A NONLOCAL PARABOLIC SYSTEM In LINEAR THERMOELASTICITY

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Abstract A system of parabolic integro-differential equations arising in the study of thermoelastic contact of two rods is considered. The questions of existence, uniqueness and continuous dependence of the solution upon the data are demonstrated via the theoretical potential solution representation technique and theory of Volterra equations of second kind. An application to linear thermoelasticity is also given.

Keywords: Parabolic, integro-differential, non-local, thermoelastic contact.

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1. Introduction

In this paper we consider the following parabolic system of finding $U = (u^1(x,t), u^2(x,t))$ such that

$$u_t^1 - c_1 u_{xx}^1 = \frac{b_1}{l_1}\frac{d}{dt}\max\{I(U),0\}, \quad \text{in } Q_T^1 = (0,l_1)\times J, \tag{1.1}$$

$$u_t^2 - c_2 u_{xx}^2 = \frac{b_2}{1-l_2}\frac{d}{dt}\max\{I(U),0\}, \quad \text{in } Q_T^2 = (l_2,1)\times J, \tag{1.2}$$

$$u^1(x,0) = \phi_1(x), \qquad 0 < x < l_1, \tag{1.3}$$

$$u^2(x,0) = \phi_2(x), \qquad l_2 < x < 1, \tag{1.4}$$

$$\mu_1 u_x^1(0,t) + \nu_1 u^1(0,t) = f_1(t), \qquad 0 < t < T, \tag{1.5}$$

$$\mu_2 u_x^2(1,t) + \nu_2 u^2(1,t) = f_2(t), \qquad 0 < t < T, \tag{1.6}$$

$$k_1 u_x^1(l_1,t) = k_2 u_x^2(l_2,t), \qquad 0 < t < T, \tag{1.7}$$

$$-k_1 u_x^1(l_1,t) = k(u^1(l_1,t) - u^2(l_2,t)), \qquad 0 < t < T, \tag{1.8}$$

where the integral operator $I(\cdot)$ is defined by

$$I(U) = a_1 \int_0^{l_1} u^1(x,t)dx + a_2 \int_{l_2}^1 u^2(x,t)dx + g(t), \qquad 0 < t < T, \tag{1.9}$$

and $c_i > 0$, $a_i > 0$, $k_i > 0$, b_i, μ_i, ν_i $\mu_i^2 + \nu_i^2 \neq 0, (i = 1, 2)$ and k are constants related to the physical parameters, $f_i(t)$, $\phi_i(x)$, $(i = 1, 2)$ and $g(t)$ are known functions, and $0 < l_1 \leq l_2 < 1$, $J = (0, T)$ for $T > 0$.

This system of equations is derived in [2, 6] in the study of quasi-static thermoelastic contact of two rods due to the decoupling of the temperatures and displacements. Under some appropriate assumptions on the data (smooth) and the physical constants (small), the authors of [2] shown that the unique global strong solution $U = (u^1, u^2)$ exist in Sobolev space $W^{2,1}(Q_T^1) \times W^{2,1}(Q_T^2)$. A simple version of nonlocal parabolic integro-differential equation modelling the thermoelastic contact of one rod at one end has been received a considerable attention recently. We refer the readers to [1, 7, 12, 14] for mathematical modelling and the well-posedness of the problems using variational formulations, [3, 5, 6] for numerical approximations by finite difference and finite element, [5] for using monotonic operator theory approach and treating periodic solutions, [14] for using a priori estimates and Sobolev embedding inequalities, [4] for using solution representation technique via an equivalent nonlocal reformulation. although the techniques employed in [4] are elementary, the results obtained there are more general than those in [1, 5, 7, 12, 14]. That is, all results except the periodic solutions in [1, 5, 7, 12, 14] are special cases of [4]. In addition, the results of [4] can not be improved with respect to the conditions imposed upon the critical physical constants. See [4] for details. A similar nonlocal boundary value problem was considered in [5, 7, 10, 11].

In this paper we follows the idea in [4] to study the problem (1.1)-(1.8). First problem (1.1)-(1.8) is transformed into an equivalent system of parabolic equations with nonlocal and nonlinear boundary conditions, and then the existence, uniqueness and continuous dependence of the solutions upon the data are proved for the equivalent system using classical solution representation technique and theory of Volterra equations of second kind with weakly singular kernels. The well-posedness of system (1.1)-(1.8) thus follows from the equivalence of the two systems.

Some remarks are as follows: First the system (1.1)-(1.8) has been studied in [2]. The method employed there requires that $f_i(t) \in C^1(0,T), i = 1,2$, $\phi_1(x) \in C^1(0,l_1)$, $\phi_2 \in C^1(l_2,1)$ and some compatibility conditions at $t = 0$, which in fact is due to the weak formulation and analysis in $W^{2,1}(Q_T^1) \times W^{2,1}(Q_T^2)$. Second our results in this paper not only take those of [2] as a special case, but also allow the non-smooth data, ie, f_i, ϕ_i, $i = 1,2$, can be piecewise continuous functions or even f_i can be singular at $t = 0$. Moreover our results are optimal with respect to the parameters a_i, b_i, and c_i, $i = 1,2$.

Definition 1.1: A pair (u^1, u^2) is said to be a solution of problem (1.1)-(1.8) if $u^i \in C^2(Q_T^i) \cap L^\infty(Q_T^i)$, $(i = 1,2)$, and satisfy (1.1)-(1.2), and initial and boundary conditions almost everywhere.

Remark: If the data is non-smooth, say piecewise continuous or $f_i(t) = O(t^{-\alpha})$ at $t = 0$, $i = 1$ or 2, then solutions u^1 and u^2 are allowed to have similar jumps or singularities at the boundaries.

This paper is organized in the following way. In section 2 some necessary results are given which are used throughout this paper. In section 3 problem (1.1)-(1.8) is transformed into an equivalent system of parabolic equations with nonlocal boundary conditions, then the problem of well-posedness are considered. Application to thermoelastic contact of two rods is given in section 4, which suggests that the solutions of equations in [2] for thermoelastic contact of two rods always exist and are unique for all physical parameters.

Finally our method in this paper can also be applied in principle to study the thermoelastic contact of two discs subject to axially symmetric temperature fields and displacements. Since the decoupling procedure and analysis are much more complicated the complete mathematical analysis will be given in a separate paper.

2. Preliminaries

In this section we list several classical results from [3, 8, 9] which are used throughout this paper.

Let $K(x,t)$ be the heat kernel [3] defined by

$$K(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \qquad t > 0 \tag{2.1}$$

and θ-function [3] be defined by

$$\theta(x,t)=\sum_{n=-\infty}^{\infty} K(x+2n,t), \quad t>0.$$

Let $\phi \in C[0,1]$ and

$$V(x,t,\phi)=\int_0^1 [\theta(x+\xi,t)+\theta(x-\xi,t)]\phi(\xi)d\xi. \tag{2.2}$$

Then $V(x,t,\phi)$ solves [3]

$$\begin{aligned} &V_t = V_{xx}, \quad 0<x<1, \quad t>0, \\ &V(x,0)=\phi(x), \quad 0<x<1, \\ &V_x(0,t)=V_x(1,t)=0, \quad t>0. \end{aligned} \tag{2.3}$$

Lemma 2.1. Let u be the solution of

$$\begin{aligned} &u_t = u_{xx}, \quad 0<x<1, \quad t>0, \\ &u(x,0)=\phi(x), \quad 0<x<1, \\ &u(0,t)=f(t), \quad u_x(1,t)=g(t), \quad t>0, \end{aligned} \tag{2.4}$$

where $f,\ g, \in C[0,T]$ and $\phi \in C[0,1]$. Then u can

$$\begin{aligned} u &= V(x,t,\phi)-2\int_0^t \frac{\partial K}{\partial x}(x,t-s)\psi_1(s)ds \\ &\quad +2\int_0^t K(x-1,t-s)\psi_2(s)ds, \end{aligned} \tag{2.5}$$

where ψ_1 and ψ_2 in $C[0,T]$ are the unique solutions of Volterra system

$$f(t) = V(0,t,\phi)+\psi_1(t)+2\int_0^t K(-1,t-s)\psi_2(s)ds, \tag{2.6}$$

$$g(t) = 2\int_0^t \frac{\partial^2 K}{\partial x^2}(1,t-s)\psi_1(s)ds+\psi_2(t), \tag{2.7}$$

where $V(x,t,\phi)$ is defined in (2.2).

Proof: See [3]. Q.E.D.

Remark: (i) If f and g are piecewise continuous and bounded, then Lemma 2.1 is still valid with $\psi_i(t), i=1,2$ being piecewise continuous and bounded.

(ii) If f and g have singularity $f(t)=g(t)=O(t^{-\alpha})$ at $t=0$ with $0<\alpha<1$, then the solutions $\psi_i(t), i=1,2$, have the same singularity at $t=0$. (iii) If $f,g\in L^1(0,T)$, then $\psi_i(t)\in L^1(0,T), i=1,2$ and solution u in (2.5) will be in $L^1((0,1)\times(0,T))$.

Corollary 2.1. Under assumption of Lemma 2.1. If u solves for $0<l<1$,

$$\begin{aligned}
&u_t = cu_{xx}, \quad 0<x<l, \quad t>0,\\
&u(x,0)=\phi(x), \quad 0<x<l,\\
&u(0,t)=f(t), \quad u_x(l,t)=g(t), \quad t>0.
\end{aligned}$$

where $c>0$. Then u can be represented by

$$\begin{aligned}
u = V^1(x,t,\phi,c,l) \quad &- \quad 2\int_0^t \frac{\partial K}{\partial x}(\frac{x}{l},\frac{c}{l^2}(t-s))\frac{c}{l}\psi_1(s)ds \qquad (2.8)\\
&+ \quad 2\int_0^t K(\frac{x-l}{l},\frac{c}{l^2}(t-s))\frac{c}{l^2}\psi_2(s)ds
\end{aligned}$$

where ψ_1 and ψ_2 in $C[0,T)$ are the unique solutions of Volterra system

$$\begin{aligned}
f(t) &= V^1(0,t,\phi,c,l)+\psi_1(t)+2\int_0^t K(-1,\frac{c}{l^2}(t-s))\frac{c}{l^2}\psi_2(s)ds, \qquad (2.9)\\
lg(t) &= -2c\int_0^t \frac{\partial^2 K}{\partial x^2}(1,\frac{c}{l^2}(t-s))\psi_1(s)ds+\psi_2(t), \qquad (2.10)
\end{aligned}$$

where

$$V^1(x,t,\phi,c,l)=\frac{1}{l}\int_0^l\left\{\theta(\frac{x+\xi}{l},\frac{c}{l^2}t)+\theta(\frac{x-\xi}{l},\frac{c}{l^2}t)\right\}\phi(\xi)d\xi.$$

Proof: Let $y=x/l$ and $\tau=ct/l^2$, then it follows from a simple calculation that

$$\begin{aligned}
&u_\tau = u_{yy}, \quad 0<y<1, \quad \tau>0,\\
&u(y,0)=\phi(ly), \quad 0<y<1,\\
&u(0,\tau)=f(\frac{l^2}{c}\tau), \quad u_y(1,\tau)=lg(\frac{l^2}{c}\tau), \quad \tau>0.
\end{aligned}$$

Then, Corollary follows from Lemma 2.1. Q.E.D.

Corollary 2.2. Under assumption of Lemma 2.1. If u solves for $0 < l < 1$,

$$u_t = cu_{xx}, \quad l < x < 1, \quad t > 0,$$
$$u(x,0) = \phi(x), \quad l < x < 1,$$
$$u_x(l,t) = f(t), \quad u(1,t) = g(t), \quad t > 0,$$

where $c > 0$. Then u can be represented by

$$\begin{aligned} u &= V^2(x,t,\phi,c,l) - 2\int_0^t K(\frac{x-l}{1-l}, \frac{c}{(1-l)^2}(t-s))\frac{c}{(1-l)^2}\psi_1(s)ds \\ &\quad +2\int_0^t \frac{\partial K}{\partial x}(\frac{x-l}{1-l} - 1, \frac{c}{(1-l)^2}(t-s))\frac{c}{1-l}\psi_2(s)ds, \end{aligned} \tag{2.11}$$

where ψ_1 and ψ_2 in $C[0,T)$ are the unique solutions of Volterra system

$$(1-l)f(t) = \psi_1(t) + 2c\int_0^t \frac{\partial^2 K}{\partial x^2}(-1, \frac{c}{(1-l)^2}(t-s))\psi_2(s)ds, \tag{2.12}$$

$$\begin{aligned} g(t) &= V^2(1,t,\phi,c,l) + \psi_2(t) \\ &\quad -2\int_0^t K(1, \frac{c}{l^2}(t-s))\frac{c}{(1-l)^2}\psi_1(s)ds \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} V^2(x,t,\phi,c,l) &= \frac{1}{1-l}\int_l^1 \left\{\theta(\frac{x+\xi-2l}{1-l}, \frac{c}{(1-l)^2}t)\right. \\ &\quad \left. + \theta(\frac{x-\xi}{1-l}, \frac{c}{(1-l)^2}t)\right\}\phi(\xi)d\xi. \end{aligned}$$

Proof: Let $y = (x-l)/(1-l)$ and $\tau = ct/l^2$. The rest of proof is similar to the proof of Corollary 2.1. Q.E.D.

Lemma 2.2. Let $G(t) = (g_1(t), \cdots, g_N(t))^T$, $H(t) = (h_1(t), \cdots, h_N(t))^T \in (L^1(0,T))^N$, $M(t) = (m_{ij}(t))$, $A(t) = (a_{i,j}(t))$ and $B(t) = (b_{i,j}(t))$ are $N \times N$ matrices such that for some $C_a, C_b, C_M > 0$

$$||A(t)|| = \max_{1\le i,j\le N} |a_{i,j}(t)| \le C_a t^{-\alpha},$$

$$||M(t)|| = \max_{1\le i,j\le N} |m_{i,j}(t)| \le C_M, \tag{2.14}$$

$$||B(t)|| = \max_{1\le i,j\le N} |b_{i,j}(t)| \le C_b t^{-\alpha}, \quad 0 < \alpha < 1, \quad t > 0. \tag{2.15}$$

Then the following Volterra system

$$
\begin{aligned}
Y(t) &= G(t) + \int_0^t A(t-s)Y(s)ds \qquad (2.16)\\
&\quad + M \max\{H(t) + \int_0^t B(t-s)Y(s)ds, 0\}, \quad t > 0
\end{aligned}
$$

has a unique solution $Y(t) \in (L^1(0,T))^N$. Here $\max\{\Psi, 0\} = (\max\{\psi_1, 0\}, \cdots, \max\{\psi_N, 0\})^T$ for any vector $\Psi \in R^N$. If $Y_i(t)$ are two solutions with the data G_i and H_i, $i = 1, 2$, then there exists a positive constant $C = C(C_a, C_b, C_M, \alpha, T) > 0$ such that

$$
\int_0^T ||Y_1 - Y_2|| dt \leq C \int_0^T \{||G_1 - G_2|| + ||H_1 - H_2||\} \, dt, \qquad (2.17)
$$

where $||\cdot||$ is the vector norm, $||\psi|| = \sum_{i=1,N} |\psi_1|$, associated with the matrix norm (2.14)-(2.15).

Proof: Let $Y(t) \in (L^1(0,T))^N$. We define mapping $S : (L^1(0,T))^N \to (L^1(0,T))^N$,

$$
SY = G(t) + \int_0^t A(t-s)Y(s)ds + M \max\{H(t) + \int_0^t B(t-s)Y(s)ds, 0\}.
$$

Then it follows that for any $X(t), Y(t) \in (L^1(0,T))^N$,

$$
\begin{aligned}
\int_0^T ||(SX - SY)(t)|| dt &\leq \int_0^T || \int_0^t A(t-s)(X(s) - Y(s))ds|| dt \\
&+ C_M \int_0^T || \int_0^t B(t-s)(X(s) - Y(s))ds|| dt \\
&\leq \frac{(C_a + C_b C_M)T^{1-\alpha}}{1-\alpha} \int_0^T ||X(t) - Y(t)|| dt.
\end{aligned}
$$

Here, the change of order of integration was used. It is easy to see that the mapping S is a contraction when T is small, which in turn implies that the local solution exists and is unique for small T. As the smallness of T is dependent upon C_a, C_b, C_M and α only we find from the standard bootstrap argument that (2.16) has unique solution in $(L^1(0,T))^N$ for any $T > 0$.

Now if Y_i are solutions for the data G_i and H_i, $i = 1, 2$. Then it is easy to see that

$$
\int_0^T ||Y_1 - Y_2|| dt \leq \int_0^T \{||G_1 - G_2|| + C_M ||H_1 - H_2||\} \, dt
$$

$$+\frac{(C_a + C_b C_M)T^{1-\alpha}}{1-\alpha}\int_0^T ||Y_1 - Y_2||dt.$$

This implies that (2.17) is true for small $T > 0$. The standard bootstrap argument (finite many steps) shows (2.17) is also true for any $T > 0$. Q.E.D.

3. Existence and uniqueness

Following [4] we transform problem (1.1)-(1.8) into an equivalent system of parabolic equations with nonlocal and nonlinear boundary conditions.

Let $W = (w^1(x,t), w^2(x,t))$ be defined by

$$w^i(x,t) = u^i(x,t) - \frac{b_i}{L_i}\max\{I(U), 0\}, \quad i = 1, 2, \tag{3.1}$$

where $U = (u^1, u^2)$ is the solution of (1.1)-(1.8) and $L_1 = l_1$ and $L_2 = 1 - l_2$. Thus w^i satisfies $w^i_t - w^i_{xx} = 0$ in Q^i_T, $(i = 1, 2)$. In order to obtain the corresponding boundary conditions for $W = (w^1, w^2)$, it is necessary to express u^i in terms of w^i, $i = 1, 2$.

Lemma 3.1. Assume that $Q = 1 - a_1 b_1 - a_2 b_2 > 0$. Then

$$u^i(x,t) = w^i(x,t) + \frac{b_i}{L_i Q}\max\{I(W), 0\}, \quad i = 1, 2, \tag{3.2}$$

is the unique inverse of (3.1).

Proof: If $I(U) > 0$, then (3.1) becomes

$$w^i(x,t) = u^i(x,t) - \frac{b_i}{L_i} I(U), \quad i = 1, 2. \tag{3.3}$$

It follows from applying operator $I(\cdot)$ of (1.9) on (3.3) that

$$I(W) = I(U) - (a_1 b_1 + a_2 b_2)I(U) = QI(U),$$

which implies that

$$I(U) > 0 \quad \text{if and only if} \quad I(W) > 0 \tag{3.4}$$

provided that $Q = 1 - a_1 b_1 - a_2 b_2 > 0$. Hence the formula (3.2) is the inverse of (3.1). Q.E.D.

We are now ready to state the following equivalence result.

Theroem 3.1. If $Q = 1 - a_1b_1 - a_2b_2 > 0$, then problem (1.1)-(1.8) is equivalent to the problem of finding $W = (w^1, w^2)$ such that

$$w_t^1 - c_1 w_{xx}^1 = 0, \text{ in } Q_T^1 = (0, l_1) \times J, \tag{3.5}$$

$$w_t^2 - c_2 w_{xx}^2 = 0, \text{ in } Q_T^2 = (l_2, 1) \times J, \tag{3.6}$$

$$w^1(x,0) = \phi_1(x) - \frac{b_1}{L_1} \max\{I(\Phi), 0\}, \quad 0 < x < l_1, \tag{3.7}$$

$$w^2(x,0) = \phi_2(x) - \frac{b_2}{L_2} \max\{I(\Phi), 0\}, \quad l_2 < x < 1, \tag{3.8}$$

$$\mu_1 w_x^1(0,t) + \nu_1 \left\{ w^1(0,t) + \frac{b_1}{QL_1} \max\{I(W), 0\} \right\} = f_1(t), \tag{3.9}$$

$$\mu_2 w_x^2(1,t) + \nu_2 \left\{ w^2(1,t) + \frac{b_2}{QL_2} \max\{I(W), 0\} \right\} = f_2(t), \tag{3.10}$$

$$k_1 w_x^1(l_1,t) = k \left\{ w^2(l_2,t) - w^1(l_1,t) + (\frac{b_2}{L_2Q} - \frac{b_1}{L_1Q}) W^* \right\}, \tag{3.11}$$

$$k_2 u_x^1(l_2,t) = k \left\{ w^2(l_2,t) - w^1(l_1,t) + (\frac{b_2}{L_2Q} - \frac{b_1}{L_1Q}) W^* \right\}, \tag{3.12}$$

where $\Phi = (\phi_1, \phi_2)$ and $W^* = \max\{I(W), 0\}$.

Proof: See Lemma 3.1. Q.E.D.

Theorem 3.2. Assume that $Q = 1 - a_1b_1 - a_2b_2 > 0$ and $f_i(t)$, $\phi_i(x)$, $i = 1, 2$, and $g(t)$ are continuous and bounded. Then the unique solution $W = (w^1, w^2)$ of (3.5)-(3.12) exists and is continuously dependent upon the data. That is, if w_j^i are the solutions for the data f_i^j, ϕ_i^j, g^j, $i, j = 1, 2$, then there exists a positive constant $C > 0$, independent of the data, such that

$$\begin{aligned} &\int_0^T \left\{ \int_0^{l_1} |w_1^1(x,t) - w_2^1(x,t)| dx + \int_{l_2}^1 |w_1^2(x,t) - w_2^2(x,t)| dx \right\} dt \\ \le\ & C \left\{ \sum_{i=1}^2 \int_0^T |f_i^1(t) - f_i^2(t)| dt + \int_0^T |g^1(t) - g^2(t)| dt \right. \\ & \left. + \int_0^{l_1} |\phi_1^1(x) - \phi_1^2(x)| dx + \int_{l_2}^1 |\phi_2^1(x) - \phi_2^2(x)| dx \right\}. \end{aligned} \tag{3.13}$$

Proof: We only give the proof for the case of $\mu_1 = \mu_2 = 0$ and $\nu_1 = \nu_2 = 1$, the other cases can be shown in the exact same fashion [3].

We seek the solutions $w^i, i=1,2,$ of (3.5)-(3.12) in the following form:

$$\begin{aligned} w^1 &= V^1(x,t,\phi_1,c_1,l_1) \\ &- 2\int_0^t \frac{\partial K}{\partial x}(\frac{x}{l_1},\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1}\psi_1(s)ds \\ &+ 2\int_0^t K(\frac{x-l_1}{l_1},\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1^2}\psi_2(s)ds \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} w^2 &= V^1(x,t,\phi_2,c_2,l_2) \\ &- 2\int_0^t K(\frac{x-l_2}{1-l_2},\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{(1-l_2)^2}\psi_3(s)ds \\ &+2\int_0^t \frac{\partial K}{\partial x}(\frac{x-l_2}{1-l_2}-1,\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{1-l_2}\psi_4(s)ds, \end{aligned} \tag{3.15}$$

where $\psi_i, i=1,2,3,4$ are unique solutions of the following Volterra system

$$\begin{aligned} f_1(t) &= V^1(0,t,\phi_1,c_1,l_1)+\psi_1(t)+\frac{b_1}{L_1Q}\max\{I^*(\Psi),0\} \\ &\quad +2\int_0^t K(-1,\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1^2}\psi_2(s)ds \qquad (3.16) \\ \frac{l_1k}{k_1}J^*(\psi) &= -2c_1\int_0^t \frac{\partial^2 K}{\partial x^2}(1,\frac{c_1}{l_1^2}(t-s))\psi_1(s)ds+\psi_2(s), \qquad (3.17) \\ \frac{(1-l_2)k}{k_2}J^*(\Psi) &= 2c_2\int_0^t \frac{\partial^2 K}{\partial x^2}(-1,\frac{c_2}{(1-l_2)^2}(t-s))\psi_4(s)ds, \qquad (3.18) \\ &\quad +\psi_3(t) \\ f_2(t) &= V^2(1,t,\phi_2,c_2,l_2)+\frac{b_2}{L_2Q}\max\{I^*(\Psi),0\} \qquad (3.19) \\ &\quad -2\int_0^t K(1,\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{(1-l_2)^2}\psi_3(s)ds, \end{aligned}$$

where $\Psi=(\psi_1,\psi_2,\psi_3,\psi_4)$ and $J^*(\Psi)$ is given by

$$\begin{aligned} &J^*(\Psi) \\ = &\left\{V^2(l_2,t,\phi_2,c_2,l_2)-2\int_0^t K(0,\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{(1-l_2)^2}\psi_3(s)ds\right. \\ &\left.+2\int_0^t \frac{\partial K}{\partial x}(-1,\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{1-l_2}\psi_4(s)ds\right\} \end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&-\left\{V^1(l_1,t,\phi_1,c_1,l_1)-2\int_0^t\frac{\partial K}{\partial x}(1,\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1}\psi_1(s)ds\right.\\
&\left.+2\int_0^t K(0,\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1^2}\psi_2(s)ds\right\}+\left\{\frac{b_2}{L_2Q}-\frac{b_1}{L_1Q}\right\}\max\{I^*(\Psi),0\},
\end{aligned}
$$

and where $I^*(\Psi)$ is given by

$$
\begin{aligned}
&I^*(\Psi)\\
=\;& a_1\int_0^{l_1}\left\{V^1(x,t,\phi_1,c_1,l_1)+2\int_0^t K(\frac{x-l_1}{l_1},\frac{c_1}{l_1^2}(t-s))\frac{c_1}{l_1^2}\psi_2(s)ds\right\}dx\\
&-2c_1a_1\int_0^t\left\{K(1,\frac{c_1}{l_1^2}(t-s))-K(0,\frac{c_1}{l_1^2}(t-s))\right\}\psi_1(s)ds \qquad (3.21)\\
&+a_2\int_{l_2}^1\left\{V^2(x,t,\phi_2,c_1,l_2)+g(t)\right.\\
&\left.-2\int_0^t K(\frac{x-l_2}{1-l_2}-1,\frac{c_2}{(1-l_2)^2}(t-s))\frac{c_2}{(1-l_2)^2}\psi_3(s)ds\right\}dx\\
&+2c_2a_2\int_0^t\left\{K(-1,\frac{c_2}{(1-l_2)^2}(t-s))-K(0,\frac{c_2}{(1-l_2)^2}(t-s))\right\}\psi_4(s)ds.
\end{aligned}
$$

In the following we first show that Volterra system (3.16)-(3.21) possesses a unique solution Ψ. For this purpose, we rewrite system (3.16)-(3.21) into the form of (2.16). Thus we see from (3.16) that

$$
\begin{aligned}
G_1 &= f_1(t)-V^1(0,t,\phi_1,c_1,l_1),\quad b_{22}=\frac{2a_1c_1}{l_1^2}\int_0^{l_1}K(\frac{x-l_1}{l_1},\frac{c_1}{l_1^2}t)dx,\\
a_{11} &= 0,\quad a_{12}=\frac{-2c_1}{l_1^2}K(-1,\frac{c_1}{l_1^2}t),\quad a_{13}=a_{14}=0,\quad M_{11}=-\frac{b_1}{L_1Q},\\
b_{11} &= -2a_1c_1\left\{K(1,\frac{c_1}{l_1^2}t)-K(0,\frac{c_1}{l_1^2}t)\right\},\\
b_{33} &= -\frac{2a_2c_2}{(1-l_2)^2}\int_{l_2}^1 K(\frac{x-l_2}{1-l_2}-1,\frac{c_2}{(1-l_2)^2}t)dx,\\
b_{44} &= 2a_2c_2\left\{K(-1,\frac{c_2}{(1-l_2)^2}t)-K(0,\frac{c_2}{(1-l_2)^2}t)\right\},\\
H^* &= a_1\int_0^{l_1^2}V^1(x,t,\phi_1,c_1,l_1)dx+a_2\int_{l_2}^1V^2(x,t,\phi_2,c_2,l_2)dx+g(t),
\end{aligned}
$$

and similarly from (3.17)-(3.21) that

$$
G_2 = \frac{l_1k}{k_1}\left\{V^2(l_2,t,\phi_2,c_2,l_2)-V^1(l_1,t,\phi_1,c_1,l_1)\right\},
$$

$$
\begin{aligned}
a_{21} &= \frac{2c_1k}{k_1}\frac{\partial K}{\partial x}(1,\frac{c_1}{l_1^2}t)+2c_1\frac{\partial^2 K}{\partial x^2}(1,\frac{c_1}{l_1^2}t), \qquad a_{22}=\frac{-2kc_1}{k_1l_1}K(0,\frac{c_1}{l_1^2}t)\\
a_{23} &= \frac{-2l_1kc_2}{k_1(1-l_2)^2}K(0,\frac{c_2}{(1-l_1)^2}t), \quad a_{32}=\frac{-2c_1(1-l_2)k}{k_2l_1^2}K(0,\frac{c_1}{l_1^2}t),\\
G_3 &= \frac{(1-l_2)k}{k_2}\left\{V^2(l_2,t,\phi_2,c_2,l_2)-V^1(l_1,t,\phi_1,c_1,l_1)\right\},\\
a_{31} &= \frac{2c_1k(1-l_2)}{k_2l_1}\frac{\partial K}{\partial x}(1,\frac{c_1}{l_1^2}t), \quad a_{24}=\frac{2kl_1c_2}{k_1(1-l_2)}\frac{\partial K}{\partial x}(-1,\frac{c_2}{(l-l_2)^2}t),\\
a_{33} &= \frac{-2c_2k}{k_2(1-l_2)}K(0,\frac{c_2}{(1-l_2)^2}t),\\
a_{34} &= -2c_2\frac{\partial^2 K}{\partial x^2}(-1,\frac{c_2}{(1-l_2)}t)+\frac{2c_2k}{k_2}\frac{\partial K}{\partial x}(-1,\frac{c_2}{(1-l_2)^2}t),\\
G_4 &= f_2(t)-V^2(1,t,\phi_2,c_2,l_2), \quad M_{44}=\frac{-b_2}{L_2Q}\\
a_{41} &= a_{42}=a_{44}=0, \qquad a_{43}=\frac{2c_2}{(1-l_2)^2}K(1,\frac{c_2}{(1-l_2)^2}t),\\
M_{22} &= \frac{l_1k}{k_1}\left\{\frac{b_2}{L_2Q}-\frac{b_1}{L_1Q}\right\}, \qquad M_{33}=\frac{(1-l_2)k}{k_2}\left\{\frac{b_2}{L_2Q}-\frac{b_1}{L_1Q}\right\}, \qquad .
\end{aligned}
$$

Then Volterra system of (3.16)-(3.21) can be written in the form of (2.16) with $A=(a_{ij})$, $G=(G_1,G_2,G_3,G_4)^T$, $B=diag(b_{11},b_{22},b_{33},b_{44})$, $H=(H^*,H^*,H^*,H^*)^T$ and $M=diag(M_{11},M_{22},M_{33},M_{44})$ defined above. It is clear from the heat potential estimates of $K(x,t)$ and $\theta(x,t)$ [3, 8, 9] that the matrices A, B and M satisfy assumption (2.14)-(2.15) for some $C_a, C_b, C_M > 0$ with $\alpha=1/2$, and H, $G\in(L^1(0,T))^4$. Therefore Lemma 2.2 implies that Volterra system (3.16)-(3.21) has a unique solution $\Psi\in(L^1(0,T))^4$.

Now substitute the solution Ψ of Volterra system (3.16)-(3.21) into (3.14)-(3.15), we see easily from [3], Corollary 2.1 and Corollary 2.2 that $w^i, i=1,2$, given in (3.14)-(3.15) are the solutions of (3.5)-(3.12). On the other hand it follows from Corollary 2.1 and Corollary 2.2 that any solutions of (3.5)-(3.12) can be represented by (3.14)-(3.15) with Ψ satisfying (3.16)-(3.21), thus the existence and uniqueness of the solution for problem (3.5)-(3.12) are also proved.

For the continuous dependence of the solution upon the data we have from the estimates for the heat potentials $K(x,t)$ and $\theta(x,t)$ [3], Lemma 2.2 and (3.16)-(3.21) that if $\psi_m^j(t)$ are the solutions for the data f_i^j, ϕ_i^j, g^j, $i,j=1,2$, $m=1,2,3,4$, then there exists a positive constant $C>0$, independent of the data, such that

$$\begin{aligned}
&\sum_{i=1}^{4}\int_0^T |\psi_i^1(t)-\psi_i^2(t)|dt \\
\leq\ & C\left\{\sum_{i=1}^{2}\int_0^T |f_i^1(t)-f_i^2(t)|dt+\int_0^T |g^1(t)-g^2(t)|dt\right. \\
&\left.+\int_0^{l_1}|\phi_1^1(x)-\phi_1^2(x)|dx+\int_{l_2}^{1}|\phi_2^1(x)-\phi_2^2(x)|dx\right\},
\end{aligned}$$

which in turn implies from (3.14)-(3.15) that the solution of (3.5)-(3.12) satisfies (3.13). Q.E.D.

Theorem 3.3. Under the assumptions of Theorem 3.2 the problem (1.1)-(1.8) possesses a unique solution $U=(u^1,u^2)$, which is continuously dependent upon the data. That is, if u_j^i are the solutions for the data f_i^j, ϕ_i^j, g^j, $i,j=1,2$, then there exists a positive constant $C>0$, independent of the data, such that

$$\begin{aligned}
&\int_0^T\left\{\int_0^{l_1}|u_1^1(x,t)-u_2^1(x,t)|dx+\int_{l_2}^{1}|u_1^2(x,t)-u_2^2(x,t)|dx\right\}dt \\
\leq\ & C\left\{\sum_{i=1}^{2}\int_0^T |f_i^1(t)-f_i^2(t)|dt+\int_0^T |g^1(t)-g^2(t)|dt\right. \qquad (3.22)\\
&\left.+\int_0^{l_1}|\phi_1^1(x)-\phi_1^2(x)|dx+\int_{l_2}^{1}|\phi_2^1(x)-\phi_2^2(x)|dx\right\}
\end{aligned}$$

Proof: The existence and uniqueness of the solutions follows from the equivalence results of Lemma 3.1 and Theorem 3.2, and continuous dependence of the solutions upon the data follows from Theorem 3.2, (3.2) and (3.13). Q.E.D.

Remark: If k in (1.8) is a function which is Lipschitz continuous, then Theorem 3.2 and Theorem 3.3 are still valid. The proofs are almost identical to the proofs given above.

4. Application

In [2] the authors consider the quasi-static thermoelastic contact of two rods of finding $\{u^i(x,t),\theta^i(x,t)\}$, $i=1,2$, displacements and temperatures of two dissimilar rods, respectively, such that

$$\theta_t^1 - \theta_{xx}^1 = -a_1 u_{xt}^1 \ \text{ in } \ Q_T^1 = (0, l_1) \times J, \tag{4.1}$$
$$u_{xx}^1 = a_1 \theta_x^1 \ \text{ in } \ Q_T^1 = (0, l_1) \times J, \tag{4.2}$$
$$d\theta_t^2 - \theta_{xx}^2 = -a_2 u_{xt}^2 \ \text{ in } \ Q_T^1 = (l_2, 1) \times J, \tag{4.3}$$
$$u_{xx}^2 = a_2 \theta_x^2 \ \text{ in } \ Q_T^1 = (l_2, 1) \times J, \tag{4.4}$$
$$\theta^1(x,0) = \phi_1(x), \ 0 < x < l_1, \ ; \ \theta^2(x,0) = \phi_2(x), \ l_2 < x < 1, \tag{4.5}$$
$$\theta^1(0,t) = f_1(t), \qquad \theta^2(1,t) = f_2(t), \qquad 0 < t < T, \tag{4.6}$$
$$u^1(0,t) = u^2(1,t) = 0, \qquad 0 < t < T, \tag{4.7}$$
$$k_1 \theta_x^1(l_1, t) = k_2 \theta_x^2(l_2, t), \qquad 0 < t < T, \tag{4.8}$$
$$-k_1 \theta_x^1(0,t) = k(\theta^1(l_1,t) - \theta^2(l_2,t)), \qquad 0 < t < T, \tag{4.9}$$

and

$$u^1(l_1,t) \leq u^2(l_2,t) + g(t), \qquad 0 < t < T, \tag{4.10}$$
$$\sigma_1(l_1,t) = \sigma_2(l_2,t) \leq 0, \qquad 0 < t < T, \tag{4.11}$$
$$\sigma_1(l_1,t) \left\{ u^2(l_2,t) + g(t) - u^1(l_1,t) \right\} = 0, \qquad 0 < t < T, \tag{4.12}$$

where

$$\sigma_i(x,t) = \beta_i \left\{ u_x^i(x,t) - a_i \theta^1(x,t) \right\}, \qquad i = 1, 2, \tag{4.13}$$

and $a_i > 0, k_i > 0, \beta_i > 0, d > 0, k$, are physical constants with $g = l_2 - l_1$, and f_i and ϕ_i are functions for $i = 1,2$, and $0 < l_1 \leq l_2 < 1$.

After the decoupling [2, 6] of θ^i from u^i, $i = 1, 2$, we find that (θ^1, θ^2) satisfy system (1.1)-(1.9) with

$$b_1 = \frac{a_1 \lambda}{(1 + a_1^2)(1 + \lambda)}, \qquad c_1 = \frac{1}{1 + a_1^2}, \tag{4.14}$$
$$b_2 = \frac{a_2}{(d + a_2^2)(1 + \lambda)}, \qquad c_1 = \frac{1}{d + a_1^2}, \qquad \lambda = \frac{\beta_2 l_1}{\beta_1 (1 - l_2)}, \tag{4.15}$$

and once $\theta^i, i = 1, 2$, are founded, $u^i, i = 1, 2$, can be obtained by

$$u^1 = a_1 \int_0^x \theta^1(x,t) dx - \frac{\lambda x}{(1 + \lambda) l_1} \max\{I(\Theta), 0\}, \tag{4.16}$$
$$u^2 = -a_2 \int_x^1 \theta^2(x,t) dx + \frac{1 - x}{(1 + \lambda)(1 - l_2)} \max\{I(\Theta), 0\}, \tag{4.17}$$

where $\Theta = (\theta^1, \theta^2)$ and $I(\cdot)$ is defined in (1.9). See [2, 6] for the detailed derivations.

Remark: In [2, 6], $b_2 = (a_2\lambda)/\{(d + a_2^2)(1 + \lambda)\}$ is not correct which is in fact a typing error, and this error appeared in many places in [2] and [6]. b_2 given in (4.15) is correct.

The authors of [2] show using fixed point theorem that if the data is continuously differentiable and parameters satisfy

$$\begin{aligned} \frac{a_1^4}{l_1^2(1+a_1^2)^2} \quad &+ \quad \frac{a_1^2 a_2^2 k_2}{k_1 l_1^2 (1+a_1^2)(d+a_2^2)} + \frac{a_1^2 a_2^2 k_1}{k_2 (1-l_2)^2 (1+a_1^2)(d+a_2^2)} \\ &+ \frac{a_2^4}{(1-l_2)^2 (d+a_1^2)^2} \le \frac{1}{2}\left\{1 + \frac{1}{\lambda}\right\}^2, \end{aligned} \tag{4.18}$$

then, the unique solutions of (4.1)-(4.13) exist in Sobolev space $W^{2,1}(Q_T^1) \times W^{2,1}(Q_T^2)$. It is easy to see that (4.18) is very restrictive assumption since it requires not only a_1 and a_2 being small, but also a certain balance among l_1, l_2 and the other parameters. Moreover the smoothness condition on the data can not be relaxed in general since the solutions are sought in $W^{2,1}(Q_T^1) \times W^{2,1}(Q_T^2)$.

But, it is clear to see from (4.14)-(4.15) that

$$\begin{aligned} a_1 b_1 + a_2 b_2 \quad &= \quad \frac{a_1^2}{(1+a_1^2)(1+\lambda)} + \frac{a_2^2 \lambda}{(d+a_1^2)(1+\lambda)} \\ &< \quad \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} = 1. \end{aligned} \tag{4.19}$$

We therefore have the following result.

Theorem 4.1. The problem (4.1)-(4.13) is well-posed. That is, the solutions $(u^i, \theta^i), i = 1, 2$, exist, are unique and continuously dependent upon the data.

Proof: Since $Q = 1 - a_1 b_1 - a_2 b_2 > 0$, where a_i, b_i and λ are defined by (4.14)-(4.15). Thus Theorem 3.2, Theorem 3.3 and the equivalent Thoerem 3.1 indicate that problem (4.1)-(4.13) possesses unique solutions for all physical parameters. Moreover the solutions are continuously dependent upon the data. Q.E.D.

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