Automorphisms and twisted forms of Lie conformal superalgebras

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Outlines

- Twisted forms of finite dimensional algebras.
- Differential conformal superalgebras and their twisted forms.
- The case of the $N = 1, 2, 3$ Lie conformal superalgebras.
- The case of the $N = 4$ Lie conformal superalgebras.
Twisted affine Kac-Moody algebras

Let

- \( g \) a finite dimensional simple Lie algebra over \( k \),
- \( \sigma \) an automorphism of \( g \) of order \( m \).

The twisted loop algebra \( L(g, \sigma) \) is the subalgebra

\[
L(g, \sigma) = \bigoplus_{i \in \mathbb{Z}} g_i \otimes_k k t^i \subseteq g \otimes_k k[t^{\pm \frac{1}{m}}],
\]

where \( g_i = \{ x \in g | \sigma(x) = \zeta^i m x \} \), \( \zeta_m \) is a \( m \)-th primitive root of unit.

In particular, \( L(g, \text{id}) = g \otimes_k k[t^{\pm 1}] \).
Twisted affine Kac-Moody algebras (continued)

Facts:

- Every affine Kac-Moody algebra is an extension of a twisted loop algebra.
- $L(\mathfrak{g}, \sigma)$ is not only a Lie algebra over $k$, but also a Lie algebra over the ring $k[t^{\pm 1}]$.
- $L(\mathfrak{g}, \sigma) \otimes_{k[t^{\pm 1}]} k[t^{\pm \frac{1}{m}}] \cong L(\mathfrak{g}, \text{id}) \otimes_{k[t^{\pm 1}]} k[t^{\pm \frac{1}{m}}] \cong \mathfrak{g} \otimes_k k[t^{\pm \frac{1}{m}}]$. $L(\mathfrak{g}, \sigma)$ is a twisted form of $L(\mathfrak{g}, \text{id})$ with respect to the ring extension $k[t^{\pm 1}] \rightarrow k[t^{\pm \frac{1}{m}}]$. 
Twisted forms of algebras

Setting:

- \( R \) is a commutative ring.
- \( A \) is an algebra over \( R \).
- \( R \to S \) a ring homomorphism.

Change of base ring from \( R \) to \( S \):

An \( S \)--algebra structure on \( A \otimes_R S \) given by

\[
(a \otimes r)(b \otimes s) = ab \otimes rs, \quad a, b \in A, r, s \in S
\]

\( S/R \)-form of \( A \):

An \( R \)--algebra \( B \) is called a \( S/R \)--form of \( A \) if

\[
A \otimes_R S \cong B \otimes_R S
\]

as \( S \)--algebras.
Classification of twisted forms

Theorem

Let $A$ be an algebra over $R$ and $R \rightarrow S$ a faithfully flat ring extension, then

the set of isomorphism classes of $S/R$–forms of $A$ 

is one to one $\leftrightarrow$ $H^1(S/R, \text{Aut}(A))$.


The automorphism group functor

\textbf{Aut}(A):

a functor from the category of \( R \)-algebras to the category of groups

\[ S \mapsto \text{Aut}_S(A \otimes_R S). \]

For a finite dimensional algebra \( \mathcal{A} \) over \( k \),

- \( \text{Aut}(\mathcal{A}) \) is a linear algebraic group.
- \( \text{Aut}(\mathcal{A}) \) is a representable functor:

\[ R \mapsto \text{Aut}_R(\mathcal{A} \otimes_k R). \]

If \( A = \mathcal{A} \otimes_k R \), then

\[ \text{Aut}(A) = \text{Aut}(\mathcal{A})_R. \]
Outlines

- Twisted forms of finite dimensional algebras.
- Differential conformal superalgebras and their twisted forms.
- The case of the $N = 1, 2, 3$ Lie conformal superalgebras.
- The case of the $N = 4$ Lie conformal superalgebras.
Lie conformal algebras over $k$

Definition

A $k$–Lie conformal algebra is $k[\partial]$–module $\mathcal{A}$ equipped with a $k$–bilinear product $-(n)- : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ for each $n \in \mathbb{N}$ satisfying: for $a, b, c \in \mathcal{A}$,

(C0) $a_{(n)} b = 0$ for $n \gg 0$,

(C1) $\partial_{\mathcal{A}} (a)_{(n)} b = -na_{(n-1)} b$,

(C2) $a_{(n)} b = - \sum_{j \in \mathbb{N}} (-1)^{j+n} \partial_{\mathcal{A}}^{(j)} (b_{(n+j)} a)$,

(C3) $a_{(m)} (b_{(n)} c) = \sum_{j=0}^{m} \binom{m}{j} (a_{(j)} b)_{(m+n-j)} c + b_{(n)} (a_{(m)} c)$,
The $\lambda$–bracket

Let $\mathcal{A}$ be a $k$–Lie conformal algebra. For $a, b \in \mathcal{A}$, define

$$[a_\lambda b] = \sum_{n \in \mathbb{N}} \lambda^{(n)} a(n)b,$$

where $\lambda^{(n)} = \frac{1}{n!} \lambda^n$.

The axioms (C0)-(C3) are equivalent respectively to

(C0)$_\lambda$ $[a_\lambda b]$ is a polynomial in $\lambda$,

(C1)$_\lambda$ $[(\partial a)_\lambda b] = -\lambda[a_\lambda b]$,

(C2)$_\lambda$ $[a_\lambda b] = -[b_{-\lambda-\partial} a]$,

(C3)$_\lambda$ $a_\lambda[b_\mu c] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]$. 
Example

The centreless Virasoro Lie algebra:

- basis: \( \{l_n, n \in \mathbb{Z}\} \),
- \([l_m, l_n] = (m - n)l_{m+n}, m, n \in \mathbb{Z}\).

Form a formal series:

\[
L(z) = \sum_{n \in \mathbb{Z}} l_n z^{-n-2}.
\]

The operator product expansion:

\[
[L(z), L(w)] := \sum_{m,n \in \mathbb{Z}} [l_m, l_n] z^{-m-2} w^{-n-2}
\]

\[
= (\partial_w L(w)) \delta(z - w) + 2L(w) \partial_w \delta(z - w)
\]

where

\[
\delta(z - w) = \frac{1}{z} \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n.
\]
Example (continued)

The Virasoro Conformal algebra:

- the $k[\partial]$–module $\mathcal{V} = k[\partial]L$.
- the $n$-th product:

\[
L_{(0)}L = \partial L, \\
L_{(1)}L = 2L, \\
L_{(n)}L = 0, \quad n \geq 2.
\]

The $\lambda$–bracket

\[
[L_\lambda L] = (\partial + 2\lambda)L.
\]
Differential rings

To define conformal superalgebra over a ring,

\[
\text{rings} \quad \mapsto \quad \text{differential rings}
\]

A $k$–differential ring is a pair $\mathcal{R} = (R, \delta)$ consisting of

- a commutative associative $k$–algebra $R$, and
- a $k$–linear derivation $\delta : R \to R$.

$k$–differential rings form a category $k\text{-drg}$, where a morphism

\[
f : \mathcal{R} = (R, \delta_R) \to \mathcal{S} = (S, \delta_S)
\]

is a homomorphism of $k$–algebra $f : R \to S$ such that

\[
f \circ \delta_R = \delta_S \circ f.
\]
Differential Lie conformal superalgebras

Definition
Let $\mathcal{R} = (R, \delta)$ be a $k$–differential ring, an $\mathcal{R}$–Lie conformal superalgebra is a triple $(\mathcal{A}, \partial\mathcal{A}, -(\cdot)(\cdot))_{n \in \mathbb{N}}$ consisting of

- a $\mathbb{Z}/2\mathbb{Z}$–graded $R$–module $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$,
- $\partial\mathcal{A} \in \text{End}_k(\mathcal{A})$ stabilizing the even and odd parts of $\mathcal{A}$, and
- a $k$–bilinear product $(a, b) \mapsto a(\cdot)b$, $a, b \in \mathcal{A}$ for each $n \in \mathbb{N}$,

satisfying the following axioms for $r \in D, a, b, c \in \mathcal{A}, m, n \in \mathbb{N}$:

(CS0) $a(\cdot)b = 0$ for $n \gg 0$,
(CS1) $\partial\mathcal{A}(a)(\cdot)b = -na_{(n-1)}b$ and $a(\cdot)\partial\mathcal{A}(b) = \partial\mathcal{A}(a(\cdot)b) + na_{(n-1)}b$,
(CS2) $\partial\mathcal{A}(ra) = r\partial\mathcal{A}(a) + \delta(r)a$,
(CS3) $a(\cdot)(rb) = r(a(\cdot)b)$ and $(ra)(\cdot)b = \sum_{j \in \mathbb{N}} \delta(j)(r)(a_{(n+j)}b)$,
(CS4) $a(\cdot)b = -p(a, b) \sum_{j \in \mathbb{N}} (-1)^{i+n} \partial^{(j)}\mathcal{A}(b_{(n+j)}a)$,
(CS5) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}c + p(a, b)b_{(n)}(a_{(m)}c)$,
Change the base ring

Let

- $\mathcal{R} = (R, \delta_R)$ be a $k$–differential ring,
- $\mathcal{A}$ an $\mathcal{R}$-Lie conformal superalgebra,
- $\mathcal{R} = (R, \delta_R) \to \mathcal{S} = (S, \delta_S)$ an extension of $k$–differential rings.

Then we can form a $\mathcal{S}$–Lie conformal superalgebra $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{S}$:

- The underlying $\mathcal{S}$–module: $\mathcal{A} \otimes_R \mathcal{S}$,
- $\partial_{\mathcal{A} \otimes_R \mathcal{S}} = \partial_{\mathcal{A}} \otimes \text{id} + \text{id} \otimes \delta_S$,
- The $n$-th product for $n \in \mathbb{N}$:

\[
(a \otimes f)(n)(b \otimes g) = \sum_{j \in \mathbb{N}} a_{(n+j)} b \otimes \delta_S^{(j)}(f) g,
\]

where $\delta_S^{(j)} = \frac{1}{j!} \delta_S^j$. 

Let

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- $\mathcal{A}$ an $\mathcal{R}$-Lie conformal superalgebra,
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\]

where $\delta_S^{(j)} = \frac{1}{j!} \delta_S^j$. 

Twisted forms

**Definition (Kac, Lau, Pianzola, 2009)**

Let

- $\mathcal{R} \to S$ be an extension of $k$–differential rings,
- $\mathcal{A}$ an $\mathcal{R}$–Lie conformal superalgebra.

An $S/\mathcal{R}$–form of $\mathcal{A}$ is

an $\mathcal{R}$–Lie conformal superalgebra $\mathcal{B}$ such that

$$\mathcal{B} \otimes_{\mathcal{R}} S \cong \mathcal{A} \otimes_{\mathcal{R}} S$$

as $S$–Lie conformal superalgebras.
Automorphisms

Let $\mathcal{A}$ be an $\mathcal{R}$–Lie conformal superalgebra, a map $\phi : \mathcal{A} \to \mathcal{A}$ is called an $\mathcal{R}$–automorphism of $\mathcal{A}$ if:

- $\phi$ is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$–graded $\mathcal{R}$–modules,
- $\phi \circ \partial_\mathcal{A} = \partial_\mathcal{A} \circ \phi$,
- $\phi$ preserves all $n$-th products.

All $\mathcal{R}$–automorphisms of $\mathcal{A}$ form a group, denoted by $\text{Aut}_{\mathcal{R}\text{-conf}}(\mathcal{A})$. 
The automorphism group functor $\text{Aut}(A)$:

$$\begin{array}{ccc}
\text{the category of } & \rightarrow & \text{the category of } \\
\mathcal{R}-\text{extensions} & & \text{groups} \\
S & \mapsto & \text{Aut}_{S\text{-conf}}(A \otimes \mathcal{R} S)
\end{array}$$
Classification of twisted forms

Theorem (Kac, Lau, Pianzola, 2009)

Let

- $\mathcal{R} \to S$ be a faithfully flat extension of $k$–differential rings,
- $A$ an $\mathcal{R}$–conformal superalgebra.

Then

the set of

isomorphism classes of $S/\mathcal{R}$–forms of $A$

(up to $\mathcal{R}$–automorphism)

is one to one

$\longleftrightarrow$

$H^1(S/\mathcal{R}, \text{Aut}(A))$
Twisted loop conformal superalgebras

Let

- $\mathcal{A}$ be a Lie conformal superalgebra over $k$,
- $\sigma$ an automorphism of $\mathcal{A}$ of order $m$,
- $\mathcal{D} = (k[t^\pm 1], \frac{d}{dt})$, $\mathcal{D}_m = (k[t^\pm \frac{1}{m}], \frac{d}{dt})$,
- $\hat{\mathcal{D}} = \lim_{\rightarrow} \mathcal{D}_m = (k[t^q], q \in \mathbb{Q}, \frac{d}{dt})$.

A twisted loop Lie conformal algebra

$$\mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \otimes k t^i \subseteq \mathcal{A} \otimes_k \mathcal{D}_m$$

where $\mathcal{A}_i = \{ a \in \mathcal{A} | \sigma(a) = \zeta_m^i a \}$, $i \in \mathbb{Z}$, $\zeta_m$ is a primitive $m$-th root of unit.
Twisted loop conformal superalgebras (continued)

Facts:

- $\mathcal{L}(\mathcal{A}, \sigma)$ is a $\mathcal{D}$–Lie conformal superalgebra;
- $\mathcal{L}(\mathcal{A}, \sigma)$ is a $\mathcal{D}_m/\mathcal{D}$–form of $\mathcal{L}(\mathcal{A}, \text{id}) = \mathcal{A} \otimes_k \mathcal{D}$.
- $\mathcal{L}(\mathcal{A}, \sigma)$ is a $\hat{\mathcal{D}}/\mathcal{D}$–form of $\mathcal{L}(\mathcal{A}, \text{id})$. 
Finiteness condition

Proposition

Let $A$ an $\mathcal{D}$–conformal superalgebras. If $A$ satisfies:

There exist $a_1, \cdots, a_n \in A$ such that

\[ \{ \partial^\ell_A(ra_i) | r \in \mathcal{D}, \ell \geq 0 \} \text{ spans } A. \]

Then

\[ H^1(\hat{\mathcal{D}}/\mathcal{D}, \text{Aut}(A)) \cong H^1_{ct}(\pi_1(D), \text{Aut}(A)(\hat{\mathcal{D}})), \]

where $\pi_1(D)$ is the algebraic fundamental group of $\text{Spec}(D)$ at the geometric point $a = \text{Spec}(\mathbb{C}(t))$ and $H^1_{ct}$ denotes the continuous non-abelian cohomology of the profinite group $\pi_1(D) = \hat{\mathbb{Z}}$ acting (continuously) on the group $\text{Aut}(A)(\hat{\mathcal{D}})$. 
Centroid trick method

The centroid of $\mathcal{A}$ is

$$\text{Ctd}_{\mathcal{R}}(\mathcal{A}) = \{ \chi \in \text{End}_{R\text{-smod}}(\mathcal{A}) | \chi(a_{(n)}b) = a_{(n)}\chi(b), \forall a, b \in \mathcal{A}, n \in \mathbb{Z}_+ \},$$

**Proposition**

Let $\mathcal{A}_i, i = 1, 2$ be two $\mathcal{R}$–conformal superalgebras. Assume that $\text{Aut}_k(\mathcal{R}) = 1$. If $\mathcal{R} \to \text{Ctd}_k(\mathcal{A}_i), i = 1, 2$ are both $k$–algebra isomorphisms. Then

$$\mathcal{A}_1 \cong \mathcal{A}_2 \text{ as } k\text{–Lie conformal superalgebras}$$

if and only if

$$\mathcal{A}_1 \cong \mathcal{A}_2 \text{ as } \mathcal{R}\text{–Lie conformal superalgebras}.$$
Outlines for completed works

- Automorphisms and twisted forms of $N = 1, 2, 3$ Lie conformal superalgebras
- Automorphisms of the $N = 4$ Lie conformal superalgebras
$N = 1, 2, 3$ Lie conformal superalgebras $\mathcal{K}_N$

$$\mathcal{K}_N = k[\partial] \otimes_k \Lambda(N),$$

where $\Lambda(N)$ is the Grassmann superalgebra over $k$ in $N$ variables $\xi_1, \ldots, \xi_N$.

For $n \in \mathbb{N}$, the $n$-th product on $\mathcal{K}_N$ is given by

$$f_{(0)}g = \left(\frac{1}{2} |f| - 1\right) \partial \otimes fg + \frac{1}{2} (-1)^{|f|} \sum_{i=1}^{N} (\partial_i f)(\partial_i g)$$

$$f_{(1)}g = \left(\frac{1}{2} (|f| + |g|) - 2\right) fg,$$

$$f_{(n)}g = 0, n \geq 2.$$
A subgroup functor

\[ \text{GrAut}(\mathcal{K}_N)(\mathcal{R}) = \{ \phi \in \text{Aut}(\mathcal{K}_N)(\mathcal{R}) \middle| \phi(k \otimes_k \Lambda(N) \otimes_k R) \subseteq k \otimes_k \Lambda(N) \otimes_k R \}. \]

Theorem

Let \( \mathcal{R} = (R, \delta) \) be an arbitrary \( k \)-differential ring.

For \( N = 1, 2, 3 \),

\[ \text{GrAut}(\mathcal{K}_N)(\mathcal{R}) \cong O_N(R), \]

where \( O_N \) is the group scheme of \( N \times N \)-orthogonal matrices.

The isomorphism is functorial in \( \mathcal{R} \).
The construction of the isomorphism.

Given $A \in O_N(R)$, it defines an automorphism $\varphi_A$ of $K_N \otimes_k R$

- $N = 1, A = a \in R,$
  $\varphi_A(1) = 1, \varphi_A(\xi_1) = \xi_1 \otimes a.$

- $N = 2, A = (a_{ij}),$
  $\phi_A(1) = 1 + \xi_1 \xi_2 \otimes r, \quad \phi_A(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21},$
  $\phi_A(\xi_1 \xi_2) = \xi_1 \xi_2 \otimes \det(A), \quad \phi_A(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$

where $\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = 2\delta(A)A^T.$
The construction of the isomorphism (continued)

For $N = 3$,

$$
\phi_A(1) = 1 + \sum_{l=1}^{3} \epsilon_{mnl} \xi_m \xi_n \otimes r_l,
$$

$$
\phi_A(\xi_j) = \sum_{l=1}^{3} \xi_l \otimes a_{lj} + \xi_1 \xi_2 \xi_3 \otimes s_j,
$$

$$
\phi_A(\xi_i \xi_j) = \epsilon_{ijl} \sum_{l'=1}^{3} \epsilon_{mnl'} \xi_m \xi_n \otimes A_{l'l},
$$

$$
\phi_A(\xi_1 \xi_2 \xi_3) = \xi_1 \xi_2 \xi_3 \otimes \text{det}(A),
$$

$i, j = 1, 2, 3, i \neq j$, where $A_{l'l}$ is the cofactor of $a_{l'l}$ in $A$ and

$$
\begin{bmatrix}
0 & r_3 & -r_2 \\
-r_3 & 0 & r_1 \\
-2 & -r_1 & 0
\end{bmatrix} = 2\delta(A)A^T,

\begin{bmatrix}
0 & s_3 & -s_2 \\
-s_3 & 0 & s_1 \\
s_2 & -s_1 & 0
\end{bmatrix} = 2(\text{det}A)A^T\delta(A).$$
**Theorem**

Let $\mathcal{R} = (R, \delta)$ be a $k$–differential ring. If $R$ is an integral domain, then for $N = 1, 2, 3$,  

$$\text{GrAut}(K_N)(\mathcal{R}) = \text{Aut}(K_N)(\mathcal{R}).$$
Classification up to $\mathcal{R}$–isomorphism

**Theorem**

Let $N = 1, 2, 3$. There are exactly two $\hat{D}/D$–forms of $\mathcal{K}_N \otimes_k D$ (up to isomorphism of $D$–Lie conformal superalgebras):

$$\mathcal{L}(\mathcal{K}_N, \text{id}) \text{ and } \mathcal{L}(\mathcal{K}_N, \omega_N).$$
Classification up to $k$–isomorphism

**Proposition (Chang, Pianzola, 2011)**

Let $\mathcal{A} = \mathcal{L}(\mathcal{K}_N, \sigma)$ where $\sigma$ is an automorphism of $\mathcal{K}_N$ of order $m$, and $N = 1, 2, 3$. Then the canonical map

$$D \rightarrow \text{Ctd}_k(\mathcal{A})$$

is a $k$–algebra isomorphism.

**Theorem (Chang, Pianzola, 2011)**

There are exactly two twisted loop Lie conformal superalgebra (up to $k$–isomorphism) based on each $\mathcal{K}_N, N = 1, 2, 3$:

$\mathcal{L}(\mathcal{K}_N, \text{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$. 
Corresponding Lie superalgebras

Let $\mathcal{A}$ be a $k$–Lie conformal superalgebra, every twisted loop algebra $\mathcal{L}(\mathcal{A}, \sigma)$ corresponds to a Lie superalgebra over $k$: 

$$\text{Alg}(\mathcal{A}, \sigma) = \frac{\mathcal{L}(\mathcal{A}, \sigma)}{\left(\partial + \frac{d}{dt}\right)\mathcal{L}(\mathcal{A}, \sigma)},$$

where the super bracket comes from the 0-th product on $\mathcal{L}(\mathcal{A}, \sigma)$.

For $\mathcal{K}_N$, $N = 1, 2, 3$, $\mathcal{L}(\mathcal{K}_N, \text{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$ yield $\text{Alg}(\mathcal{K}_N, \text{id})$ and $\text{Alg}(\mathcal{K}_N, \omega_N)$.

Theorem (Chang, Pianzola, 2011)

$\text{Alg}(\mathcal{K}_N, \text{id})$ and $\text{Alg}(\mathcal{K}_N, \omega_N)$ are non-isomorphic.
Outlines for completed works

- Automorphisms and twisted forms of $N = 1, 2, 3$ Lie conformal superalgebras
- Automorphisms of the $N = 4$ Lie conformal superalgebras
\[ N = 4 \text{ Lie conformal superalgebra } \mathcal{F} \]

\[ \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = (k[\partial] \otimes V_0) \oplus (k[\partial] \otimes V_1), \]

where

\[ V_0 = kL \oplus kT^1 \oplus kT^2 \oplus kT^3, \]
\[ V_1 = kG^1 \oplus kG^2 \oplus \bar{k}^{-1} \oplus \bar{k}^{-2}. \]

The \( \lambda \)–bracket on \( \mathcal{L} \) is given by

\[
\begin{align*}
[L_\lambda L] &= (\partial + 2\lambda)L, & [L_\lambda T^i] &= (\partial + \lambda)T^i, \\
[L_\lambda G^p] &= (\partial + \frac{3}{2}\lambda)G^p, & [T^i_\lambda T^j] &= i\epsilon_{ijk}T^k, \\
[L_\lambda \bar{G}^p] &= (\partial + \frac{3}{2}\lambda)\bar{G}^p, & [G^p_\lambda G^q] &= 0, \\
[T^i_\lambda G^p] &= -\frac{1}{2} \sum_{q=1}^{2} \sigma_{pq}^i G^q, & [\bar{G}^p_\lambda \bar{G}^q] &= 0, \\
[T^i_\lambda \bar{G}^p] &= \frac{1}{2} \sum_{q=1}^{2} \sigma_{qp}^i \bar{G}^q, \\
[G^p_\lambda \bar{G}^q] &= 2\delta_{pq}L - 2(\partial + 2\lambda) \sum_{i=1}^{3} \sigma_{pq}^i T^i,
\end{align*}
\]
A subgroup functor

\[ \text{GrAut}(\mathcal{F})(\mathcal{R}) = \{ \phi \in \text{Aut}(\mathcal{F})(\mathcal{R}) | \phi(k \otimes_k V \otimes_k R) \subseteq k \otimes_k V \otimes_k R \}, \]

where \( V = V_0 \oplus V_1 \) and \( \mathcal{R} = (R, \delta) \) is a \( k \)-differential ring.
GrAut(\mathcal{F})

**Theorem**

For every $k$–differential ring $\mathcal{R} = (R, \delta)$, there is an exact sequence of groups:

$$1 \to \mu_2(R_0) \to \text{SL}_2(R) \times \text{SL}_2(R_0) \xrightarrow{\iota_{\mathcal{R}}} \text{GrAut}(\mathcal{F})(\mathcal{R}),$$

where

- $R_0 = \ker \delta$,
- $\text{SL}_2$ is the group scheme of matrices with determinant 1,
- $\mu_2$ is the group scheme of square roots of 1.

The exact sequence is functorial in $\mathcal{R}$. 
The “surjectivity” of $\iota_\mathcal{R}$

**Proposition**

For every $k$–differential ring $\mathcal{R} = (R, \delta_R)$ with $R$ an integral domain, and $\varphi \in \text{GrAut}(\mathcal{F})(\mathcal{R})$, there is

- an étale extension $S = (S, \delta_S)$ of $\mathcal{R}$, and
- an element $(A, B) \in \text{SL}_2(S) \times \text{SL}_2(S_0)$

such that $\iota_S(A, B) = \varphi_S$, where $\varphi_S$ is the image of $\varphi$ under $\text{GrAut}(\mathcal{F})(\mathcal{R}) \to \text{GrAut}(\mathcal{F})(S)$.

**Proposition**

Let $\mathcal{R} = (R, \delta)$ be a $k$–differential ring. If

- $R$ is an integral domain, and
- the étale cohomology set $H^1_{\text{ét}}(R, \mu_2)$ is trivial,

then $\iota_\mathcal{R}$ is surjective.
Theorem

Let $\mathcal{R} = (R, \delta)$ be a $k$–differential ring such that $R$ is an integral domain. Then

$$\text{GrAut}(\mathcal{F})(\mathcal{R}) = \text{Aut}(\mathcal{F})(\mathcal{R}).$$
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Thank You!