Differential Conformal Superalgebras
and
Their Twisted Forms

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Outline

**Affine Kac-Moody algebras**

- Lie algebras and Lie conformal algebras
- Differential conformal algebras and their forms
- The \( N = 1, 2, 3 \) conformal superalgebras
- The small \( N = 4 \) conformal superalgebra
- The large \( N = 4 \) conformal superalgebra
Twisted loop realization of affine KM algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $\sigma$ an automorphism of $\mathfrak{g}$ of order $m$. Then

$$\mathfrak{g} = \bigoplus_{\ell=1}^{m} \mathfrak{g}_\ell,$$

where $\mathfrak{g}_\ell = \{x \in \mathfrak{g} \mid \sigma(x) = \zeta_\ell^\ell x\}$, $\zeta_m = e^{\frac{2\pi i}{m}}$.

One can construct a Lie algebra

$$L(\mathfrak{g}, \sigma) = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_\ell \otimes \mathbb{C}t^\ell.$$

Fact: every affine Kac-Moody algebra is isomorphic to a Lie algebra of the form

$$L(\mathfrak{g}, \sigma) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$
Affine algebras as twisted forms

Key observations by A. Pianzola and his collaborators:

- $L(g, \sigma)$ is a Lie algebra over $D := \mathbb{C}[t^{\pm 1}]$.
- $D \hookrightarrow D_m := \mathbb{C}[t^{\pm \frac{1}{m}}]$ is a finite Galois ring extension.
- $L(g, \sigma) \otimes_D D_m \cong (g \otimes_{\mathbb{C}} D) \otimes_D D_m \cong g \otimes_{\mathbb{C}} D_m$, i.e., $L(g, \sigma)$ is a $D_m/D$–twisted form of $g \otimes_{\mathbb{C}} D = L(g, \text{id})$.

Well known result on twisted forms:
The isomorphism classes of $D_m/D$–twisted form of $g \otimes_{\mathbb{C}} D$ bijectively correspond to elements of $H^1(D_m/D, \text{Aut}(g \otimes_{\mathbb{C}} D))$. 
Researches motivated by the above viewpoint

- $H^1_{\text{ét}}(D, G)$ for a reductive group scheme $G$ over $D$.
  c.f. [P 2004], [GP 2007-2008], [CGP2012], [GP2013].

- Application of descent theory to Lie theory:
  - central extension:
    c.f. [PPS 2007], [S 2009].
  - derivation:
    c.f. [P 2010].
  - conjugacy of Cartan subalgebras:
    c.f. [P 2004], [CGP 2011], [CEGP 2012].
  - finite dimensional irreducible representation:
    c.f. [L 2010], [LP 2013].
  - invariant bilinear form:
    c.f. [NPP 20??].
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Differential conformal algebras and their forms

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The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
Motivating Example:
The Centreless Virasoro Algebra

\( \mathbb{k} \): an algebraically closed field of characteristic zero
\( \mathfrak{v} \): the centreless Virasoro algebra

- \( \mathfrak{v} \) has a basis \( \{ L_n | n \in \mathbb{Z} \} \) satisfying \( [L_m, L_n] = (m - n)L_{m+n} \).

Consider the formal series \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \).

The Lie bracket on \( \mathfrak{v} \) yields the OPE

\[
[L(z), L(w)] : = \sum_{m,n \in \mathbb{Z}} [L_m, L_n] z^{-m-2} w^{-n-2}
\]

\[
= (\partial_w L(w)) \delta(z - w) + 2L(w) \partial_w \delta(z - w),
\]

where \( \delta(z - w) = \sum_{n \in \mathbb{Z}} w^n z^{-n-1} \).
Lie Algebra $\Rightarrow$ Conformal Algebra

Let $\mathcal{V} := \text{span}_k \{ \partial_{\ell}^L(z) | \ell \geq 0 \}$.

- For $a(z), b(z) \in \mathcal{V}$, we have
  
  $$[a(z), b(w)] = \sum_{\ell \geq 0} c_{\ell}(w) \cdot \partial_w^{(\ell)} \delta(z - w),$$

  where $c_{\ell}(z) \in \mathcal{V}$.

- We can define a product on $\mathcal{V}$ for each $\ell \geq 0$ by
  
  $$a(z)_{(\ell)} b(z) := c_{\ell}(z).$$

Notation: $\lambda$–bracket

$$[a(z) \lambda b(z)] = \sum_{\ell \geq 0} \lambda^{(\ell)} (a(z)_{(\ell)} b(z)).$$

Summarizing: $\mathcal{V}$ is a $k[\partial]$–module generated by $L(z)$, equipped with a $\lambda$–bracket on $\mathcal{V}$ given by

$$[L(z) \lambda L(z)] := (\partial_z + 2\lambda)L(z).$$
Lie conformal algebras
axiomatic definition

Due to V. G. Kac.

A Lie conformal algebra over $\mathbb{k}$ is a $\mathbb{k}[\partial]$–module $\mathcal{A}$ equipped with a $\lambda$–bracket

$$[-\lambda-]: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[\lambda],$$

satisfying

(C1) $[(\partial a)_\lambda b] = -\lambda[a_\lambda b],$

(C2) $[b_\lambda a] = -[a_{-\partial-} \lambda b],$

(C3) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_\lambda + \mu c] + [b_\mu [a_\lambda c]],$

for all $a, b, c \in \mathcal{A}.$
Lie conformal algebra $\Rightarrow$ Lie algebra

affinization

$\mathcal{A}$: a Lie conformal algebra over $\mathbb{k}$.

Define a conformal algebra structure on $\mathcal{A} \otimes_{\mathbb{k}} \mathbb{k}[t^\pm 1]$ by

$$\hat{\partial}(a \otimes r) = \partial(a) \otimes r + a \otimes \frac{d}{dt}(r)$$

and

$$(a \otimes r)(\ell)(b \otimes s) := \sum_{j \geq 0} (a(\ell+j)b) \otimes (\frac{d}{dt})^j(r)s,$$

for $a, b \in \mathcal{A}$ and $r, s \in \mathbb{k}[t^\pm 1]$.

Terminology: the (untwisted) loop conformal algebra based on $\mathcal{A}$.

Notation: $\mathcal{A} \otimes_{\mathbb{k}} \mathcal{D}$, where $\mathcal{D} = (\mathbb{k}[t^\pm 1], \frac{d}{dt})$. 
Lie conformal algebra $\Rightarrow$ Lie algebra

The conformal algebra $\mathcal{A} \otimes_k \mathcal{D}$ determines a Lie algebra

$$\text{Alg}(\mathcal{A}) := (\mathcal{A} \otimes_k \mathcal{D}) \big/ \hat{\partial} (\mathcal{A} \otimes_k \mathcal{D}),$$

with Lie bracket induced by the 0–th product on $\mathcal{A} \otimes_k \mathcal{D}$. 
Twisted loop Lie conformal algebras

Given

- $\mathcal{A}$: a Lie conformal algebra over $\mathbb{k}$
- $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ an automorphism of $\mathcal{A}$ of order $m$

We know that

- $\mathcal{A} \otimes_{\mathbb{k}} D_m$ is a Lie conformal algebra over $\mathbb{k}$, where $D_m = (\mathbb{k}[t^{\pm \frac{1}{m}}], \frac{d}{dt})$.

\[
\mathcal{A} = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell,
\]

where $\mathcal{A}_\ell = \{a \in \mathcal{A} | \sigma(a) = \zeta_{m}^{\ell} a\}$ and $\zeta_{m} = e^{\frac{2\pi i}{m}}$. 

small $N = 4$

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Twisted loop Lie conformal algebras

Facts:

- The $k$–subspace
  \[
  \mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell \otimes t^\ell \mathbb{K}[t^{\pm 1}] \subseteq \mathcal{A} \otimes_k \mathcal{D}_m,
  \]
  is a Lie conformal subalgebra of $\mathcal{A} \otimes_k \mathcal{D}_m$.

  (the twisted loop conformal algebra based on $\mathcal{A}$ w.r.t $\sigma$)

- \[
  \mathcal{L}(\mathcal{A}, \sigma) = (\mathcal{A} \otimes \mathcal{D}_m)^\Gamma,
  \]
  where $\Gamma$ is a finite cyclic group of automorphisms of
  $\mathcal{A} \otimes \mathcal{D}_m$ generated by

  \[
  \sigma \otimes \psi : \mathcal{A} \otimes_k \mathcal{D}_m \to \mathcal{A} \otimes_k \mathcal{D}_m,
  \]
  \[
  a \otimes t^\frac{n}{m} \mapsto \sigma(a) \otimes \zeta^m_{-n} t^\frac{n}{m}.
  \]

- In particular, $\mathcal{L}(\mathcal{A}, \text{id}) = \mathcal{A} \otimes_k \mathcal{D}$. 
The associated Lie algebras

- The conformal algebra $\mathcal{L}(\mathcal{A}, \sigma)$ determines a Lie algebra

$$\text{Alg}(\mathcal{A}, \sigma) := \mathcal{L}(\mathcal{A}, \sigma)/\hat{\partial}\mathcal{L}(\mathcal{A}, \sigma)$$

with Lie bracket induced by the 0-th product on $\mathcal{L}(\mathcal{A}, \sigma)$.

- Central extensions of Lie superalgebras of this form are indeed the twisted superconformal Lie algebras which appear in physics literature.
Lie algebras ⇐⇐ Lie conformal algebras

Summary:

\[ \mathcal{L}(\mathcal{A}, \text{id}) \quad \text{Alg}(\mathcal{A}, \text{id}) \]

\[ \mathfrak{g} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A} \otimes_k \mathcal{D} \longrightarrow \text{Alg}(\mathcal{A}) \quad \longrightarrow \mathfrak{g} \]

\[ \mathcal{L}(\mathcal{A}, \sigma) \longrightarrow \text{Alg}(\mathcal{A}, \sigma) \]
Question

Given a conformal algebra $\mathcal{A}$ over $\mathbb{k}$, how can we classify all twisted loop conformal algebras based on $\mathcal{A}$?

The theory of differential conformal (super)algebras was developed in

[KLP] V. G. Kac, M. Lau, and A. Pianzola,

*Differential conformal superalgebras and their forms*,
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**Differential conformal algebras and their forms**

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Key observation I in [KLP]

\( \mathcal{L}(\mathcal{A}, \sigma) \) is not only a conformal algebra over \( \mathbb{k} \), but also a differential conformal algebra over \( \mathcal{D} := (\mathbb{k}[t^{\pm 1}], \frac{d}{dt}) \).

Let \( \mathcal{R} = (R, d) \) be a \( \mathbb{k} \)-differential ring.

A differential Lie conformal algebra over \( \mathcal{R} \) consists of

- an \( R \)-module \( \mathcal{A} \),
- a \( \mathbb{k} \)-linear operator \( \partial : \mathcal{A} \to \mathcal{A} \) such that
  \[ \partial(ra) = d(r)a + r\partial(a), \]
- a \( \mathbb{k} \)-bilinear product \( -_{(n)} - \) for each \( n \in \mathbb{Z}_+ \) satisfying
  \[ (1) \quad a_{(n)}(rb) = r(a_{(n)}b), \]
  \[ (2) \quad (ra)_{(n)}b = \sum_{j \in \mathbb{Z}_+} d^{(j)}(r)(a_{(n+j)}b). \]

which satisfy axioms (C1), (C2) and (C3).

Terminology: \( \mathcal{R} \)-conformal algebra.
Key observation II in [KLP]

\( \mathcal{L}(\mathcal{A}, \sigma) \) is a \( \mathcal{D}_m/\mathcal{D} \)-form of \( \mathcal{L}(\mathcal{A}, \text{id}) \), i.e.,

\[
\mathcal{L}(\mathcal{A}, \sigma) \otimes_{\mathcal{D}} \mathcal{D}_m \cong \mathcal{L}(\mathcal{A}, \text{id}) \otimes_{\mathcal{D}} \mathcal{D}_m \cong \mathcal{A} \otimes_{\mathbb{k}} \mathcal{D}_m
\]

as \( \mathcal{D}_m \)-conformal algebras, where \( \mathcal{D}_m := (\mathbb{k}[t^{\pm \frac{1}{m}}], \frac{d}{dt}) \).

**Base Change**

an \( \mathcal{R} \)-conformal algebra \( \mathcal{A} \)
\( \mathcal{R} = (R, d_R) \to S = (S, d_S) \) \( \implies \) an \( S \)-conformal algebra \( \mathcal{A} \otimes_{\mathcal{R}} S \)

- the underlying \( S \)-module: \( \mathcal{A} \otimes_{\mathcal{R}} S \).
- \( \hat{\partial} := \partial \otimes \text{id} + \text{id} \otimes d_S \)
- the \( n \)-th product

\[
(a \otimes r)_{(n)}(b \otimes s) = \sum_{j \geq 0} (a_{(n+j)}b) \otimes d_{(j)}^S (r)s,
\]

for \( a, b \in \mathcal{A}, r, s \in S \).
Classification of twisted forms

\( \mathcal{A} \): an \( \mathcal{R} \)–conformal algebra

**Automorphism group functor:**

\[ \text{Aut}(\mathcal{A}) : S \mapsto \text{Aut}_{S \text{-conf}}(\mathcal{A} \otimes \mathcal{R} S). \]

**Theorem (Kac, Lau, Pianzola, 2009)**

Let \( \mathcal{R} \to S \) be a faithfully flat extension of \( k \)–differential rings. Then

\[ \text{the set of isomorphism classes of } S / \mathcal{R} \text{–forms of } \mathcal{A} \text{ (up to } \mathcal{R} \text{–isomorphism)} \leftrightarrow H^1(S / \mathcal{R}, \text{Aut}(\mathcal{A})). \]
Classification of twisted loop conformal algebras

Recall: $\mathcal{L}(\mathcal{A}, \sigma)$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathcal{A} \otimes_k \mathcal{D}$.

Take

$$\hat{\mathcal{D}} := \lim_{\to} \mathcal{D}_m.$$ 

We know: $\mathcal{L}(\mathcal{A}, \sigma)$ is a $\hat{\mathcal{D}}/\mathcal{D}$–form of $\mathcal{A} \otimes_k \mathcal{D}$.

Fact:

\[
\text{the set of isomorphism classes of twisted loop conformal algebras based on } \mathcal{A} \\
\text{(up to } \mathcal{D} \text{–isomorphism)} \\
\longleftrightarrow \text{ one to one } \\
H^1(\hat{\mathcal{D}}/\mathcal{D}, \text{Aut}(\mathcal{A}))
\]
\[ H^1 \Rightarrow H^1_{\text{ct}} \Rightarrow H^1_{\text{ét}} \]

**Proposition (Kac, Lau, Pianzola, 2009)**

*If \( \mathcal{A} \) is a finitely generated \( \mathbb{k}[\partial] \)-module, then*

\[ H^1(\hat{D}/D, \text{Aut}(\mathcal{A})) = H^1_{\text{ct}} \left( \hat{\mathbb{Z}}, \text{Aut}_{\text{ct}}(\mathcal{A} \otimes_{\mathbb{k}} \hat{D}) \right). \]

**Proposition (Gille, Pianzola, 2008)**

*Let \( G \) be an affine group scheme over \( D \). If \( G \) is an extension of a twisted finite constant group by a reductive group, then*

\[ H^1_{\text{ct}}(\hat{\mathbb{Z}}, G(\hat{D})) = H^1_{\text{ét}}(D, G). \]

**Proposition (Pianzola 2005)**

*Let \( G \) be a reductive group scheme over \( D \). Then \( H^1_{\text{ét}}(D, G) = 1 \).*
Centroid trick

Question:
Given two twisted loop conformal algebras $\mathcal{L}(\mathcal{A}, \sigma_i), i = 1, 2,$

$$\mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_D \mathcal{L}(\mathcal{A}, \sigma_2) \Rightarrow \mathcal{L}(\mathcal{A}, \sigma_1) \not\cong_k \mathcal{L}(\mathcal{A}, \sigma_2)$$

Centroid: the centroid of an $\mathcal{R}$–conformal algebra $\mathcal{B}$ is

$$\text{Ctd}_R(\mathcal{B}) = \{ \chi \in \text{End}_{R\text{-mod}}(\mathcal{B}) \mid \chi(a_n b) = a_n \chi(b), \forall a, b \in \mathcal{B}, n \in \mathbb{Z}_+ \}$$

Proposition (Kac, Lau, Pianzola, 2009)

If the canonical maps $D \rightarrow \text{Ctd}_k(\mathcal{L}(\mathcal{A}, \sigma_i)), i = 1, 2$ are both $k$–algebra isomorphisms, then

$$\mathcal{L}(\mathcal{A}, \sigma_1) \cong_D \mathcal{L}(\mathcal{A}, \sigma_2) \iff \mathcal{L}(\mathcal{A}, \sigma_1) \cong_k \mathcal{L}(\mathcal{A}, \sigma_2)$$
Centroid: special case

Proposition (Chang 2013)

Let

- $\mathcal{A}$ be a $\mathbb{k}$–conformal superalgebra and
- $\sigma$ an automorphism of $\mathcal{A}$ of finite order.

Suppose $\mathcal{A}$ satisfies all of the following conditions:

1. There are $a_1, \cdots, a_{n_0} \in \mathcal{A}$ such that
   - $\mathcal{A}_0 = \mathbb{k}[\partial]a_1 \oplus \cdots \oplus \mathbb{k}[\partial]a_{n_0}$,
   - $L := a_1$ satisfies $[L_\lambda L] = (\partial + 2\lambda)L$ and $\sigma(L) = L$,
   - $[L_\lambda a_i] = (\partial + \lambda)a_i$, for $i = 2, \cdots, n_0$.

2. There are $b_1, \cdots, b_{n_1} \in \mathcal{A}_\overline{1}$ generating $\mathcal{A}_\overline{1}$ as a $\mathbb{k}[\partial]$–module such that $[L_\lambda b_i] = (\partial + \Delta'_i \lambda)b_i$ with $\Delta'_i \neq 0$ for $i = 1, \cdots, n_1$.

Then $\text{Ctd}_{\mathbb{k}}(\mathcal{L}(\mathcal{A}, \sigma)) = D$. 
Centroid: special case

Corollary

Let $\mathcal{A}$ be one of the $N = 1, 2, 3$ and (small or large) $N = 4$ conformal superalgebras over $\mathbb{k}$, and $\mathcal{L}(\mathcal{A}, \sigma)$ be an arbitrary twisted loop conformal superalgebra based on $\mathcal{A}$.
Then the canonical map

$$\mathbb{k}[t^{\pm 1}] \to \text{Ctd}_\mathbb{k}(\mathcal{L}(\mathcal{A}, \sigma))$$

is an isomorphism.
Summary

Given a conformal superalgebra $\mathcal{A}$ over $k$, the twisted loop conformal superalgebras based on $\mathcal{A}$ can be classified using the following steps:

1. Compute the automorphism group
   
   $$\text{Aut}_{\hat{D}-\text{conf}}(\mathcal{A} \otimes_k \hat{D});$$

2. Compute the non-abelian cohomology set
   
   $$H^1_{\text{ct}} \left( \hat{\mathbb{Z}}, \text{Aut}_{\hat{D}-\text{conf}}(\mathcal{A} \otimes_k \hat{D}) \right);$$

3. Compute the centroid $\text{Ctd}_k(\mathcal{L}(\mathcal{A}, \sigma))$ for all twisted loop conformal superalgebra $\mathcal{L}(\mathcal{A}, \sigma)$. 
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The $N = 1, 2, 3$ Lie Conformal Superalgebras $\mathcal{K}_N$

\[ \mathcal{K}_N := \mathbb{k}[\partial] \otimes_{\mathbb{k}} \Lambda(N), \]

where $\Lambda(N)$ is the exterior superalgebra over $\mathbb{k}$ in $N$–variables $\xi_1, \ldots, \xi_N$.

For $n \in \mathbb{Z}_+$, the $n$-th product on $\mathcal{K}_N$ is given by

\[ f(0)g = (\frac{1}{2}|f| - 1) \partial \otimes fg + \frac{1}{2}(-1)^{|f|} \sum_{i=1}^{N} (\partial_if)(\partial_ig), \]

\[ f(1)g = (\frac{1}{2}(|f| + |g|) - 2) fg, \]

\[ f(n)g = 0, \quad n \geq 2, \]

where $f$ and $g$ are monomials in $\xi_1, \ldots, \xi_N$ of degree $|f|$ and $|g|$, respectively.
Automorphism group functors: $N = 1, 2, 3$

For a $k$–differential ring $\mathcal{R} = (R, d)$, we define

$$\text{GrAut}(\mathcal{H}_N)(\mathcal{R}) := \{ \phi \in \text{Aut}(\mathcal{H}_N)(\mathcal{R}) | \phi(\Lambda(N) \otimes R) \subseteq \Lambda(N) \otimes R \}.$$ 

**Fact:** $\text{GrAut}(\mathcal{H}_N)$ is a subgroup functor of $\text{Aut}(\mathcal{H}_N)$.

**Proposition (Chang, Pianzola, 2011)**

For $N = 1, 2, 3$, the following hold:

- $\text{GrAut}(\mathcal{H}_N) \cong O_N \circ f$ as functors $k$-drng $\rightarrow$ grp, where

  $$f : k$-drng $\rightarrow$ k-rng, \quad \mathcal{R} = (R, d) \mapsto R.$$ 

- If $\mathcal{R} = (R, d)$ where $R$ is an integral domain, then

  $$\text{GrAut}(\mathcal{H}_N)(\mathcal{R}) = \text{Aut}(\mathcal{H}_N)(\mathcal{R}).$$
Sketch of Proof:

Explicit construction of automorphisms

Let $\mathcal{R} = (R, d)$ be an object in $\mathbb{k}$-\textbf{drng}.
For each $N$, we construct a group homomorphism

$$O_N(R) \rightarrow \text{GrAut}(\mathcal{H}_N)(\mathcal{R}) \subseteq \text{Aut}(\mathcal{H}_N)(\mathcal{R})$$

$$A \mapsto \phi_A.$$ 

$N = 1$:

$\triangleright$ $A = (a)$ where $a^2 = 1$.

$\triangleright$ $\phi_A$ is given by

$$\phi_A(1) = 1, \text{ and } \phi_A(\xi_1) = \xi_1 \otimes a$$
Sketch of Proof:

Explicit construction of automorphisms

\( N = 2: \)

- \( A = (a_{ij})_{2 \times 2}. \)
- \( \phi_A \) is given by

\[
\begin{align*}
\phi_A(1) &= 1 + \xi_1 \xi_2 \otimes r, \\
\phi_A(\xi_1) &= \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \\
\phi_A(\xi_1 \xi_2) &= \xi_1 \xi_2 \otimes \text{det}(A), \\
\phi_A(\xi_2) &= \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},
\end{align*}
\]

where

\[
\begin{pmatrix}
0 & r \\
-r & 0
\end{pmatrix} = 2d(A)A^T.
\]
Sketch of Proof:

Explicit construction of automorphisms

\( N = 3: \)

- For \( A = (a_{ij})_{3 \times 3} \in O_3(R) \), \( \phi_A \) is given by

\[
\phi_A(1) = 1 + \sum_{l=1}^{3} \epsilon_{mnl} \xi_m \xi_n \otimes r_l,
\]

\[
\phi_A(\xi_j) = \sum_{l=1}^{3} \xi_l \otimes a_{lj} + \xi_1 \xi_2 \xi_3 \otimes s_j,
\]

\[
\phi_A(\xi_i \xi_j) = \epsilon_{ijl} \sum_{l'=1}^{3} \epsilon_{mnl'} \xi_m \xi_n \otimes A_{l'l},
\]

\[
\phi_A(\xi_1 \xi_2 \xi_3) = \xi_1 \xi_2 \xi_3 \otimes \det(A),
\]

\( i, j = 1, 2, 3, i \neq j \), where \( A_{l'l} \) is the cofactor of \( a_{l'l} \) in \( A \) and

\[
\begin{pmatrix}
0 & r_3 & -r_2 \\
-r_3 & 0 & r_1 \\
r_2 & -r_1 & 0
\end{pmatrix} = 2d(A)A^T, \quad \begin{pmatrix}
0 & s_3 & -s_2 \\
-s_3 & 0 & s_1 \\
s_2 & -s_1 & 0
\end{pmatrix} = 2(\det A)A^T d(A).
\]
Twisted loop conformal superalgebras: $N = 1, 2, 3$

**Theorem (Chang, Pianzola, 2011)**

There are exactly two twisted loop conformal superalgebras (up to isomorphism of $\mathbb{k}$–conformal superalgebras) based on each $\mathcal{H}_N$, $N = 1, 2, 3$, namely, $L(\mathcal{H}_N, \text{id})$ and $L(\mathcal{H}_N, \omega_N)$, where $\omega_N : \mathcal{H}_N \to \mathcal{H}_N$ is given by

- $\omega_1 : 1 \mapsto 1$, $\xi_1 \mapsto -\xi_1$,
- $\omega_2 : 1 \mapsto 1$, $\xi_1 \mapsto -\xi_1$, $\xi_2 \mapsto \xi_2$, $\xi_1\xi_2 \mapsto -\xi_1\xi_2$,
- $\omega_3 : 1 \mapsto 1$, $\xi_j \mapsto -\xi_j$, $j = 1, 2, 3$, $\xi_i\xi_j \mapsto \xi_i\xi_j$, $i \neq j$, $\xi_1\xi_2\xi_3 \mapsto -\xi_1\xi_2\xi_3$. 
Sketch of Proof

- $H^1_{\text{ct}}(\hat{\mathbb{Z}}, \text{Aut}_{\hat{D}-\text{conf}}(\mathcal{K}_N, \hat{D})) \cong H^1_{\text{ct}}(\hat{\mathbb{Z}}, O_N(\hat{D}))$.

- $H^1_{\text{ct}}(\hat{\mathbb{Z}}, O_N(\hat{D}))$ has exactly two elements.
  - There is a split exact sequence of $\hat{\mathbb{Z}}$–groups
    
    \[
    1 \to \text{SO}_N(\hat{D}) \to O_N(\hat{D}) \xrightarrow{\text{det}} \mathbb{Z}_2 \to 1,
    \]

    which induces an exact sequence of pointed sets

    \[
    H^1_{\text{ct}}(\hat{\mathbb{Z}}, \text{SO}_N(\hat{D})) \to H^1_{\text{ct}}(\hat{\mathbb{Z}}, O_N(\hat{D})) \xrightarrow{\psi} H^1_{\text{ct}}(\hat{\mathbb{Z}}, \mathbb{Z}_2).
    \]

    - $\psi$ is surjective and each fiber of $\psi$ contains exactly one point.

- Centroid trick.
The associated Lie superalgebras

For each of $N = 1, 2, 3$, the two non-isomorphic twisted loop conformal superalgebras $\mathcal{L}(\mathcal{K}_N, \text{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$ yield two Lie superalgebras

$$\text{Alg}(\mathcal{K}_N, \text{id}) \text{ and } \text{Alg}(\mathcal{K}_N, \omega_N).$$

Proposition (Chang, Pianzola, 2011&2013)

For each $N = 1, 2, 3$, $\text{Alg}(\mathcal{K}_N, \text{id})$ and $\text{Alg}(\mathcal{K}_N, \omega_N)$ are not isomorphic as Lie superalgebras over $\mathbb{k}$.
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The small $N = 4$ conformal superalgebra $\mathcal{W}$

$\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1$, where

$\mathcal{W}_0 = \mathbb{k}[\partial]L \oplus \mathbb{k}[\partial]T^1 \oplus \mathbb{k}[\partial]T^2 \oplus \mathbb{k}[\partial]T^3,$

$\mathcal{W}_1 = \mathbb{k}[\partial]G^1 \oplus \mathbb{k}[\partial]G^2 \oplus \mathbb{k}[\partial]\overline{G}^1 \oplus \mathbb{k}[\partial]\overline{G}^2.$

The $\lambda$–bracket on $\mathcal{W}$ is given by

\[
\begin{align*}
[L_\lambda L] &= (\partial + 2\lambda)L, \\
[L_\lambda T^i] &= (\partial + \lambda)T^i, \\
[L_\lambda G^p] &= (\partial + \frac{3}{2}\lambda) G^p, \\
[L_\lambda \overline{G}^p] &= (\partial + \rac{3}{2}\lambda) \overline{G}^p, \\
[T^i_\lambda T^j] &= i\epsilon_{ijk} T^k, \\
[T^i_\lambda G^p] &= -\frac{1}{2} \sum_{q=1}^{2} \sigma^i_{pq} G^q, \\
[T^i_\lambda \overline{G}^p] &= \frac{1}{2} \sum_{q=1}^{2} \sigma^i_{qp} \overline{G}^q, \\
[G^p_\lambda G^q] &= 2\delta_{pq}L - 2(\partial + 2\lambda) \sum_{i=1}^{3} \sigma^i_{pq} T^i,
\end{align*}
\]

for $i, j = 1, 2, 3$ and $p, q = 1, 2$, where

\[
\begin{align*}
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]
**Known results**

**Proposition (Kac, Lau, Pianzola, 2009)**

\[
\text{Aut}_{\hat{D}\text{-conf}}(\mathcal{W}_{\hat{D}}) \cong \frac{\text{SL}_2(\hat{D}) \times \text{SL}_2(\mathbb{k})}{\langle (-I_2, -I_2) \rangle}.
\]

**Proposition (Kac, Lau, Pianzola, 2009)**

The set of isomorphism classes of twisted loop conformal algebras based on \( \mathcal{W} \) (up to \( \mathbb{k} \)-isomorphism) is one to one with the set of conjugacy classes of elements of finite order in \( \text{PGL}_2(\mathbb{k}) \).
Automorphism group functor

Fix a subspace

\[ V = \text{span}_k \{ L, T^1, T^2, T^3, G^1, G^2, \overline{G}^1, \overline{G}^2 \} \].

Then \( \mathcal{W} = k[\partial] \otimes k V \).

For each \( \mathcal{R} = (R, d) \in k\text{-drng} \), we define a subgroup

\[ \text{GrAut}(\mathcal{W})(\mathcal{R}) = \{ \phi \in \text{Aut}(\mathcal{W})(\mathcal{R}) \mid \phi(V \otimes R) \subseteq V \otimes R \} \],

which is functorial in \( \mathcal{R} \).
Automorphism group functor

Theorem (Chang, 2013)

Let $\mathcal{R} = (R, d) \in \kappa$-drng.

- $\text{GrAut}(\mathcal{W})(\mathcal{R}) = \text{Aut}(\mathcal{W})(\mathcal{R})$ if $R$ is an integral domain.
- There is an exact sequence of groups

$$1 \to \mu_2(R) \to \text{SL}_2(R) \times \text{SL}_2(R_0) \overset{\iota_{\mathcal{R}}}{\to} \text{GrAut}(\mathcal{W})(\mathcal{R}),$$

where $R_0 = \ker d$. The sequence is functorial in $\mathcal{R}$.

- Assume $R$ is an integral domain. Then, for every $\phi \in \text{GrAut}(\mathcal{W})(\mathcal{R})$,

  there is an étale extension $S$ of $\mathcal{R}$ such that

$$\phi_S \in \text{Im}(\iota_S : \text{SL}_2(S) \times \text{SL}_2(S_0) \overset{\iota_{S}}{\to} \text{GrAut}(\mathcal{W})(S)).$$
Sketch of Proof I:
review the definition relations for $\mathcal{W}$

$$\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1,$$
where

$$\mathcal{W}_0 = k[\partial]L \oplus k[\partial]T^1 \oplus k[\partial]T^2 \oplus k[\partial]T^3,$$
$$\mathcal{W}_1 = k[\partial]G^1 \oplus k[\partial]G^2 \oplus k[\partial]\overline{G}^1 \oplus k[\partial]\overline{G}^2.$$  

The $\lambda$–bracket on $\mathcal{W}$ is given by

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda T^i] = (\partial + \lambda)T^i,$$
$$[L_\lambda G^p] = (\partial + \frac{3}{2}\lambda)G^p, \quad [T^i_\lambda T^j] = i\epsilon_{ijk}T^k,$$
$$[L_\lambda \overline{G}^p] = (\partial + \frac{3}{2}\lambda)\overline{G}^p, \quad [G^p_\lambda G^q] = [\overline{G}^p_\lambda \overline{G}^q] = 0,$$
$$[T^i_\lambda G^p] = -\frac{1}{2}\sum_{q=1}^{2}\sigma^i_{pq}G^q, \quad [T^i_\lambda \overline{G}^p] = \frac{1}{2}\sum_{q=1}^{2}\sigma^i_{qp}\overline{G}^q,$$
$$[G^p_\lambda \overline{G}^q] = 2\delta_{pq}L - 2(\partial + 2\lambda)\sum_{i=1}^{3}\sigma^i_{pq}T^i,$$  

for $i, j = 1, 2, 3$ and $p, q = 1, 2$, where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Sketch of Proof II: simplification

Notation:

\[ T(x) := (x_{12} + x_{21}) T^1 + i(x_{12} - x_{21}) T^2 + 2x_{11} T^3, \]
\[ G(u) := u_{22} G^1 + u_{11} \overline{G}^1 - u_{12} G^2 + u_{21} \overline{G}^2. \]

for \( x = (x_{ij})_{2 \times 2} \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u = (u_{ij})_{2 \times 2} \in \text{Mat}_2(\mathbb{k}) \).

The \( \lambda \)-bracket on \( \mathcal{W} \) is rewritten as follows:

\[
\begin{align*}
[L_\lambda L] &= (\partial + 2\lambda)L, \\
[L_\lambda T(x)] &= (\partial + \lambda)T(x), \quad [T(x) \lambda T(y)] = T([x, y]), \\
[L_\lambda G(u)] &= (\partial + \frac{3}{2}\lambda)G(u), \quad [T(x) \lambda G(u)] = G(xu), \\
[G(u) \lambda G(v)] &= 2\text{tr}(uv^\dagger)L + (\partial + 2\lambda)T(uv^\dagger - vu^\dagger).
\end{align*}
\]

where \( x, y \in \mathfrak{sl}_2(\mathbb{k}) \), and \( u, v \in \text{Mat}_2(\mathbb{k}) \).

\[ \dagger : \text{Mat}_2(\mathbb{k}) \to \text{Mat}_2(\mathbb{k}), \quad u = u_{ij} \mapsto u^\dagger := \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}. \]
Sketch of Proof III: construction of automorphisms

Let \( \mathcal{R} = (R, \text{d}) \in \mathbb{k}\text{-drng} \) and \( R_0 = \ker \text{d} \).
Then every element \((A, B) \in \text{SL}_2(R) \times \text{SL}_2(R_0)\) defines an automorphism \( \theta_{A,B} \) of \( \mathcal{W} \):

\[
\begin{align*}
\theta_{A,B}(L) &= L + T(d(A)A^{-1}), \\
\theta_{A,B}(T(x)) &= T(AxA^{-1}), \\
\theta_{A,B}(G(u)) &= G(AuB^{-1}).
\end{align*}
\]
Outline

Affine Kac-Moody algebras

Lie algebras and Lie conformal algebras

Differential conformal algebras and their forms

The $N = 1, 2, 3$ conformal superalgebras

The small $N = 4$ conformal superalgebra

The large $N = 4$ conformal superalgebra
The large $N = 4$ superconformal Lie algebras

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3-m}{12} \delta_{m,-n}c, \]
\[ [L_m, U_n] = -nU_{m+n}, \]
\[ [T^{+i}_m, T^{+j}_n] = \epsilon_{ijk} T^{+k}_{m+n} - \frac{m}{12 \gamma} \delta_{ij} \delta_{m,-n}c, \]
\[ [T^{-i}_m, T^{-j}_n] = \epsilon_{ijk} T^{-k}_{m+n} - \frac{m}{12(1-\gamma)} \delta_{ij} \delta_{m,-n}c, \]
\[ [L_m, Q^p_{n'}] = -\left( \frac{1}{2} m + n' \right) Q^p_{m+n'}, \]
\[ [L_m, G^p_{n'}] = \left( \frac{1}{2} m - n' \right) G^p_{m+n'}, \]
\[ [Q^p_{m'}, Q^q_{n'}] = -\frac{\delta_{pq} \delta_{m',-n'}}{12\gamma(1-\gamma)} c, \]
\[ [T^{+i}_m, G^p_{n'}] = \alpha^{+i}_{pq} (G^q_{m+n'} - 2(1-\gamma)mQ^q_{m+n'}), \]
\[ [T^{-i}_m, G^p_{n'}] = \alpha^{-i}_{pq} (G^q_{m+n'} + 2\gamma mQ^q_{m+n'}), \]
\[ [Q^p_{m'}, G^q_{n'}] = \delta_{pq} U_{m'+n'} + 2(\alpha^{+i}_{pq} T^{+i}_{m'+n'} - \alpha^{-i}_{pq} T^{-i}_{m'+n'}), \]
\[ [G^p_{m'}, G^q_{n'}] = 2\delta_{pq} L_{m'+n'} + 4(n' - m')(\gamma \alpha^{+i}_{pq} T^{+i}_{m'+n'} + (1-\gamma) \alpha^{-i}_{pq} T^{-i}_{m'+n'}) \]
\[ + \frac{1}{3} \delta_{pq} \delta_{m',-n'} (m'^2 - 1/4)c, \]

for $i, j = 1, 2, 3, p, q = 1, 2, 3, 4, m, n \in \mathbb{Z}$, $m', n' \in \frac{1}{2} + \mathbb{Z}$.
The large $N = 4$ conformal superalgebra

the $\mathbb{k}[\partial]$–module

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_\bar{1},$$

where

$$\mathcal{M}_0 = \mathbb{k}[\partial] \otimes_{\mathbb{k}} (\mathbb{k}L \oplus \mathfrak{sl}_2(\mathbb{k}) \oplus \mathfrak{sl}_2(\mathbb{k}) \oplus \mathbb{k}U),$$

$$\mathcal{M}_\bar{1} = \mathbb{k}[\partial] \otimes_{\mathbb{k}} (\text{Mat}_2(\mathbb{k}) \oplus \text{Mat}_2(\mathbb{k})).$$

Notation:

$$T^+(x) = 1 \otimes (0 \oplus x \oplus 0 \oplus 0) \in \mathcal{M}_0,$$

$$T^-(x) = 1 \otimes (0 \oplus 0 \oplus x \oplus 0) \in \mathcal{M}_0,$$

$$G(u) = 1 \otimes (u \oplus 0) \in \mathcal{M}_\bar{1},$$

$$Q(u) = 1 \otimes (0 \oplus u) \in \mathcal{M}_\bar{1},$$

for $x \in \mathfrak{sl}_2(\mathbb{k})$ and $u \in \text{Mat}_2(\mathbb{k}).$
The large $N = 4$ conformal superalgebra defining relations

\[
\begin{align*}
[L_\lambda L] &= (\partial + 2\lambda)L, \\
[L_\lambda U] &= (\partial + \lambda)U, \\
[T^\pm (x)_\lambda T^\pm (y)] &= T^\pm ([x, y]), \\
[L_\lambda Q(u)] &= (\partial + \frac{1}{2} \lambda) Q(u), \\
[L_\lambda G(u)] &= (\partial + \frac{3}{2} \lambda) G(u), \\
[T^+ (x)_\lambda G(u)] &= G(xu) - \lambda Q(xu), \\
[T^- (x)_\lambda G(u)] &= -G(ux) - \lambda Q(ux), \\
[Q(u)_\lambda G(v)] &= 2\text{tr}(uv^\dagger)U - T^+(uv^\dagger - vu^\dagger) + T^- (u^\dagger v - v^\dagger u), \\
[G(u)_\lambda G(v)] &= 4\text{tr}(uv^\dagger)L + (\partial + 2\lambda) \left( T^+(uv^\dagger - vu^\dagger) + T^- (u^\dagger v - v^\dagger u) \right),
\end{align*}
\]

for $x, y \in \mathfrak{sl}_2(\mathbb{K})$ and $u, v \in \text{Mat}_2(\mathbb{K})$. 
Automorphism group

Proposition (Chang, Pianzola, 2013)

\[ \text{Aut}_{\tilde{D}-\text{conf}}(\mathcal{M}_{\tilde{D}}) \cong \left( \frac{\text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D})}{\langle (-I_2, -I_2) \rangle} \times \mathbb{G}_a(\hat{D}) \right) \rtimes \mathbb{Z}_2. \]
Sketch of Proof I

There is a group homomorphism

$$\mathbf{SL}_2(\hat{D}) \times \mathbf{SL}_2(\hat{D}) \to \text{Aut}_{\hat{D}\text{-conf}}(\mathcal{M}_{\hat{D}}), \quad (A, B) \mapsto \theta_{A,B},$$

where $\theta_{A,B}$ is defined by

$$L \mapsto L + T^+(d_t(A)A^{-1}) + T^{-1}(d_t(B)B^{-1}),$$

$$T^+(x) \mapsto T^+(AxA^{-1}),$$

$$T^-(y) \mapsto T^-(ByB^{-1}),$$

$$U \mapsto U,$$

$$G(u) \mapsto G(AuB^{-1}) - Q(d_t(A)uB^{-1} - Aud_t(B^{-1})),$$

$$Q(u) \mapsto Q(AuB^{-1}),$$

for $x \in \mathfrak{sl}_2(\mathbb{k})$ and $u \in \text{Mat}_2(\mathbb{k}).$
Sketch of Proof II

- There is a group homomorphism

\[ G_a(\widehat{D}) \to \text{Aut}_{\widehat{D}\text{-conf}}(\mathcal{M}_{\widehat{D}}), \quad s \mapsto \tau_s, \]

where \( \tau_s \) is defined by

- \( L \mapsto L + U \otimes s \),
- \( T^\pm(x) \mapsto T^\pm(x) \),
- \( U \mapsto U \),
- \( G(u) \mapsto G(u) + Q(su) \),
- \( Q(u) \mapsto Q(u) \),

for \( x \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u \in \text{Mat}_2(\mathbb{k}) \).
Sketch of Proof III

There is an element \( \omega \in \text{Aut}_{\hat{D}\text{-conf}}(\hat{\mathcal{M}}) \) of order 2 given by

\[
\begin{align*}
\omega(L) &= L, \\
\omega(T^{\pm}(x)) &= T^{\mp}(x), \\
\omega(U) &= -U, \\
\omega(G(u)) &= G(u^\dagger), \\
\omega(Q(u)) &= -Q(u^\dagger),
\end{align*}
\]

for \( x \in \mathfrak{sl}_2(\mathbb{k}) \) and \( u \in \text{Mat}_2(\mathbb{k}) \).

These automorphisms satisfy

\[
\begin{align*}
\theta_{A,B} \circ \tau_s &= \tau_s \circ \theta_{A,B}. \\
\omega \circ \theta_{A,B} \circ \omega &= \theta_{B,A}. \\
\omega \circ \tau_s \circ \omega &= \tau_{-s}.
\end{align*}
\]

Hence, we get the group homomorphism

\[
(\text{SL}_2(\hat{D}) \times \text{SL}_2(\hat{D}) \times G_a(\hat{D})) \rtimes \mathbb{Z}_2 \rightarrow \text{Aut}_{\hat{D}\text{-conf}}(\hat{\mathcal{M}}),
\]

which is surjective and has kernel \( \langle (-I_2, -I_2, 0, 0) \rangle \).
Twisted loop conformal superalgebras based on $\mathcal{M}$

Theorem (Chang, Pianzola, 2013)

There are exactly two twisted loop conformal superalgebras based on $\mathcal{M}$ (up to isomorphism of $\mathbb{k}$–conformal superalgebras), namely, $\mathcal{L}(\mathcal{M}, \text{id})$ and $\mathcal{L}(\mathcal{M}, \omega)$.

Sketch of Proof.

- $H^1_{\text{ct}} \left( \widehat{\mathbb{Z}}, \left( \frac{\text{SL}_2(\widehat{D}) \times \text{SL}_2(\widehat{D})}{\langle (-I_2, -I_2) \rangle} \right) \rtimes \mathbb{G}_a(\widehat{D}) \right) \rtimes \mathbb{Z}_2$ has exactly two elements.
- Centroid trick.
The associated Lie superalgebras

The two non-isomorphic Lie conformal superalgebras

\[ \mathcal{L}(\mathcal{M}, \text{id}) \text{ and } \mathcal{L}(\mathcal{M}, \omega) \]

yield two Lie superalgebras

\[ \text{Alg}(\mathcal{M}, \text{id}) \text{ and } \text{Alg}(\mathcal{M}, \omega). \]

Proposition (Chang, Pianzola, 2013)

*The two Lie superalgebras* \( \text{Alg}(\mathcal{M}, \text{id}) \text{ and } \text{Alg}(\mathcal{M}, \omega) \) *are not isomorphic.*
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Thank You!