A Nonlocal Parabolic System modelling Axially Symmetric Thermoelastic Contact of Two Discs[1]

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Abstract

A system of parabolic integro-differential equations modelling homogeneous and isotropic elastic thin discs subject to axially symmetric temperature fields and displacements is studied. Under some appropriate assumptions the questions of existence, uniqueness and continuous dependence of the solutions upon the data is demonstrated. The main feature of our method in this paper is that the difficulties associated with this physical problem can been seen clearly and overcome easily via the decoupling procedures and transformations.

1. Introduction.

We consider two dissimilar homogeneous and isotropic thin discs subject to axially symmetric temperature fields θ^i and displacements $u^i (i = 1, 2)$. In addition it is assumed that the thermoelastic stresses $\sigma^i (i = 1, 2)$ of the bodies are also axially symmetric, thus θ^i, u^i and $\sigma^i (i = 1, 2)$ are functions of the radius r and time t, relative to the reference configuration of elastic disc $0 < r < r_1$ and annual disc $r_2 < r < 1$ with $r_1 \leq r_2$. The fundamental equations for linear thermoelasticity problems (ignoring the inertial effects) are reduced to the following axially symmetric form [12, 17] :

$$\left(u^i_r + \frac{u^i}{r}\right)_r = a_i \theta^i_r \ \text{ in } \ Q^i_T, \qquad i = 1, 2, \tag{1.1}$$

$$\left(\theta^i_{rr} + \frac{\theta^i_r}{r}\right) = c_i \theta^i_t + d_i \left(u^i_r + \frac{u^i}{r}\right)_t, \ \text{ in } \ Q^i_T, \qquad , i = 1, 2, \tag{1.2}$$

where $Q^1_T = (0, r_1) \times (0, T)$ and $Q^2_T = (r_2, 1) \times (0, T)$, a_i, c_i and d_i are positive constants related to the physical parameters (Lame constants λ_i and μ_i, thermoelasticity coefficients η_i, thermal conductivity coefficients k^0_i, mass densities of the solids ρ_i, specific heats C^s_i and the reference temperatures T^0_i, $i = 1, 2$) in the following forms:

$$a_i = \frac{\eta_i}{\lambda_i + 2\mu_i}, \qquad c_i = \frac{\rho_i\, C^s_i}{k^0_i}, \qquad \text{and} \qquad d_i = \frac{\eta_i\, T^0_i}{k^0_i}.$$

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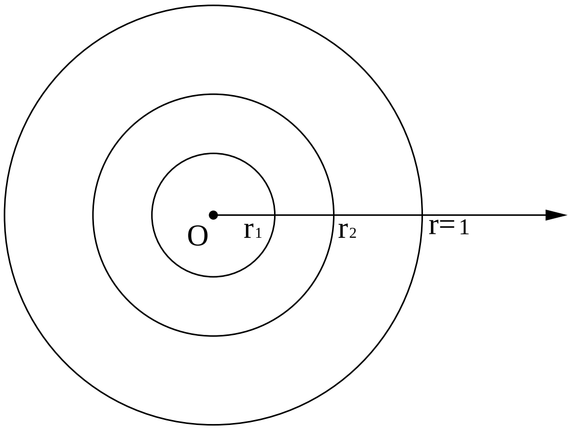

Figure 0.1: The physical setting with the initial gap $g = r_2 - r_1$.

In order to close the system we impose the following initial and boundary conditions:

$$\theta^1(r,0) = \phi_1(r), \quad 0 < r < r_1, \quad \theta^2(r,0) = \phi_2(r), \quad r_2 < r < 1, \tag{1.3}$$

$$\theta^1_r(0,t) = 0, \quad \mu\theta^2_r(1,t) + \nu\theta^2(1,t) = f(t), \quad 0 < t < T, \tag{1.4}$$

$$k_1\theta^1_r(r_1,t) = k_2\theta^2_r(r_2,t), \quad 0 < t < T, \tag{1.5}$$

$$-k_1\theta^1_r(r_1,t) = k(\theta^2(r_2,t) - \theta^1(r_1,t)), \quad 0 < t < T, \tag{1.6}$$

$$u^1(0,t) = 0, \quad u^2(1,t) = 0, \quad 0 < t < T, \tag{1.7}$$

where $\theta^1_r(0,t) = 0$ in (1.4) is due to the symmetry of $\theta^1(r,t) = \theta^1(-r,t)$, ϕ_i, f, $(i = 1,2)$ are known functions, $k_i (i = 1,2)$, k, μ and ν ($\mu^2 + \nu^2 \neq 0$) are constants related to the physical parameters, and (1.5)-(1.6) represent the heat exchange conditions at $r = r_1$ and $r = r_2$.

The thermoelastic stresses are given [12, 17]

$$\sigma^i(r,t) = (\lambda_i + 2\mu_i)\left\{u^i_r(r,t) + \frac{u^i(r,t)}{r}\right\} - 2\frac{\mu_i}{r}u^i(r,t) - \eta_i\theta^i(r,t), \quad i = 1,2,$$

which can be rewritten as

$$\sigma^i(r,t) = \beta_i\left\{u^i_r(r,t) + \alpha_i\frac{u^i(r,t)}{r} - a_i\theta^i(r,t)\right\}, \quad i = 1,2, \tag{1.8}$$

where $0 < \alpha_i < 1$ and $\beta_i > 0$ are constants. If there is no contact at the free surfaces we find that $\sigma^1(r_1,t) = \sigma^2(r_2,t) = 0$, and if there is a contact we also find that $\sigma^1(r_1,t) = \sigma^2(r_2,t)$. Thus the stresses are equal at the free surfaces for all time. We therefore have the Signorini type of contact conditions at the free surfaces:

$$u^1(r_1,t) \leq u^2(r_2,t) + g, \quad 0 < t < T, \tag{1.9}$$

$$\sigma^1(r_1,t) = \sigma^2(r_2,t) \leq 0, \quad 0 < t < T, \tag{1.10}$$

$$\left\{u^1(r_1,t) - u^2(r_2,t) - g\right\}\sigma^1(r_1,t) = 0, \quad 0 < t < T, \tag{1.11}$$

where $g = r_2 - r_1$ is the initial gap of the two discs, and (1.9) implies that there is no penetration at the free surfaces.

The thermoelastic contacts are very common and of important in industrial applications, which arise naturally in the processes of castings and moldings, in connection with the operation of thermostats and valves, and in the thermal expansion of cylinders and engine heads. The above problem (1.1)-(1.11) is a mathematically simplified version of thermoelastic contact model in prestressed duplex tubes of a heat-exchange in the Experimental Breeder Reactor II [15]. We refer the readers to [14, 15] and the references therein for the details of mathematical modeling and related issues in applications.

Mathematically, the thermoelastic contact on one unit disc was studied in [17], where the author formulated the problem as a singular parabolic equation for the temperature field θ after the decoupling, and then shown that the problem is well-posed in a weighted Sobolev space provided that physical parameters are small enough. The problem is also considered recently in [4] where the authors found that the problem is well-posed for all physical paramters.

The one dimensional quasi-static thermoelastic contact of two rods similar to the problem (1.1)-(1.11) has been investigated recently in [2] and some numerical experiments are given in [6]. As for the one rod contact problem the authors of [2] decoupled temperatures from the displacements so that the temperatures alone satisfy a nonlocal nonlinear parabolic system, based on which the well-posedness of the problem was studied under very restricted assumptions on the data (smooth) and physical parameters (small and certain balance with the sizes of the solid bars considered). The author of [10] employed a different method and obtained optimal results concerning the existence, uniqueness and continuous dependence of the solution upon the data, that is the smoothness of the data and the smallness of the parameters and balance with the sizes of the solid bars are not needed.

In this paper we first follow the idea of [2] and [6] to decouple the system (1.1)-(1.11) into one nonlocal and nonlinear parabolic system for the temperature fields $\theta^i(r,t), i=1,2$, and give the explicit expressions of the displacements $u^i(r,t), i=1,2$, in term of the temperature fields $\theta^i(r,t), i=1,2$. Second a transformation is introduced to reduce the nonlocal system for the temperatures into a standard homogeneous parabolic system with nonlocal and nonlinear boundary conditions. The well-posedness of the resultant system is then proved using the standard procedure [3, 8, 9], and thus the well-posedness of θ^i (and u^i) follows from the equivalence of two systems and decoupling procedures.

For the thermoelastic contact of one rod we refer the readers to [1, 5, 7, 13, 16, 17] for the details of mathematical formulations and analysis. A similar problem can also be found in [7, 11].

2. The decoupled system

In this section we will decouple the temperatures θ^i from the displacements $u^i (i=1,2)$ in the system (1.1)-(1.11). In doing so we follow the idea initiated in [2].

For $i=1$, we integrate (1.1) from r to r_1 to obtain that

$$\frac{1}{r}(ru^1(r,t))_r = \left\{\frac{1}{r}(ru^1)_r - a_1\theta^1\right\}_{r=r_1} + a_1\theta^1(r,t), \tag{2.1}$$

and then, integrate after multiplying (2.1) by r from 0 to r_1 that

$$r_1 u^1(r_1,t) = R_1\left\{\frac{1}{r}(ru^1)_r - a_1\theta^1\right\}_{r=r_1} + a_1\int_0^{r_1} r\theta^1(r,t)dr, \tag{2.2}$$

where $R_1 = r_1^2/2$. It is now easy to see from substituting (2.2) into (2.1) that

$$\frac{1}{r}(ru^1(r,t))_r = \frac{1}{R_1}\left\{r_1u^1(r_1,t) - a_1\int_0^{r_1} r\theta^1(r,t)dr\right\} + a_1\theta^1(r,t). \tag{2.3}$$

Thus, one find from (1.8), (1.10) and (2.3) that

$$\sigma^1(r_1,t) = \frac{\beta_1}{R_1}\left\{\frac{(1+\alpha_1)r_1}{2}u^1(r_1,t) - a_1\int_0^{r_1} r\theta^1(r,t)dr\right\} \le 0, \qquad 0<t<T, \tag{2.4}$$

which indicates that

$$u^1(r_1,t) \le \frac{2a_1}{(1+\alpha_1)r_1}\int_0^{r_1} r\theta^1(r,t)dr, \qquad 0<t<T. \tag{2.5}$$

From (2.3) and (1.11), it follows that

$$(u^1(r_1,t)-u^2(r_1,t)-g)\left\{u^1(r_1,t) - \frac{2a_1}{(1+\alpha_1)r_1}\int_0^{r_1} r\theta^1(r,t)dr,\right\} = 0, \qquad 0<t<T, \tag{2.6}$$

and hence, we have from (1.9) and (2.5)-(2.6) that

$$u^1(r_1,t) = \min\left\{\frac{2a_1}{(1+\alpha_1)r_1}\int_0^{r_1} r\theta^1(r,t)dr,\ \ u^2(r_1,t)+g\right\}, \qquad 0<t<T. \tag{2.7}$$

Here $A \le B, A \le C$ and $(A-B)(A-C)=0 \Longleftrightarrow A=\min\{B,C\}$ was used.

Similarily, for $i=2$, it follows from integrating (1.1) from r to 1 that

$$\frac{1}{r}(ru^2(r,t))_r = \left\{\frac{1}{r}(ru^2)_r - a_2\theta^2\right\}_{r=1} + a_2\theta^2(r,t), \tag{2.8}$$

and then from r_2 to 1, upon multiplying (2.8) by r, that

$$-r_2u^2(r_2,t) = R_2\left\{\frac{1}{r}(ru^2)_r - a_2\theta^2\right\}_{r=1} + a_2\int_{r_2}^1 r\theta^2(r,t)dr, \tag{2.9}$$

where $R_2=(1-r_2^2)/2$. Again, we find from (2.8)-(2.9) and (1.8)-(1.10) that

$$\sigma_2(r_1,t) = \frac{-\beta_2}{R_2}\left\{\frac{(1-\alpha_2)R_2+r_2^2}{r_2}u^2(r_2,t) + a_2\int_{r_2}^1 r\theta^2(r,t)dr\right\} \le 0, \qquad 0<t<T, \tag{2.10}$$

or equivalently,

$$u^2(r_2,t) \ge -\frac{a_2r_2}{r_2^2+(1-\alpha_2)R_2}\int_{r_2}^1 r\theta^2(r,t)dr. \tag{2.11}$$

Thus, it follows from (1.11) and (2.10) that

$$\{u^1(r_1,t)-u^2(r_1,t) \quad - \quad g\}\left\{u^2(r_2,t) + \frac{a_2r_2}{r_2^2+(1+\alpha_2)R_2}\int_{r_2}^1 r\theta^2(r,t)dr,\right\} = 0, \tag{2.12}$$

and then,

$$u^2(r_2,t) = \max\left\{u^1(r_1,t)-g, \frac{-a_2r_2}{r_2^2+(1-\alpha_2)R_2}\int_{r_2}^1 r\theta^2(r,t)dr,\right\}, \qquad 0<t<T. \tag{2.13}$$

Here $A \ge B, A \ge C$ and $(A-B)(A-C)=0 \Longleftrightarrow A=\max\{B,C\}$ was used.

Now it is easy to see from (1.10), (2.4) and (2.10) that

$$\begin{aligned}(2.14)\qquad u^1(r_1,t) &- \frac{2a_1}{(1+\alpha_1)r_1}\int_0^{r_1} r\theta^1(r,t)dr\\ &= -\lambda\left\{u^2(r_2,t)+\frac{a_2r_2}{r_2^2+(1-\alpha_2)R_2}\int_{r_2}^1 r\theta^2(r,t)dr,\right\},\qquad 0<t<T,\end{aligned}$$

where

$$(2.15)\qquad \lambda=\frac{\beta_2R_1}{\beta_1R_2}\left\{\frac{(1-\alpha_2)R_2+r_2^2}{r_1r_2(1+\alpha_1)}\right\}>0.$$

If let

$$(2.16)\qquad A_1=\frac{2a_1}{(1+\alpha_1)r_1},\qquad A_2=\frac{a_2r_2}{r_2^2+(1-\alpha_2)R_2}$$

then (2.7), (2.13)-(2.14) can be written in the compact form:

$$(2.17)\qquad u^1(r_1,t)=\min\left\{u^2(r_1,t)+g,A_1\int_0^{r_1} r\theta^1(r,t)dr\right\},\qquad 0<t<T,$$

$$(2.18)\qquad u^2(r_2,t)=\max\left\{u^1(r_1,t)-g,-A_2\int_{r_2}^1 r\theta^2(r,t)dr\right\},\qquad 0<t<T,$$

$$(2.19)\qquad u^1(r_1,t)-A_1\int_0^{r_1} r\theta^1(r,t)dr=-\lambda\left\{u^2(r_2,t)+A_2\int_{r_2}^1 r\theta^2(r,t)dr,\right\},\qquad 0<t<T.$$

Now we substitute (2.3) and (2.17) into (1.2) for $i=1$ to obtain that

$$\begin{aligned}(2.20)\quad (c_1+d_1a_1)\theta_t^1-\left(\theta_{rr}^1+\frac{\theta_r^1}{r}\right) &= \frac{-d_1}{R_1}\frac{d}{dt}\left\{r_1u^1(r_1,t)-a_1\int_0^{r_1} r\theta^1(r,t)dr\right\}\\ &= \frac{d_1a_1}{R_1}\frac{d}{dt}\int_0^{r_1} r\theta^1(r,t)dr\\ &\quad -\frac{d_1r_1}{R_1}\frac{d}{dt}\min\left\{u^2(r_2,t)+g,A_1\int_0^{r_1} r\theta^1(r,t)dr\right\}\\ &= \frac{d_1(a_1-A_1r_1)}{R_1}\frac{d}{dt}\int_0^{r_1} r\theta^1(r,t)dr\\ &\quad -\frac{d_1r_1}{R_1}\frac{d}{dt}\min\left\{u^2(r_2,t)+g-A_1\int_0^{r_1} r\theta^1(r,t)dr,0\right\}.\end{aligned}$$

Here $\min\{C,D\}=D+\min\{C-D,0\}$ was used.

Now, if $u^2(r_2,t)+g-A_1\int_0^{r_1} r\theta^1(r,t)dr\le 0$, then we find from (2.17) that $u^1(r_1,t)=u^2(r_2,t)+g$, which together with (2.19) implies that

$$\begin{aligned}(2.21)\qquad & u^2(r_2,t)+g-A_1\int_0^{r_1} r\theta^1(r,t)dr\\ =\ & \frac{\lambda}{1+\lambda}\left\{g-A_1\int_0^{r_1} r\theta^1(r,t)dr-A_2\int_{r_2}^1 r\theta^2(r,t)dr\right\},\qquad 0<t<T,\end{aligned}$$

where λ is defined in (2.15). On the other hand, if $u^2(r_2,t)+g-A_1\int_0^{r_1} r\theta^1(r,t)dr > 0$, we have from (2.17)-(2.19) that $u^1(r_1,t)=A_1\int_0^{r_1} r\theta^1(r,t)dr$ and $u^2(r_2,t)=-A_2\int_{r_2}^1 r\theta^2(r,t)dr$ and $u^1(r_1,t)=A_1\int_0^{r_1} r\theta^1(r,t)dr$. We therefore conclude that

$$\begin{aligned}(2.22)\qquad & \min\left\{u^2(r_2,t)+g-A_1\int_0^{r_1} r\theta^1(r,t)dr, 0\right\}\\ = & \frac{\lambda}{1+\lambda}\min\left\{g-A_1\int_0^{r_1} r\theta^1(r,t)dr - A_2\int_{r_2}^1 r\theta^2(r,t)dr, 0\right\}, \qquad 0<t<T.\end{aligned}$$

Similarly, we find from (1.2), (2.17)-(2.19) for $i=2$ that

$$\begin{aligned}(2.23)\qquad (c_2+d_2a_2)\theta_t^2-\left(\theta_{rr}^2+\frac{\theta_r^2}{r}\right) = & \frac{d_2(a_2-A_2r_2)}{R_2}\frac{d}{dt}\int_{r_2}^1 r\theta^2(r,t)dr\\ & +\frac{d_2r_2}{R_2}\frac{d}{dt}\max\left\{u^1(r_1,t)-g+A_2\int_{r_2}^1 r\theta^2(r,t)dr, 0\right\}\end{aligned}$$

and

$$\begin{aligned}(2.24)\qquad & \max\left\{u^1(r_1,t)-g+A_2\int_{r_2}^1 r\theta^2(r,t)dr, 0\right\}\\ = & \frac{-1}{1+\lambda}\min\left\{g-A_1\int_0^{r_1} r\theta^1(r,t)dr - A_2\int_{r_2}^1 r\theta^2(r,t)dr, 0\right\}, \qquad 0<t<T.\end{aligned}$$

Hence, we see now from the above derivations that θ^1 and θ^2 satisfy

$$\begin{aligned}(2.25)\qquad (c_1+d_1a_1)\theta_t^2-\left(\theta_{rr}^1+\frac{\theta_r^1}{r}\right) = & \frac{d_1(a_1-A_1r_1)}{R_1}\frac{d}{dt}\int_0^{r_1} r\theta^1(r,t)dr\\ & +\frac{d_1r_1\lambda}{R_1(1+\lambda)}\frac{d}{dt}\max\{I(\Theta),0\}, \text{ in } Q_T^1,\end{aligned}$$

$$\begin{aligned}(2.26)\qquad (c_2+d_2a_2)\theta_t^2-\left(\theta_{rr}^2+\frac{\theta_r^2}{r}\right) = & \frac{d_2(a_2-A_2r_2)}{R_2}\frac{d}{dt}\int_{r_2}^1 r\theta^2(r,t)dr\\ & +\frac{d_2r_2}{R_2(1+\lambda)}\frac{d}{dt}\max\{I(\Theta),0\}, \text{ in } Q_T^2,\end{aligned}$$

$$(2.27)\qquad \theta^1(r,0)=\phi_1(r), \quad 0<r<r_1, \quad \theta^2(r,0)=\phi_2(r), \quad r_2<r<1,$$

$$(2.28)\qquad \theta_r^1(0,t)=0, \qquad \mu\theta_r^2(1,t)+\nu\theta^2(1,t)=f(t), \quad 0<t<T,$$

$$(2.29)\qquad k_1\theta_r^1(r_1,t)=k_2\theta_r^2(r_2,t), \quad 0<t<T,$$

$$(2.30)\qquad -k_1\theta_r^1(r_1,t)=k(\theta^2(r_2,t)-\theta^1(r_1,t)), \quad 0<t<T,$$

where the integral operator $I(\cdot)$ is defined for $\Theta=(\theta^1,\theta^2)$ by

$$(2.31)\qquad I(\Theta)=A_1\int_0^{r_1} r\theta^1(r,t)dr + A_2\int_{r_2}^1 r\theta^2(r,t)dr - g, \quad 0<t<T.$$

Once θ^1 and θ^2 are found, it is easy to see from (2.17) and (2.22) that

$$(2.32)\qquad u^1(r_1,t)=A_1\int_0^{r_1} r\theta^1(r,t)dr - \frac{\lambda}{1+\lambda}\max\{I(\Theta),0\},$$

and from (2.3) for $0 < r < r_1$ and $0 < t < T$ that

$$(2.33)\qquad u^1(r,t) = \frac{a_1}{r}\int_0^r r\theta^1(r,t)dr + \frac{r(r_1A_1 - a_1)}{r_1^2}\int_0^{r_1} r\theta^1(r,t)dr - \frac{r\lambda}{r_1(1+\lambda)}\max\{I(\Theta),0\}.$$

Similarily, we have from (2.18) and (2.24) that

$$(2.34)\qquad u^2(r_2,t) = -A_2\int_{r_2}^1 r\theta^2(r,t)dr + \frac{1}{1+\lambda}\max\{I(\Theta),0\},$$

and from (2.8) for $r_2 < r < 1$ and $0 < t < T$ that

$$(2.35)\qquad u^2(r,t) = \frac{-a_2}{r}\int_r^1 r\theta^2(r,t)dr + \frac{1}{r}\left(a_2 - \frac{r_2(1-r^2)A_2}{1-r_2^2}\right)\int_{r_2}^1 r\theta^2(r,t)dr - \frac{r_2(1-r^2)}{r(1+\lambda)(1-r_2^2)}\max\{I(\Theta),0\}.$$

3. An Equivalent System

Following the idea of [4, 10], we will in this section, instead of studying the system (2.25) - (2.30) directly, transform the system (2.25)-(2.30) into a system of parabolic equations with nonlocal and nonlinear boundary conditions. For this purpose we let

$$(3.1)\qquad \rho_i = \frac{1}{c_i + d_ia_i},\qquad i=1,2,\qquad C_1 = \frac{d_1r_1\lambda}{(c_1+d_1a_1)(1+\lambda)},\qquad C_2 = \frac{d_2r_2}{(c_2+d_2a_2)(1+\lambda)},$$

$$(3.2)\qquad B_1 = \frac{d_1(a_1 - A_1r_1)}{c_1+d_1a_1},\qquad B_2 = \frac{d_2(a_2 - A_2r_2)}{c_2+d_2a_2},$$

where A_1 and A_2 are defined in (2.16). Then, one find from (2.25)-(2.30) that $\theta^i (i=1,2)$ satisfy

$$(3.3)\qquad \theta^i_t - \rho_i\left(\theta^i_{rr} + \frac{\theta^i_r}{r}\right) = \frac{B_i}{R_i}\frac{d}{dt}\int_{I_i} r\theta^1(r,t)dr + \frac{C_i}{R_i}\frac{d}{dt}\max\{I(\Theta),0\},\ \text{ in }\ Q^i_T,$$

where $I_1 = (0,r_1)$ and $I_2 = (r_2,1)$.

Let now W^i $(i=1,2)$ be defined by

$$(3.4)\qquad W^i(r,t) = \theta^i(r,t) - \frac{B_i}{R_i}\int_{I_i} r\theta^i(r,t)dr - \frac{C_i}{R_i}\frac{d}{dt}\max\{I(\Theta),0\},$$

then, we see that $W^i_t - \rho_i(W^i_{rr} + W^i/r) = 0$, in Q^i_T, $i=1,2$. In order to find the corresponding initial and boundary conditions for W^1 and W^2, we need the following lemma.

Lemma 3.1. Assume that A_i, B_i and C_i $(i=1,2)$ are such that

$$(3.5)\qquad B_i \neq 1,\qquad i=1,2,\qquad Q = 1 - \sum_{i=1}^{2}\frac{A_iC_i}{1-B_i} > 0.$$

Then the unique inverse of (3.4) is given by

$$\theta^i(r,t) = W^i - \frac{B_i}{(1-B_i)R_i}\int_{I_i} rW^i(r,t)dr - \frac{C_i}{Q(1-B_i)R_i}\max\{J(W),0\}, \tag{3.6}$$

for $i = 1, 2$, where the integral operator $J(\cdot)$ is defined for $W = \{W^1, W^2\}$ by

$$J(W) = \frac{A_1}{1-B_1}\int_{I_1} rW^1(r,t)dr + \frac{A_2}{1-B_2}\int_{I_2} rW^2(r,t)dt - g. \tag{3.7}$$

Proof: First, it follows from multiplying (3.4) by r and intgerating the resultant equations on I_i that

$$\int_{I_i} rW^i dt = (1-B_i)\int_{I_i} r\theta^1(r,t)dr - C_i\max\{I(\Theta),0\}$$

which implies that (3.4) is equivalent to

$$W^i(r,t) + \frac{B_i}{(1-B_i)R_i}\int_{I_i} rW^i(r,t)dr = \theta^i(r,t) - \frac{C_i}{(1-B_i)R_i}\max\{I(\Theta),0\}, \tag{3.8}$$

provided that $B_i \neq 1$ for $i = 1, 2$.

Now assuming that $I(\Theta) > 0$, we find from applying the operator $I(\cdot)$ to (3.8) that

$$I(W) + \sum_{i=1}^{2}\frac{A_iB_i}{1-B_i}\int_{i_i} rW^i(r,t)dr = \left(1 - \sum_{i=1}^{2}\frac{A_iC_i}{1-B_i}\right)I(\Theta), \tag{3.9}$$

or equivalently with a simple calculation that $J(W) = QI(\Theta)$. Thus we find that

$$I(\Theta) > 0 \quad \text{if and only if} \quad J(W) > 0$$

provided that $Q > 0$. Hence Lemma 3.1 is porved. Q.E.D.

Theorem 3.1. Under assumption (3.5), the system (2.25)-(2.30) is equivalent to problem of finding $W = \{W^1, W^2\}$ such that

$$W^i_t = \rho_i(W^i_{rr} + W^i/r), \text{ in } Q^i_T,\ i = 1,2, \tag{3.10}$$

$$W^i(r,0) = \phi_i(r) - \frac{B_i}{R_i}\int_{I_i} r\phi_i(r)dr - \frac{C_i}{R_i}\max\{I(\Phi),0\}, \qquad r \in I_i, \tag{3.11}$$

$$W^1_r(0,t) = 0, \quad \mu W^2_r(1,t) + \nu J_2(W,1) = f(t), \quad 0 < t < T, \tag{3.12}$$

$$k_1W^1_r(r_1,t) = k(J_1(W,r_1) - J_2(W,r_2)), \quad 0 < t < T, \tag{3.13}$$

$$k_2W^2_r(r_2,t) = k(J_1(W,r_1) - J_2(W,r_2)), \quad 0 < t < T, \tag{3.14}$$

where $\Phi = (\phi_1, \phi_2)$ and $J_i(\cdot, r), i = 1, 2$, are defined for $W = \{W^1, W^2\}$ by

$$J_i(W,r) = W^i(r,t) - \frac{B_i}{(1-B_i)R_i}\int_{I_i} rW^i(r,t)dr - \frac{C_i}{Q(1-B_i)R_i}\max\{J(W),0\}, \tag{3.15}$$

and W^i and θ^i $(i = 1, 2)$ are related by (3.4) and (3.6).

Proof: See Lemma 3.1. Q.E.D.

If we let

$$L = \max_{i=1,2}\left\{1 + \left|\frac{B_i}{(1-B_i)R_i}\right| + \left|\frac{A_iC_i}{(1-B_i)QR_i}\right|\right\} \tag{3.16}$$

then, we see for any $W_j = (W_j^1, W_j^2), j = 1, 2$ that

$$|J_i(W_1, r) - J_i(W_2, r)| \le L \sum_{j=1}^{2} \left\{ |W_1^j(r,t) - W_2^j(r,t)| + \int_{I_j} r|W_1^j - W_2^j| dr \right\} \tag{3.17}$$

for $i = 1, 2$.

The equation (3.17) and system (3.10)-(3.14) motivate us to consider a more general problem which takes the system (3.10)-(3.14) as a special case. This will be given in section 4.

4. Existence and Uniqueness

In this section we shall study a more general system of parabolic equations which in fact takes the system (3.10)-(3.14) as a special case. The well-posedness of system (2.25)-(2.30) follows once the existence and uniqueness of the general system are proved.

We consider the problem of finding $U = \{u_k\}_{k=1}^M$ such that

$$\frac{\partial u_k}{\partial t} + A^k u_k = 0, \text{ in } Q_T^k, \tag{4.1}$$

$$u_k(x,t) = \phi_k(x), \quad x \in \Omega_k \tag{4.2}$$

$$\frac{\partial u_k}{\partial n_k} + B_k u_k = F_k(x, t, U(\Lambda^k x), \int_\Omega U dx), \text{ on } \partial\Omega_k \times (0,T) \tag{4.3}$$

where $Q_T^k = \Omega_k \times (0,T)$, $T > 0$, $\Omega_k \subset R^n$ $(n \ge 1)$ with smooth boundary $\partial\Omega_k$, ϕ_k, B_k and $F_k(x,t,p,q)$ are known smooth functions with $F_k(x,t,p,q)$ Lipschitz continuous in $p, q \in R^n$,

$$\int_\Omega U dx = \left\{ \int_{\Omega_k} u_k dx \right\}_{k=1}^M, \qquad U(\Lambda^k x) = \left\{ u_j(\Lambda_j^k x) \right\}_{j=1}^M, \tag{4.4}$$

where $\Lambda_j^k : \partial\Omega_k \to \partial\Omega_j$ are continuous mappings for all $j, k = 1, \cdots, M$, and

$$A^k = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{i,j}^k \frac{\partial}{\partial x_j} + a^k, \qquad k = 1, \cdots, M, \tag{4.5}$$

are strongle elliptic operators with smooth coefficients, and

$$\frac{\partial u_k}{\partial n_k} = -\sum_{i,j=1}^{n} a_{i,j}^k \frac{\partial u_k}{\partial x_j} cos(x_i, n_k) + a^k, \qquad k = 1, \cdots, M, \tag{4.6}$$

where n_k is the outer-normal directions on $\partial\Omega_k$.

Remark: In (4.3), if some of $\partial\Omega_k$ have two piece boundaries: $\partial\Omega_k = \partial\Omega_k^D \cup \partial\Omega_k^N$, $k = 1, 2, \cdots, m$, such that $\partial\Omega_k^D \cap \partial\Omega_k^N = \{\phi\}$, then the following boundary conditions can be imposed for u_k:

$$u_k(x,t) = G_k(x, t, \int_\Omega U dx), \text{ on } \partial\Omega_k^D \times (0,T), \tag{4.7}$$

$$\frac{\partial u_k}{\partial n_k} + B_k u_k = F_k(x, t, U(\Lambda^k x), \int_\Omega U dx), \text{ on } \partial\Omega_k^N \times (0,T), \tag{4.8}$$

where $G_k(x,t,q)$ is Lipschitz continuous for $k = 1, 2, \cdots, m$.

We now have the following general result.

Theorem 4.1. Assume that the data is smooth and satisfy the compatibility conditions at $t = 0$. Then the problem (4.1)-(4.3) (or (4.1)-(4.2), (4.3) for $k = m, m+1, \cdots, M$ and (4.7)-(4.8) for $k = 1, 2, \cdots, m$) possesses unique solution $u_k \in C^{2,1}(Q_T^k), k = 1, 2, \cdots, M$, which is continuously dependent upon the data.

Proof: First let us consider the problem (4.1)-(4.3). Let $U^0 = \{\phi_k(x)\}$ and $\{U^l\}$, $l = 1, 2, \cdots$ be defined by

$$\frac{\partial u_k^l}{\partial t} + A^k u_k^l = 0, \text{ in } Q_T^k, \tag{4.9}$$

$$u_k^l(x,t) = \phi_k(x), x \in \Omega_k \tag{4.10}$$

$$\frac{\partial u_k^l}{\partial n_k} + B_k u_k^l = F_k(x, t, U^{l-1}(\Lambda^k x), \int_\Omega U^{l-1} dx), \text{ on } \partial\Omega_k \times (0,T). \tag{4.11}$$

It follows from the standard theory for parabolic equations [6, 7] that $\{u_k^l\}_{k=1}^M$ is well-defined in $\prod_{k=1}^M C(\overline{Q_T^k}) \cap C^{2,1}(Q_T^k)$, and u_k^l can be represented by

$$u_k^l = \int_{\Omega_k} \Gamma^k(x,\xi,t,0)\phi_k(\xi)d\xi + \int_0^t \int_{\partial\Omega_k} \Gamma^k(x,\xi,t,\tau)\Psi_k^l(\xi,\tau)dS_\xi d\tau \text{ in } \Omega_T^k, \tag{4.12}$$

where $\Psi_k^l(x,t)$ is the solution of Volterra system

$$\Psi_k^l(x,t) = \int_0^t \int_{\partial\Omega_k} N^k(x,\xi,t,\tau)\Psi_k^l(\xi,\tau)dS_\xi d\tau + 2H^k, \text{ on } \partial\Omega_k \times (0,T), \tag{4.13}$$

with

$$N^k(x,t,\xi,\tau) = \frac{\partial\Gamma^k}{\partial n_k}(x,t,\xi,\tau) + B_k\Gamma^k(x,t,\xi,\tau) \tag{4.14}$$

$$H^k = \int_{\Omega_k} N^k(x,\xi,t,0)\phi_k(\xi)d\xi - F_k(x,t,U^{l-1}(\Lambda^k x), \int_\Omega U^{l-1}dx), \tag{4.15}$$

and where Γ^k is the foundamental solutions for the kth equation defined in [8, 9].

Let $z_k^l = u_k^l - u_k^{l-1}$ and $\Phi_k^l = \Psi_k^l - \Psi_k^{l-1}$, $k = 1, 2, \cdots, M$ and $l = 1, 2, \cdots$. we find from the standard potential estimates for Γ [6, 7] and (4.12)-(4.13) that there exists some $C > 0$, independent of k, l and the data ,

$$||\Psi_k^l(\cdot,t)||_\infty \le C\int_0^t \frac{||\Psi_k^l(\cdot,\tau)||_\infty}{\sqrt{t-\tau}}d\tau + C\sum_{k=1}^M ||z_k^{l-1}(\cdot,t)||_\infty \tag{4.16}$$

and

$$||z_k^l(\cdot,t)||_\infty \le C\int_0^t \frac{||\Phi_k^l(\cdot,\tau)||_\infty}{\sqrt{t-\tau}}d\tau \tag{4.17}$$

for $l = 2, 3, \cdots$ and $k = 1, 2, \cdots, M$, and where for $v(x,t)$ defined on $S \times (0,T)$, $||v(\cdot,t)||_\infty = \max_{x\in S}|v(x,t)|$ for $t \in (0,T)$.

It is easy to see from (4.16)-(4.17) that $\{u_k^l\}$ and $\{\Psi_k^l\}$ are Cauchy in $C(\overline{Q_T^k})$ and $C(\partial\Omega_k \times [0,T])$ for $k = 1, \cdots, M$. Hence, there exists u_k and Ψ_k such that $u_k^l \to u_k \in C(\overline{Q_T^k})$ and $\Psi_k^l \to \Psi_k \in C(\partial\Omega_k \times [0,T])$ uniformly as $l \to \infty$, and u_k and Ψ_k satisfies (4.12)-(4.15) with l removed, and also (4.9)-(4.11). The equatitions (4.16)-(4.17) also indicate that the solution depends continuously upon the data. The case of (4.1)-(4.2) and (4.3) for (4.3) for $k = m, m+1, \cdots, M$ and (4.4) for $k = 1, 2, \cdots, m$ can be proved in the same way. Q.E.D.

Remark: If the data is not compatible at $t = 0$ Theorem 3.1 still hold with solution $u_k \in C(Q_T^k) \cap L^1(Q_T^k), k = 1, 2, \cdots, M$. That is, it can be shown that $\{u^k\}_{k=1}^M$ is Cauchy in $\prod_{k=1}^{M} L^1(Q_T^k)$.

Theorem 4.2 The system (3.10)-(3.14) possesses unique solution which is continuously dependent upon the data.

Proof: If we write the system (3.10)-(3.14) in Cartesian coordinates, we find that $W_t^i - \rho_i \Delta W^i = 0$ in Ω_T^i, where $\Omega_T^1 = \Omega^1 \times (0,T)$, $\Omega_T^2 = \Omega^2 \times (0,T)$, $\Omega^1 = \{x \mid \; ||x|| < r_1\}$ and $\Omega^1 = \{x \mid \; r_2 < ||x|| < 1\}$ and $||x|| = \sqrt{x_1^2 + x_2^2}$ in R^2, and that $J_j(W, r), j = 1, 2$ are Lipschitz continuous. Thus, it is a special case of Theorem 4.1. Q.E.D.

Theorem 4.3. Th problem (1.1)-(1.11) has unique solution which is continuously dependent upon the data.

Proof: We see from Theorem Lemma 3.1, Theorem 3.1, Theorem 4.2 and the decoupling procedures in Section 2 that it is sufficient to verify that the assumption (3.5) is valid. We have from (2.16) and (3.1)-(3.2) that

$$
\begin{aligned}
&(4.18) \qquad A_1 C_1 &=& \frac{2a_1}{(1+\alpha_1) r_1} \frac{d_1 r_1 \lambda}{(c_1 + d_1 a_1)(1+\lambda)} = \frac{a_1 d_1}{c_1 + d_1 a_1} \frac{2}{1+\alpha_1} \frac{\lambda}{1+\lambda}, \\
&(4.19) \qquad A_2 C_2 &=& \frac{a_2 r_2}{r_2^2 + (1-\alpha_2) R_2} \frac{d_2 r_2}{(c_2 + d_2 a_2)(1+\lambda)} = \frac{a_2 d_2}{c_2 + d_2 a_2} \frac{r_2^2}{r_2^2 + (1-\alpha_2) R_2} \frac{1}{1+\lambda}, \\
&(4.20) \qquad 1 - B_1 &=& 1 - \frac{d_1(a_1 - A_1 r_1)}{c_1 + d_1 a_1} = 1 + \frac{d_1 a_1}{c_1 + d_1 a_1} \frac{1-\alpha_1}{1+\alpha_1}, \\
&(4.21) \qquad 1 - B_2 &=& 1 - \frac{d_2(a_2 - A_2 r_2)}{c_2 + d_2 a_2} = 1 - \frac{d_2 a_2}{c_2 + d_2 a_2} \frac{(1-\alpha_2) R_2}{r_2^2 + (1-\alpha_2) R_2},
\end{aligned}
$$

thus, we find for all parameters that $1 - B_1 > 0$, $1 - B_2 > 0$ and

$$
\begin{aligned}
(4.22) \qquad & \frac{A_1 C_1}{1 - B_1} + \frac{A_2 C_2}{1 - B_2} \\
= \; & \left\{ \frac{a_1 d_1}{c_1 + d_1 a_1} \frac{2}{1+\alpha_1} \frac{\lambda}{1+\lambda} \right\} \left\{ 1 + \frac{d_1 a_1}{c_1 + d_1 a_1} \frac{1-\alpha_1}{1+\alpha_1} \right\}^{-1} \\
& \left\{ \frac{a_2 d_2}{c_2 + d_2 a_2} \frac{r_2^2}{r_2^2 + (1-\alpha_2) R_2} \frac{1}{1+\lambda} \right\} \left\{ 1 - \frac{d_2 a_2}{c_2 + d_2 a_2} \frac{(1-\alpha_2) R_2}{r_2^2 + (1-\alpha_2) R_2} \right\}^{-1} \\
= \; & \left\{ A^* + A^* \frac{1-\alpha_1}{1+\alpha_1} \right\} \left\{ 1 + A^* \frac{1-\alpha_1}{1+\alpha_1} \right\}^{-1} \frac{\lambda}{1+\lambda} \\
& + \left\{ B^* - B^* \frac{(1-\alpha_2) R_2}{r_2^2 + (1-\alpha_2) R_2} \right\} \left\{ 1 - B^* \frac{(1-\alpha_2) R_2}{r_2^2 + (1-\alpha_2) R_2} \right\}^{-1} \frac{1}{1+\lambda} \\
< \; & \frac{\lambda}{1+\lambda} + \frac{1}{1+\lambda} = 1,
\end{aligned}
$$

where

$$A^* = \frac{a_1 d_1}{c_1 + d_1 a_1} < 1, \qquad B^* = \frac{a_2 d_2}{c_2 + d_2 a_2} < 1. \tag{4.23}$$

Hence, (3.5) is valid and Theorem 4.3 is proved. Q.E.D.

Stage I:	Parabolic-Elliptic System for θ^i and u^i, $i = 1, 2$ Signorini Contact Boundary Conditions: $u^1(r_1, t) \leq u^2(r_2, t) + g$ and etc.
Stage II:	Decoupling θ^i from u^i, $i = 1, 2$ Non-local Parabolic System for θ^i, $i = 1, 2$: Standard boundary conditions
Stage III:	Standard Parabolic System for W^i, $i = 1, 2$ (via a transformation θ^i into W^i, $i = 1, 2$) Non-local boundary conditions for W^i, $i = 1, 2$

We see from the above diagram that Stage I is the mathematical model (ignoring the inertial effects), the authors of [1, 2, 6, 14, 15, 16] introduced the decoupling procedure and studied the contact problem in Stage II, which in general simplifies the problem but does not give the best results due to the mathematical settings and methods employed therein. An additional transformation is introduced in [4] and in this paper which put the contact problem into the Stage III— homogeneous parabolic system with nonlocal boundary conditions. This last stage III (equivalent to the Stage I and Stage II) , in this author's opinion, places the problem into the best possible form so that the standard theory of parabolic problems can be applied effectively regarding the questions of existence, uniqueness and continuous dependence of the solution upon the data.

All previous work [1, 2, 6, 13, 14, 17] and the references therein have some kind restrictions on the physical parameters (a_i, c_i, d_i being small in some sense), we here have demonstrated by the technique above that all the restrictions mentioned are purely due to the methods used there. In fact the original physical model setting is well-posed with respect to all physical parameters because of (4.23) as demonstrated in this paper.

Even through the problem considered in this paper may not be as practical as any real situation in applications, the analysis above itself is of great interest mathematically.

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