{W})(\mathcal{R}) | \phi(V \otimes R) \subseteq V \otimes R \}, \]

which is functorial in \( \mathcal{R}. \)
Automorphism group functor

Theorem (Chang, 2013)

Let $\mathcal{R} = (R, d) \in \mathbb{k}\text{-}\text{drng}$.

- $\text{GrAut}(\mathcal{W})(\mathcal{R}) = \text{Aut}(\mathcal{W})(\mathcal{R})$ if $R$ is an integral domain.
Automorphism group functor

Theorem (Chang, 2013)

Let \( \mathcal{R} = (R, d) \in \k\text{-drng} \).
- \( \text{GrAut}(\mathcal{W})(\mathcal{R}) = \text{Aut}(\mathcal{W})(\mathcal{R}) \) if \( R \) is an integral domain.
- There is an exact sequence of groups

\[
1 \rightarrow \mu_2(R) \rightarrow \text{SL}_2(R) \times \text{SL}_2(R_0) \overset{\iota_R}{\rightarrow} \text{GrAut}(\mathcal{W})(\mathcal{R}),
\]

where \( R_0 = \ker d \). The sequence is functorial in \( \mathcal{R} \).
Automorphism group functor

Theorem (Chang, 2013)

Let $\mathcal{R} = (R, d) \in \mathbb{k}$-drng.

- $\mbox{GrAut}(\mathcal{W})(\mathcal{R}) = \mbox{Aut}(\mathcal{W})(\mathcal{R})$ if $R$ is an integral domain.
- There is an exact sequence of groups

$$1 \rightarrow \mu_2(R) \rightarrow \text{SL}_2(R) \times \text{SL}_2(R_0) \xrightarrow{\iota_{\mathcal{R}}} \mbox{GrAut}(\mathcal{W})(\mathcal{R}),$$

where $R_0 = \ker d$. The sequence is functorial in $\mathcal{R}$.

- Assume $R$ is an integral domain. Then, for every $\phi \in \mbox{GrAut}(\mathcal{W})(\mathcal{R})$, there is an étale extension $S$ of $\mathcal{R}$ such that

$$\phi_S \in \text{Im}(\iota_S : \text{SL}_2(S) \times \text{SL}_2(S_0) \xrightarrow{\iota_S} \mbox{GrAut}(\mathcal{W})(S)).$$
Sketch of Proof I:

review the definition relations for $\mathcal{W}$

$$\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1,$$

where

$$\mathcal{W}_0 = k[\partial]L \oplus k[\partial]T^1 \oplus k[\partial]T^2 \oplus k[\partial]T^3,$$

$$\mathcal{W}_1 = k[\partial]G^1 \oplus k[\partial]G^2 \oplus k[\partial]\overline{G}^1 \oplus k[\partial]\overline{G}^2.$$

The $\lambda$–bracket on $\mathcal{W}$ is given by

$$[L_\lambda L] = (\partial + 2\lambda)L,$$

$$[L_\lambda T^i] = (\partial + \lambda)T^i,$$

$$[L_\lambda G^p] = (\partial + \frac{3}{2}\lambda) G^p,$$

$$[T^i_\lambda T^j] = i\epsilon_{ijk} T^k,$$

$$[L_\lambda \overline{G}^p] = (\partial + \frac{3}{2}\lambda) \overline{G}^p,$$

$$[G^p_\lambda G^q] = [\overline{G}^p_\lambda \overline{G}^q] = 0,$$

$$[T^i_\lambda G^p] = -\frac{1}{2} \sum_{q=1}^{2} \sigma^i_{pq} G^q,$$

$$[T^i_\lambda \overline{G}^p] = \frac{1}{2} \sum_{q=1}^{2} \sigma^i_{qp} \overline{G}^q,$$

$$[G^p_\lambda \overline{G}^q] = 2\delta_{pq} L - 2(\partial + 2\lambda) \sum_{i=1}^{3} \sigma^i_{pq} T^i,$$

for $i, j = 1, 2, 3$ and $p, q = 1, 2$, where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Sketch of Proof II: simplification

Notation:

\[ T(x) := (x_{12} + x_{21})T^1 + i(x_{12} - x_{21})T^2 + 2x_{11}T^3, \]
\[ G(u) := u_{22}G^1 + u_{11}G^1 - u_{12}G^2 + u_{21}G^2. \]

for \( x = (x_{ij})_{2 \times 2} \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u = (u_{ij})_{2 \times 2} \in \text{Mat}_2(\mathbb{k}) \).

The \( \lambda \)-bracket on \( \mathcal{W} \) is rewritten as follows:

\[
\begin{align*}
[L_\lambda L] &= (\partial + 2\lambda)L, \\
[L_\lambda T(x)] &= (\partial + \lambda)T(x), \quad [T(x) L_{\lambda} T(y)] = T([x, y]), \\
[L_\lambda G(u)] &= (\partial + \frac{3}{2}\lambda)G(u), \quad [T(x) L_{\lambda} G(u)] = G(xu), \\
[G(u) L_{\lambda} G(v)] &= 2\text{tr}(uv^\dagger)L + (\partial + 2\lambda)T(uv^\dagger - vu^\dagger).
\end{align*}
\]

where \( x, y \in \mathfrak{sl}_2(\mathbb{k}) \), and \( u, v \in \text{Mat}_2(\mathbb{k}) \).

\[ \dagger: \text{Mat}_2(\mathbb{k}) \rightarrow \text{Mat}_2(\mathbb{k}), \quad u = u_{ij} \mapsto u^\dagger := \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}. \]
Sketch of Proof III:
construction of automorphisms

Let $\mathcal{R} = (R, d) \in k$-drng and $R_0 = \ker d$.
Then every element $(A, B) \in \text{SL}_2(R) \times \text{SL}_2(R_0)$
defines an automorphism $\theta_{A,B}$ of $\mathcal{W}$:

$$
\theta_{A,B}(L) = L + T(d(A)A^{-1}),
$$
$$
\theta_{A,B}(T(x)) = T(AxA^{-1}),
$$
$$
\theta_{A,B}(G(u)) = G(AuB^{-1}).
$$
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The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
The large $N = 4$ superconformal Lie algebras

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_m, -n c, \\
[L_m, U_n] &= -nU_{m+n},
\end{align*}
\]

\[
\begin{align*}
[T^+_m, T^+_n] &= \epsilon_{ijk} T^+_{m+n} - \frac{m}{12\gamma} \delta_{ij} \delta_m, -n c, \\
[T^-_m, T^-_n] &= \epsilon_{ijk} T^-_{m+n} - \frac{m}{12(1-\gamma)} \delta_{ij} \delta_m, -n c, \\
[L_m, Q'^p_{n'}] &= -\left(\frac{1}{2} m + n'\right) Q'^p_{m+n'}, \\
[L_m, G'^p_{n'}] &= \left(\frac{1}{2} m - n'\right) G'^p_{m+n'}, \\
[Q'^p_{m'}, Q'^q_{n'}] &= -\frac{\delta_{pq}\delta_{m', -n'}}{12\gamma(1-\gamma)} c, \\
[T^+_m, G'^p_{n'}] &= \alpha^+_pq (G'^q_{m+n'} - 2(1 - \gamma) mQ'^q_{m+n'}), \\
[T^-_m, G'^p_{n'}] &= \alpha^-iq (G'^q_{m+n'} + 2\gamma mQ'^q_{m+n'}), \\
[Q'^p_{m'}, G'^q_{n'}] &= \delta_{pq} U_{m'+n'} + 2(\alpha^+_pq T^+_m + \alpha^-iq T^-_m) - \alpha^-iq T^-_m + n' \right), \\
[G'^p_{m'}, G'^q_{n'}] &= \frac{1}{3}\delta_{pq}\delta_{m', -n'}(m'^2 - 1/4)c, \\
\end{align*}
\]

for $i, j = 1, 2, 3, p, q = 1, 2, 3, 4, m, n \in \mathbb{Z}, m', n' \in \frac{1}{2} + \mathbb{Z}$. 

\[N = 1, 2, 3, \text{ small } N = 4, \text{ large } N = 4\]
The large $N = 4$ conformal superalgebra

the $k[\partial]$–module

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1,$$

where

$$\mathcal{M}_0 = k[\partial] \otimes_k (kL \oplus \mathfrak{sl}_2(k) \oplus \mathfrak{sl}_2(k) \oplus kU),$$

$$\mathcal{M}_1 = k[\partial] \otimes_k (\text{Mat}_2(k) \oplus \text{Mat}_2(k)).$$

Notation:

$$T^+(x) = 1 \otimes (0 \oplus x \oplus 0 \oplus 0) \in \mathcal{M}_0,$$

$$T^-(x) = 1 \otimes (0 \oplus 0 \oplus x \oplus 0) \in \mathcal{M}_0,$$

$$G(u) = 1 \otimes (u \oplus 0) \in \mathcal{M}_1,$$

$$Q(u) = 1 \otimes (0 \oplus u) \in \mathcal{M}_1,$$

for $x \in \mathfrak{sl}_2(k)$ and $u \in \text{Mat}_2(k)$. 
The large $N = 4$ conformal superalgebra defining relations

\[
[L_{\lambda}L] = (\partial + 2\lambda)L, \\
[L_{\lambda}U] = (\partial + \lambda)U, \\
[T^{\pm}(x)_{\lambda}T^{\pm}(y)] = T^{\pm}([x, y]), \\
[L_{\lambda}Q(u)] = (\partial + \frac{1}{2}\lambda)Q(u), \\
[L_{\lambda}G(u)] = (\partial + \frac{3}{2}\lambda)G(u), \\
[T^{+}(x)_{\lambda}G(u)] = G(xu) - \lambda Q(xu), \\
[T^{-}(x)_{\lambda}G(u)] = -G(ux) - \lambda Q(ux), \\
[Q(u)_{\lambda}G(v)] = 2tr(uv^\dagger)U - T^{+}(uv^\dagger - vu^\dagger) + T^{-}(u^\dagger v - v^\dagger u), \\
[G(u)_{\lambda}G(v)] = 4tr(uv^\dagger)L + (\partial + 2\lambda)\left(T^{+}(uv^\dagger - vu^\dagger) + T^{-}(u^\dagger v - v^\dagger u)\right) \\
\]

for $x, y \in \mathfrak{sl}_2(\mathbb{k})$ and $u, v \in \text{Mat}_2(\mathbb{k})$.
Automorphism group

Proposition (Chang, Pianzola, 2013)

\[ \text{Aut}_{\hat{D}-\text{conf}}(\hat{\mathcal{M}}_{\hat{D}}) \cong \left( \frac{\text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D})}{\langle (-I_2, -I_2) \rangle} \times G_a(\hat{D}) \right) \rtimes \mathbb{Z}_2. \]
Sketch of Proof I

There is a group homomorphism

\[ \text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D}) \to \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{M}_{\hat{D}}), \quad (A, B) \mapsto \theta_{A,B}, \]

where \( \theta_{A,B} \) is defined by

\[
L \mapsto L + T^+(d_t(A)A^{-1}) + T^{-1}(d_t(B)B^{-1}), \\
T^+(x) \mapsto T^+(AxA^{-1}), \\
T^-(y) \mapsto T^-(ByB^{-1}), \\
U \mapsto U, \\
G(u) \mapsto G(AuB^{-1}) - Q(d_t(A)uB^{-1} - Au d_t(B^{-1})), \\
Q(u) \mapsto Q(AuB^{-1}),
\]

for \( x \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u \in \text{Mat}_2(\mathbb{k}) \).
Sketch of Proof II

There is a group homomorphism

\[ G_a(\hat{D}) \to \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{M}_{\hat{D}}), \quad s \mapsto \tau_s, \]

where \( \tau_s \) is defined by

- \( L \mapsto L + U \otimes s \),
- \( T^\pm(x) \mapsto T^\pm(x) \),
- \( U \mapsto U \),
- \( G(u) \mapsto G(u) + Q(su) \),
- \( Q(u) \mapsto Q(u) \),

for \( x \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u \in \text{Mat}_2(\mathbb{k}) \).
Sketch of Proof III

- There is an element $\omega \in \text{Aut}_{\hat{D}-\text{conf}}(\hat{\mathcal{M}}_{\hat{D}})$ of order 2 given by

$$
\omega(L) = L, \quad \omega(T^\pm(x)) = T^\mp(x), \quad \omega(U) = -U,
$$

$$
\omega(G(u)) = G(u^\dagger), \quad \omega(Q(u)) = -Q(u^\dagger),
$$

for $x \in \mathfrak{sl}_2(\mathbb{k})$ and $u \in \text{Mat}_2(\mathbb{k})$. 
Sketch of Proof III

There is an element $\omega \in \text{Aut}_{\hat{D}-\text{conf}}(\hat{M}_D)$ of order 2 given by

$$
\begin{align*}
\omega(L) &= L, \\
\omega(T^\pm(x)) &= T^\mp(x), \\
\omega(U) &= -U, \\
\omega(G(u)) &= G(u^\dagger), \\
\omega(Q(u)) &= -Q(u^\dagger),
\end{align*}
$$

for $x \in \mathfrak{sl}_2(\mathbb{k})$ and $u \in \text{Mat}_2(\mathbb{k})$.

These automorphisms satisfy

- $\theta_{A,B} \circ \tau_s = \tau_s \circ \theta_{A,B}$.
- $\omega \circ \theta_{A,B} \circ \omega = \theta_{B,A}$.
- $\omega \circ \tau_s \circ \omega = \tau_{-s}$.
Sketch of Proof III

There is an element $\omega \in \text{Aut}_{\hat{D}\text{-conf}}(\hat{\mathcal{M}})$ of order 2 given by

$$
\omega(L) = L, \quad \omega(T^{\pm}(x)) = T^{\mp}(x), \quad \omega(U) = -U,
\omega(G(u)) = G(u^\dagger), \quad \omega(Q(u)) = -Q(u^\dagger),
$$

for $x \in \mathfrak{sl}_2(\mathbb{k})$ and $u \in \text{Mat}_2(\mathbb{k})$.

These automorphisms satisfy

1. $\theta_{A,B} \circ \tau_s = \tau_s \circ \theta_{A,B}$.
2. $\omega \circ \theta_{A,B} \circ \omega = \theta_{B,A}$.
3. $\omega \circ \tau_s \circ \omega = \tau_{-s}$.

Hence, we get the group homomorphism

$$(\text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D}) \times \text{G}_a(\hat{D})) \rtimes \mathbb{Z}_2 \rightarrow \text{Aut}_{\hat{D}\text{-conf}}(\hat{\mathcal{M}}),$$

which is surjective and has kernel $\langle (-I_2, -I_2, 0, 0) \rangle$. 

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Twisted loop conformal superalgebras based on $\mathcal{M}$

Theorem (Chang, Pianzola, 2013)

There are exactly two twisted loop conformal superalgebras based on $\mathcal{M}$ (up to isomorphism of $\mathbb{k}$–conformal superalgebras), namely, $\mathcal{L}(\mathcal{M}, \text{id})$ and $\mathcal{L}(\mathcal{M}, \omega)$.

Sketch of Proof.

$\triangleright$ $H^1_{\text{ct}} \left( \hat{\mathbb{Z}}, \left( \frac{\text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D})}{\langle \langle -I_2, -I_2 \rangle \rangle} \right) \rtimes G_a(\hat{D}) \right) \rtimes \mathbb{Z}_2$ has exactly two elements.

$\triangleright$ Centroid trick.
The associated Lie superalgebras

The two non-isomorphic Lie conformal superalgebras

$\mathcal{L}(\mathcal{M}, \text{id})$ and $\mathcal{L}(\mathcal{M}, \omega)$

yield two Lie superalgebras

$\text{Alg}(\mathcal{M}, \text{id})$ and $\text{Alg}(\mathcal{M}, \omega)$. 
The associated Lie superalgebras

The two non-isomorphic Lie conformal superalgebras

\[ \mathcal{L}(\mathcal{M}, \text{id}) \text{ and } \mathcal{L}(\mathcal{M}, \omega) \]

yield two Lie superalgebras

\[ \text{Alg}(\mathcal{M}, \text{id}) \text{ and } \text{Alg}(\mathcal{M}, \omega). \]

Proposition (Chang, Pianzola, 2013)

*The two Lie superalgebras \( \text{Alg}(\mathcal{M}, \text{id}) \text{ and } \text{Alg}(\mathcal{M}, \omega) \) are not isomorphic.*
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Thank You!

KM alg
Lie ⇔ conf
diff conf
\( N = 1, 2, 3 \)
small \( N = 4 \)
large \( N = 4 \)