A Parabolic Integro-differential Equation Arising from Thermoelastic Contact*

W. Allegretto
Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1 Canada

John R. Cannon
Department of Mathematics
University of Central Florida
Orlando, Florida 32816
and
Yanping Lin
Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1 Canada

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Abstract. In this paper we consider a class of integro-differential equations of parabolic type arising in the study of a quasi-static thermoelastic contact problem involving a critical parameter α. For $\alpha < 1$, the problem is first transformed into an equivalent standard parabolic equation with non-local and non-linear boundary conditions. Then the existence, uniqueness and continuous dependence of the solution upon the data are demonstrated via solution representation techniques and the maximum principle. Finally the asymptotic behavior of the solution as $t \to \infty$ is examined, and we show that the non-local term has no impact on the asymptotic behavior for $\alpha < 1$. The paper concludes with some remarks on the case $\alpha > 1$.

1. Introduction.

This paper is devoted to the study of the following " mildly " nonlinear parabolic integro-differential equation: Find $u(x,t)$ such that

$$u_t - u_{xx} = \alpha \frac{d}{dt} \max \left\{ \int_0^1 u dx, 0 \right\} + f, \text{ in } Q_T = (0,1) \times J, \tag{1.1}$$

where $J = (0,T)$ with $0 < T \leq \infty$ and α is a parameter, and subject to Dirichlet or Neumann or mixed boundary conditions. Our motivation for studying this problem comes from [1, 13], where the above model is derived in the study of a one-dimensional quasi-static thermoelastic contact problem due to the decoupling of the temperature and stress. A simple version of (1.1) is also derived in [11], where prepore-rock pressure distribution is considered. On the questions of existence, uniqueness, and continuous dependence of the solution upon the data, Shi and his coauthors [1, 2, 3, 13, 14] recently have made substantial progress. Basically speaking, $0 < \alpha < 1$, which is crucial for their arguments, is assumed in almost all their work, and the existence of strong and weak solutions is proved via various techniques including abstract functional analysis theory. The relaxation of the parameter to $\alpha < 1$ had been made by Xie [15], where a different approach than that used in [1, 2, 3, 13, 14] was taken, but with smooth data. Another approach using monotonic operator theory has also appeared in [4], where $0 < \alpha < 1$ is assumed, and the existence of periodic solutions is shown. For numerical solutions of the above problems, there are two published papers. The authors of [14] considered a Crank-Nicolson scheme and only gave a proof of convergence without the rate of convergence being determined. The backward finite element Euler method was studied and the optimal rate of convergence was proved in [4]. In [16] Zhou considered the quasi-static contact problem on the unit disk in polar coordinates with uniformity in θ, which results in a singular parabolic non-linear integro-differential equation,

$$u_t - \left(u_{rr} + \frac{1}{r} u_r \right) = a_0 \frac{d}{dt} \left\{ a_1 \int_0^1 ru(r,t)dr, a_2 \int_0^1 ru(r,t)dr - g \right\} \tag{1.2}$$

where a_0, a_1, a_2 and g are constants related to the physical parameters. The existence of the solution is demonstrated under the weak formulation with very restrictive assumptions on the parameters a_0, a_1 and a_2. It should be pointed out that the methods used in [1, 2, 13, 14, 15, 16] are all one-dimensional and can not be generalized easily to higher dimensions.

In this paper we shall give a systematic study of the above problem under Dirichlet, Neumann or mixed (so-called Barber's heat exchange) conditions. Our approach is very simple. We employ the solution representation and the maximum principle. However, the results, which include singular and incompatible data, are very general. All previous work mentioned above will turn out to be special cases of our results. In addition our methods are valid in higher dimensions. Our approach, despite its simplicity, allows us to see the essential features of this type of nonlocal problem and its difficulties. Some remarks comparing our results with earlier work will be given throughout the paper.

We now provide some physical background for the problem studied in this paper. First, let us assume that the elastic body under consideration is homogeneous and isotropic, with the reference configuration being the unit disk. After ignoring the inertia effect in the system, which is usually the case when the acceleration of the system is sufficiently slow, one can write the field equations in the quasi-static approximation of linearized thermoelasticity as

$$\lambda \frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \nu \left(\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i} \right) = \beta \frac{\partial \theta}{\partial x_i} \tag{1.3}$$

and

$$q\Delta\theta = \rho s \frac{\partial \theta}{\partial t} + \beta c \frac{\partial^2 u_i}{\partial t \partial x_i}, \quad i = 1, 2. \tag{1.4}$$

Here the summation convention is used with $U = (u_1, u_2)$ denoting the displacement and θ denoting the temperature. The parameters λ and ν are Lame constants, β is the interaction constant, q, ρ, s and c denote, the heat conductivity, the density, the specific heat and the absolute reference temperature respectively. After employing the decoupling arguments using complex variables given in [12], we can find that the temperature θ satisfies the following heat equation

$$\frac{\partial \theta}{\partial t} - A\Delta\theta = B\frac{d}{dt}\int_\Omega K(x, y, t)\theta(y, t)dy \tag{1.5}$$

where

$$A = \frac{q}{\rho s + \beta\beta^* c/2}, \qquad B = -\frac{\beta\beta^* c}{2\rho s + \beta\beta^* c}, \qquad \beta^* = \frac{\beta}{\lambda + 2\nu}$$

Ω is the unit disk and the $K(x, y, t)$ is the kernel. Problem (1.5) is a similar problem to (1.1) or (1.2) in 2-dimensions without the contact, and it can also be derived in the study of pore pressure rocks [11]. An existence theorem was given in [12] with homogeneous Dirichlet data under the assumption that B is small. In fact the more general non-linear version in n-dimensions of (1.5) with general Dirichlet or mixed nonlinear boundary conditions has been studied [9] under the assumption that $\mu = B$ is not an eigenvalue of the integral equation

$$\mu \int_\Omega K(x, y, t) V(y, t) dy = V(x, t), \qquad t \in J, \quad x \in \Omega. \tag{1.6}$$

A finite difference scheme, the rate of convergence and some numerical results using the real physical data of the above problem (1.5) are analyzed in [10]. Next when there are some contacts at the boundaries, the displacement satisfies the nonlocal equation (1.1) in one-dimension and (1.2) in the unit disk respectively. The detailed derivation of these equations can be found in [1, 10].

This paper is organized as follows: In Section 2, problems equivalent to (1.1) or (1.2) are derived and more general non-local boundary value problems are formulated in n-dimensions. In Section 3 the existence, uniqueness and continuous dependence of the solution upon the data are demonstrated. For problem (1.1) the behavior of the solution as $t \to \infty$ is studied in Section 4. Results for $\alpha > 1$ are presented in Section 5.

2. Equivalent Problems

First let us consider the problem of finding $u = u(x,t)$ such that

$$u_t - u_{xx} = \alpha \frac{d}{dt} \max\{\int_0^1 u dx,\ 0\} + f \ \text{ in } \ Q_T, \tag{2.1}$$

$$u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \quad 0 < t < T, \tag{2.2}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{2.3}$$

where $Q_T = (0,1) \times (0,T)$, $T > 0$, f, g_1, g_2 and u_0 are known functions.

The methods used in [1, 2, 13, 14, 15] basically treat the first term on the right hand side of (2.1) as a perturbation of u_t, and thus, $|\alpha| < 1$ is needed in most cases. Our approach is entirely different and based upon a transformation into an equivalent heat equation with nonlocal boundary conditions. Thus all results for the heat equation can be used with some careful treatment of the boundary conditions.

Let u be a solution of (2.1) and let

$$v(x,t) = u(x,t) - \alpha \max\{\int_0^1 u(x,t) dx,\ 0\}, \tag{2.4}$$

then v satisfies the heat equation $v_t = v_{xx} + f$. In order to find out the corresponding initial and boundary conditions for v, it is necessary to express u in terms of v. This leads us to consider the following functional equation: Given $\psi \in L^2(0,1)$ and $\alpha \in R$, find $\phi \in L^2(0,1)$ such that

$$\phi(x) = \psi(x) + \alpha \max\{\int_0^1 \phi(x) dx,\ 0\}. \tag{2.5}$$

Lemma 2.1. Assume that $\alpha < 1$ and $\psi \in L^2(0,1)$. Then the functional equation (2.5) has a unique solution $\phi \in L^2(0,1)$ given by

$$\phi = \psi + \frac{\alpha}{1-\alpha} \max\{\int_0^1 \psi(x) dx,\ 0\}. \tag{2.6}$$

Proof: If $\int_0^1 \phi dx > 0$ then we find that $\int_0^1 \psi dx = (1-\alpha) \int_0^1 \phi dx$ which implies $\int_0^1 \psi dx > 0$

provided that $\alpha < 1$ and

$$\phi = \psi + \frac{\alpha}{1-\alpha}\int_0^1 \psi(x)dx.$$

On the other hand if $\int_0^1 \phi dx \le 0$ then $\phi = \psi$ and $\int_0^1 \psi dx \le 0$. Hence $\int_0^1 \phi dx > 0$ if and only if $\int_0^1 \psi dx > 0$, and then, ϕ given by (2.6) is the unique solution of (2.5). Q.E.D.

Lemma 2.2. Assume that $\alpha > 1$ and $\psi \in L^2(0,1)$. Then we have the following:

(I) The necessary condition for (2.5) to be solvable is that $\int_0^1 \psi \le 0$;

(II) If $\int_0^1 \psi dx = 0$ then $\phi = \psi$ is the only solution;

(III) If $\int_0^1 \psi dx < 0$, then there are two solutions:

(A): $\phi = \psi$ with $\int_0^1 \phi dx < 0$;

(B): $\phi = \psi + \frac{\alpha}{1-\alpha}\int_0^1 \psi(x)dx$ with $\int_0^1 \phi dx > 0$.

Proof: (I) If $\phi \in L^2(0,1)$ exists, then it is easy to see that

$$\begin{aligned}\int_0^1 \psi dx &= \int_0^1 \phi dx - \alpha \max\{\int_0^1 \phi dx,\ 0\} \\ &= \begin{cases} (1-\alpha)\int_0^1 \phi dx, & if \quad \int_0^1 \phi dx > 0, \\ \int_0^1 \phi dx, & if \quad \int_0^1 \phi dx \le 0, \end{cases}\end{aligned}$$

is less or equal to zero since $\alpha > 1$.

(II) If $\int_0^1 \psi dx = 0$, then $\int_0^1 \phi dx = \alpha \max\{\int_0^1 \phi dx,\ 0\}$ which is not possible for $\int_0^1 \phi dx \ne 0$, and thus $\int_0^1 \phi dx = 0$ and $\phi = \psi$.

(III) If $\int_0^1 \psi dx < 0$, then both $\phi = \psi$ and $\phi = \psi + \dfrac{\alpha}{1-\alpha}\displaystyle\int_0^1 \psi dx$ are (the only) solutions. The former means that $\int_0^1 \phi dx < 0$ and the latter means that $\int_0^1 \phi dx > 0$. Q.E.D.

Lemma 2.3. Assume that $\alpha = 1$ and $\psi \in L^2(0,1)$. Then we have the following:

(I) If $\int_0^1 \psi dx > 0$, then there is no solution for (2.5).

(II) If $\int_0^1 \psi dx < 0$, then $\phi = \psi$ is the only solution for (2.5).

(III) If $\int_0^1 \psi dx = 0$, then $\phi = \psi + C$, for any constant $C \ge 0$, is a solution of (2.5).

Proof: (I) follows immediately from Lemma 2.2. (II) If $\int_0^1 \psi dx < 0$, then $\int_0^1 \phi dx - \max\{\int_0^1 \phi dx,\ 0\} < 0$, which is possible only if $\int_0^1 \phi dx < 0$, that is, $\phi = \psi$; (III) If $\int_0^1 \psi dx = 0$, then we have $\int_0^1 \phi dx = \max\{\int_0^1 \phi dx,\ 0\}$, which implies that solutions exist and are: $\phi = \psi$ if $\int_0^1 \phi dx = 0$; $\phi = \psi + \int_0^1 \phi dx$ if $\int_0^1 \phi dx > 0$. Hence the solution is $\phi = \psi + C$ for some constant which is related to ϕ by $C = \int_0^1 \phi dx \geq 0$. Q.E.D.

An application of Lemma 2.1 then yields:

Theorem 2.1. For $\alpha < 1$, problem (2.1)-(2.3) is equivalent to the problem

$$v_t = v_{xx} + f, \quad \text{in} \quad Q_T, \tag{2.7}$$

$$v(0,t) + \frac{\alpha}{1-\alpha}\max\{\int_0^1 vdx, 0\} = g_1, \quad 0 < t < T, \tag{2.8}$$

$$v(1,t) + \frac{\alpha}{1-\alpha}\max\{\int_0^1 vdx, 0\} = g_2, \quad 0 < t < T, \tag{2.9}$$

$$v(x,0) = u_0(x) - \alpha\max\{\int_0^1 u_0(x)dx, 0\}. \tag{2.10}$$

Here u and v are related by (2.4).

Note that if the boundary conditions of (2.2) are replaced by

$$\alpha_k u_x(k,t) + \beta_k u(k,t) = g_k(t), \quad k = 0, 1, \quad 0 < t < T, \tag{2.11}$$

where α_k, β_k are constants with $\alpha_k^2 + \beta_k^2 \neq 0$, $k = 0, 1$, then problem (2.1), (2.11) and (2.3) with $\alpha < 1$ is equivalent to problem (2.7), (2.10) and

$$\alpha_k v_x(k,t) + \beta_k \left\{ v(k,t) + \frac{\alpha}{1-\alpha}\max\{\int_0^1 vdx,\ 0\} \right\} = g_k(t) \tag{2.12}$$

for $k = 0, 1, \quad 0 < t < T$.

Next, we consider the problem (1.2) and first convert (1.2) into Cartesian coordinates

$$u_t - \Delta u = \frac{a_0}{2\pi}\frac{d}{dt}\max\left\{ a_1 \int_\Omega udx,\ a_2 \int_\Omega udx - 2\pi g(t) \right\}, \tag{2.13}$$

$$u(x,t) = h(t), \qquad \text{on} \ \partial\Omega \times (0,T), \tag{2.14}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.15}$$

where Ω is the unit disk in R^2.

CASE I: If $a = a_1 = a_2$, then (2.13) becomes

$$u_t - \Delta u = \frac{a_0\ a}{2\pi}\frac{d}{dt}\int_\Omega udx + a_0\frac{d}{dt}\max\left\{ -g(t),\ 0 \right\}, \tag{2.16}$$

which is the special case studied in [8].

CASE II: If $a_1 \neq a_2$, then let

$$u(x,t) = \theta(x,t) + \lambda 2\pi g(t), \tag{2.17}$$

where λ is a constant to be determined. Multiplying (2.17) by a_1, integrating over Ω, and repeating with a_2, we find that

$$\begin{aligned} a_1 \int_\Omega u dx &= a_1 \int_\Omega \theta dx + a_1 \lambda 2\pi g(t), \\ a_2 \int_\Omega u dx - 2\pi g(t) &= a_2 \int_\Omega \theta dx + (a_2\lambda - 1) 2\pi g(t). \end{aligned}$$

Choosing $\lambda = 1/(a_2 - a_1)$, it follows that θ satisfies

$$\theta_t - \Delta\theta = \frac{a_0}{2\pi}\frac{d}{dt} \max\left\{a_1 \int_\Omega \theta dx,\ a_2 \int_\Omega \theta dx\right\} + a_3 g'(t), \tag{2.18}$$

where $a_3 = (a_0 a_1 - 2\pi)/(a_2 - a_1)$.

If we let

$$W = \theta - \frac{a_0}{2\pi} \max\left\{a_1 \int_\Omega \theta dx,\ a_2 \int_\Omega \theta dx\right\}. \tag{2.19}$$

we conclude that W satisfies

$$W_t - \Delta W = a_3\ g'(t). \tag{2.20}$$

In order to find the corresponding boundary condition that W satisfies, we need the following lemma.

Lemma 2.4. Assume that $a_1 \neq a_2$, $a_0\ a_1 < 2$ and $a_0\ a_2 < 2$. Then for given $W \in L^2(\Omega)$, the unique solution of the functional equation (2.19) is given by

$$\theta = W + F(\int_\Omega W dx) \tag{2.21}$$

where $F = F_+$ if $a_1 > a_2$ and $F = F_-$ if $a_1 < a_2$, and

$$\begin{aligned} F_+(p) &= \frac{a_0\ a_1}{\pi(2 - a_0\ a_1)} \max\{p,\ 0\} - \frac{a_0\ a_2}{\pi(2 - a_0\ a_2)} \max\{-p,\ 0\}, \\ F_-(p) &= \frac{a_0\ a_2}{\pi(2 - a_0\ a_2)} \max\{p,\ 0\} - \frac{a_0\ a_1}{\pi(2 - a_0\ a_1)} \max\{-p,\ 0\}. \end{aligned} \tag{2.22}$$

Proof: Assume that $a_1 > a_2$. First, if $a_1 \int_\Omega \theta dx > a_2 \int_\Omega \theta dx$, then we have $(a_1 - a_2)\int_\Omega \theta dx > 0$, which implies $\int_\Omega \theta dx > 0$. It follows from (2.19) that

$$\int_\Omega W dx = \left\{1 - \frac{a_0\ a_1}{2}\right\} \int_\Omega \theta dx > 0 \quad if \quad a_0 a_1 < 2,$$

and then

$$\theta = W + \frac{a_0\ a_1}{\pi(2 - a_0\ a_1)} \int_\Omega W dx. \tag{2.23}$$

Similarily, if $a_1 \int_\Omega \theta dx \leq a_2 \int_\Omega \theta dx$, we have $(a_1 - a_2)\int_\Omega \theta dx \leq 0$, and then $\int_\Omega \theta dx \leq 0$. Thus, we find by integrating (2.19) over Ω that

$$\int_\Omega W dx = \left\{1 - \frac{a_0 a_2}{2}\right\} \int_\Omega \theta dx \leq 0 \quad if \quad a_0 a_2 < 2,$$

and then

$$\theta = W + \frac{a_0\ a_2}{\pi(2 - a_0\ a_2)} \int_\Omega W dx. \tag{2.24}$$

Hence, $\int_\Omega \theta dx > 0$ if and only if $\int_\Omega W dx > 0$, and the case $a_1 > a_2$ of Lemma 2.4 follows from (2.23) and (2.24). The case $a_1 < a_2$ can be proved in the same way. Q.E.D.

The above analysis thus shows:

Theorem 2.2. Assume that $a_1 \neq a_2$, $a_0\ a_1 < 2$ and $a_0\ a_2 < 2$. Then problem (2.13)-(2.15) is equivalent to the following problem:

$$W_t - \Delta W = a_3 g'(t), \tag{2.25}$$

$$W(x,0) = \theta_0 - \frac{a_0}{2\pi} \max\{a_1 \int_\Omega \theta_0 dx,\ a_2 \int_\Omega \theta_0 dx\}, \quad x \in \Omega, \tag{2.26}$$

$$W(x,t) + F(\int_\Omega W dx) + \frac{2\pi g(t)}{a_2 - a_1} = h(t), \quad \text{on } \partial\Omega \times (0,T), \tag{2.27}$$

where

$$\theta_0(x) = u_0(x) - \frac{2\pi g(0)}{a_2 - a_1} \quad x \in \Omega. \tag{2.28}$$

and u and W are related through (2.17) and (2.19).

Remark: If the boundary condition (2.14) is replaced by

$$\mu\frac{\partial u}{\partial r}(1,t)+\nu u(1,t)=h(t),\quad 0<t<T, \tag{2.29}$$

where $r=|x|$, μ and ν are two constant with $\mu^2+\nu^2\neq 0$, then under the assumptions of Theorem (2.2), (2.27) becomes

$$\mu\frac{\partial W}{\partial n}+\nu\left\{W+F(\int_\Omega Wdx)+\frac{2\pi g(t)}{a_2-a_1}\right\}=h(t),\quad \text{on } \partial\Omega\times(0,T), \tag{2.30}$$

where n is the outer normal direction on $\partial\Omega$.

In summary, under our assumptions problem (2.7)-(2.11) or (2.25)-(2.27) are special cases of the following nonlocal parabolic boundary value problems: Find $U=U(x,t)$ such that

$$U_t+AU=G,\ \text{ in } Q_T, \tag{2.31}$$

$$U(x,0)=U_0(x),\quad x\in\Omega, \tag{2.32}$$

$$\mu\frac{\partial U}{\partial n}+\nu\left\{U(x,t)+H(\int_\Omega Udx)\right\}=h\ \text{ on } \partial\Omega\times(0,T), \tag{2.33}$$

where $\Omega\in R^d$ $d\geq 1$, G, h, U_0, H are known functions, and H is Lipschitz continuous, μ and ν are constants with $\nu\neq 0$ if $\mu=0$, and

$$A=-\sum_{i,j=1}^{d}a_{ij}\frac{\partial^2}{\partial x_i\partial x_i}+\sum_{i=1}^{d}b_i\frac{\partial}{\partial x_i}+cI$$

such that

$$C_0|\xi|^2\leq\sum_{i,j=1}^{d}a_{ij}\xi_i\xi_j\leq C_1|\xi|^2,\quad \xi\in R^d$$

for some positive constants C_0 and C_1, and a_{ij}, b_i and c are known smooth functions.

Remark: If $H=0$ then (2.31)-(2.33) is the standard linear parabolic problem for which existence, uniqueness and continuous dependence upon the data are well-known [7, 8].

3. Existence and Uniqueness

In this section we briefly consider the questions of existence, uniqueness and continuous dependence of the solution of (2.31)-(2.33) upon the data, and then treat problem (2.1)-(2.3) and (2.13)-(2.15) as special cases with a result concerning the smoothness of the solutions. Existence follows from arguments in [6] as the following theorem shows.

Theorem 3.1. Assume that $a_{ij}, b_i, c, h, G, U_0$ are smooth, H is continuous with $|H(p)-H(q)| \leq L|p-q|$ for some $L > 0$ and $p,\ q \in R$, and the compatibility conditions are satisfied at $\partial\Omega \times \{0\}$. Then the problem (2.31)-(2.33) has a unique solution $U \in C(\overline{Q_T})$ which depends continuously upon the data.

Proof: First let us consider the case $\nu = 1$ and $\mu = 0$. Let $U^0 = U_0(x)$ and $\{U^k\}$, $k = 1, 2, \cdots$ be defined by

$$U_t^k + AU^k = f \text{ in } Q_T \tag{3.1}$$

$$U^k + H(\int_\Omega U^{k-1}dx) = h \text{ on } \partial\Omega \times (0,T), \tag{3.2}$$

$$U^k(x,0) = U_0(x), \quad x \in \Omega. \tag{3.3}$$

It follows from the standard theory for parabolic equations [7, 8] that $\{U^k\}$ is well-defined in $C(\overline{Q_T})$. Let $z^k = U^k - U^{k-1}$. We find from [6, p. 520] that z^k can be estimated by

$$\left|\int_\Omega z^k dx\right|(t) \quad \leq \quad \sigma(T)L \sup_{0<s<T} \left|\int_\Omega z^{k-1}dx\right|(s), \tag{3.4}$$

where $\sigma(t) \geq 0$ is a continuous function at $t = 0$ with $\sigma(0) = 0$, independent of z^k. Thus, if T is small enough, $\{\int_\Omega U^k dx\}$ is a Cauchy sequence in $C([0,T])$, and it follows from the continuous dependence property for parabolic equations [7, 8] that $\{U^k\}$ is also Cauchy in $C(\overline{Q_T})$. Hence, there exists U such that $U^k \to U \in C(\overline{Q_T})$ uniformly as $k \to \infty$, and U satisfies (2.31)-(2.33) in Q_T and the initial and boundary conditions. Estimate (3.4) also indicates that the solution depends continuously upon the data. The case for $\mu \neq 0$ can be proved in the same way. Uniqueness once again follows by the contraction argument for small T, and global existence follows by the independence of

σ on the data. The other case can be shown in a similar way. Q.E.D.

Remark: If the data is not compatible at $t = 0$, Theorem 3.1 still holds with the solution u now in the space $C(Q_T) \cap L^\infty(Q_T)$.

We can now state our results for problem (2.1)-(2.3) and (2.13)(2.15).

Theorem 3.2. Let the assumptions of Theorem 2.1 hold and suppose further that let f, g_1, g_2 and u_0 are continuous and compatible at $t = 0$, and that $f \in C^{r,r/2}(\bar{Q_T})$ $(0 < r < 1)$. Then problem (2.1)-(2.3) has a unique solution which depends continuously upon the data.

Proof: It follows from Theorem 3.1 that v, the solution of (2.7)-(2.10) exists and is unique which implies that u defined by (2.4) is also a unique solution for (2.1)-(2.3). Q.E.D.

Theorem 3.3. Assume that $a_1 = a_2$ and $a_0 a_1 \neq 2$, or that the assumptions of Theorem 2.2 hold, and f, h and u_0 are Holder continuous and compatible at $t = 0$, Then problem (2.13)-(2.15) has a unique solution which depends continuously upon the data.

Proof: First we show our result for **CASE I**. if $a_1 = a_2$ and $a_0 a_1 \neq 2$ then $\lambda = a_0 a_1$ is not an eigenvalue of

$$\frac{\lambda}{2\pi} \int_\Omega K(x,y)\psi(y)dy = \psi(x) \quad x \in \Omega$$

with $K(x,y) = 1$, thus Theorem 2.2 of [9] implies that the solution of (2.13)-(2.15) exists and is continuously dependent upon the data. **CASE II**, which involves the assumptions of Theorem 2.2, follows from Lemma 2.4 and Theorem 3.1. Q.E.D.

Remark: Zou in [16] showed the following: If a_0, a_1 and a_2 in (1.2) satisfy

$$a_0^1(a_1^2 + a_2^2 - a_1 a_2) < \frac{8}{3}$$

then the solution of (2.13)-(2.15) exists, and is unique provided in addition that

$$\frac{a_0}{2}\left(1 + \max\{a_1^2, (a_2 - a_1)^2, a_2^2\}\right) < 1.$$

We observe that our assumptions on a_0, a_1 and a_2 in Theorem 3.3 are much weaker than Zou's, and in addition these constants need not be small. The main reason is, as stated earlier, that we do not treat the nonlocal term on the right hand side of (1.1) or (1.2) as a perturbation of u_t.

Consider next the following axially symmetric thermoelastic contact problem: find $(u,\ \theta)$ such that

$$
\begin{aligned}
&(\frac{1}{r}(ru)_r)_r = a\theta_r, \ \text{ in } \ (0,1)\times(0,T),\\
&\theta_{rr} + \frac{1}{r}\theta_r = c\theta_t + \frac{d}{r}(ru)_{tr}, \ \text{ in } \ (0,1)\times(0,T),\\
&\theta_r(0,t) = 0, \quad 0<t<T,\\
&\theta(1,0) = 0 \quad or \quad \theta_r(1,t) = -k\theta(1,t), \quad 0<t<T,\\
&\theta(r,0) = \phi(r), \quad 0<x<1,\\
&u(0,t) = 0, \quad 0<t<T,\\
&u(1,t) \le g, \quad 0<t<T,\\
&u_r(1,t) + \frac{\lambda}{\lambda+2\mu}u(1,t) - a\theta(1,t) \le, \quad 0<t<T,\\
&(u(1,t)-g)\left\{u_r(1,t) + \frac{\lambda}{\lambda+2\mu}u(1,t) - a\theta(1,t)\right\} = 0, \quad 0<t<T,
\end{aligned}
$$

where $T>0, a,\ d,\ c,\ \beta,\ \lambda,\ \mu$ are positive physical constants, u denotes the displacement and θ denotes the temperature. After decoupling the variables as in [16], we find that θ satisfies (1.2) with a_0, a_1 and a_2 defined as follows:

$$
a_0 = \frac{2d}{c+da}, \quad a_1 = a - \frac{\beta}{\lambda+\mu}, \quad a_2 = a
$$

It is easy to see that the inequalities $a_0a_1 < 2$ and $a_0a_2 < 2$, are always satisfied regardless of the parameter values. Thus, by Theorem 3.2, the above problem always has a unique solution.

Theorem 3.4— Regularity. For the solution of (2.1)-(2.3) or (2.13)-(2.15),

(i) if $f \in C^\infty(Q_T)$ then $u(x,t) \in C^\infty(\Omega)$ for any $0<t<T$ and for $x \in \Omega$, $u(x,t) \in C^\infty$ for almost every t in $(0,T)$;

(ii) if $f \in L^2(Q_T)$, $u_0 \in H^2(\Omega)$ and $h \in H^1(0,T)$ (or $g_1, g_2 \in H^1(0,T)$), then $u \in W^{2,1}(Q_T)$.

Proof: (i) that $u(x,t) \in C^\infty(\Omega)$ for any $0 < t < T$, follows from Lemma 2.1—Lemma 2.4, Theorem 2.1 and Theorem 2.2 and the standard regularity theory for parabolic equations [7, 8]. (ii) It follows from (3.2) that our assumptions imply that $\int_\Omega U dx \in H^1(0,T)$ which is the optimum result even if the data is smooth due to (2.4) and (2.22). Thus the nonlocal nature of the boundary condition implies that we have at most $U \in W^{2,1}(Q_T)$ for the solutions of (2.1)-(2.3) or (2.13)-(2.15), and (ii) follows. Q.E.D.

Remark: We see from Theorem 3.3 that the solution $u(x,t)$ of (1.1) or (1.2) is almost smooth, but, in general, the higher order Sobolev norms $W^{p,q}(Q_T)$ with $p > 2$, $q > 1$ are not bounded.

4. Asymptotic behavior as $t \to \infty$

In this section the asymptotic behavior of the solution of problem (2.1)-(2.3) as $t \to \infty$ is considered. First let $z = z(x)$ be the solution of the two-point boundary value problem

$$\begin{aligned} &z''(x) + f^\infty(x) = 0, \quad 0 < x < 1, \\ &z(0) = g_1^\infty, \qquad z(1) = g_2^\infty. \end{aligned} \tag{4.1}$$

Then it is easy to see that $z(x)$ is given by

$$z(x) = g_1^\infty + x(g_2^\infty - g_1^\infty) + x\int_0^1\int_0^s f^\infty(\tau)d\tau ds - \int_0^x\int_0^s f^\infty(\tau)d\tau ds. \tag{4.2}$$

Let

$$\bar{g} = \int_0^1 z(x)dx = \frac{g_2^\infty + g_1^\infty}{2} + \int_0^1 (s - \frac{1}{2})\int_0^s f^\infty(\tau)d\tau ds. \tag{4.3}$$

We recall:

Lemma 4.1. Let $u(x,t)$ be the solution of (2.1)-(2.3) with $\alpha = 0$, and

$$\lim_{t\to\infty} g_k(t) = g_k^\infty, \;\; k = 1,2; \quad \lim_{t\to\infty} f(x,t) = f^\infty(x), \quad 0 < x < 1. \tag{4.4}$$

Then

$$\lim_{t\to\infty} u(x,t) = z(x), \quad 0 < x < 1.$$

Remark: Lemma 4.1 is the classical result for heat equation. It will be shown below that the non-local terms present in (2.1)-(2.3) have no impact on the asymptotic behavior of the solution as $t \to \infty$.

The main result of this section is as follows.

Theorem 4.1. Let assumptions (4.4) hold and in addition suppose that one of the following is valid:

(i) $\bar{g} \le 0$, $-\infty < \alpha < 1$,

(ii) $\bar{g} > 0$, $-\infty < \alpha \le 1/2$,

(iii) $\bar{g} > 0$, $1/2 < \alpha < 1$, and $g_t \in L^2(0, \infty)$ for some $p > 1$, where g is defined in (4.13) below. Then the solution $u(x,t)$ of (2.1)-(2.3) has the following property:

$$\lim_{t \to \infty} u(x,t) = z(x), \tag{4.5}$$

where $z(x)$ is given in (4.2).

Proof: Since problem (2.1)-(2.3) is equivalent to problem (2.7)-(2.10), we consider the asymptotic behavior of the solution $v(x,t)$ of (2.7)-(2.10) as $t \to \infty$ instead of that of the solution $u(x,t)$ of (2.1)-(2.3) directly. It will be shown below that for $\alpha < 1$ in (2.7)-(2.10), the asymptotic value of $v(x,t)$ not only depends upon (4.4) but also upon the sign of (4.3) which in fact takes the non-local nature into consideration. Specifically, it follows from Lemma 4.2 below that

$$\lim_{t \to \infty} v(x,t) = z(x) + \begin{cases} 0 & \text{if} \quad \bar{g} \le 0, \\ -\alpha \bar{g} & \text{if} \quad \bar{g} > 0. \end{cases}$$

and

$$\lim_{t \to \infty} \max\{\int_0^1 v(x,t)dx,\ 0\} = \begin{cases} 0 & \text{if} \quad \bar{g} \le 0, \\ (1-\alpha)\bar{g} & \text{if} \quad \bar{g} > 0. \end{cases}$$

Thus, we find that (4.5) follows from

$$u = v + \frac{\alpha}{1-\alpha} \max\{\int_0^1 v dx\ , 0\}.$$

Hence, the proof hinges upon Lemma 4.2 and Lemma 4.3 to which we turn next. Q.E.D.

Let the solution v of (2.7)-(2.10) be decomposed into $v = w(x,t) + w^1(x,t)$, where w^1 satisfies

$$w_t^1 - w_{xx}^1 = f \ \text{ in } \ Q_\infty, \tag{4.6}$$

$$w^1(0,t) = g_1(t), \quad w^1(1,t) = g_2(t), \quad t > 0, \tag{4.7}$$

$$w^1(x,0) = u_0(x) - \alpha \max\{\int_0^1 u_0(x)dx, 0\} \tag{4.8}$$

and w satisfies

$$w_t - w_{xx} = 0 \ \text{ in } \ Q_\infty, \tag{4.9}$$

$$w(0,t) + \frac{\alpha}{1-\alpha} \max\{\int_0^1 w dx + g(t), 0\} = 0, \quad t > 0, \tag{4.10}$$

$$w(1,t) + \frac{\alpha}{1-\alpha} \max\{\int_0^1 w dx + g(t), 0\} = 0, \quad t > 0, \tag{4.11}$$

$$w(x,0) = 0, \quad 0 < x < 1, \tag{4.12}$$

where

$$g(t) = \int_0^1 w^1(x,t) dx, \qquad t > 0. \tag{4.13}$$

It follows from Lemma 4.1 that

$$\lim_{t\to\infty} w^1(x,t) = z(x), \qquad 0 < x < 1, \qquad \lim_{t\to\infty} g(t) = \bar{g}.$$

where $z(x)$ and $\bar{g}$ are given in (4.2) and (4.3).

We now see that it is sufficient to consider the asymptotic behavior of the solution of problem (4.9)-(4.12).

Lemma 4.2. Under the assumptions stated in Theorem 4.1, the solution $w(x,t)$ of problem (4.9)-(4.12) satisfies

$$\lim_{t\to\infty} w(x,t) = \begin{cases} 0 & \text{if} \quad \bar{g} \le 0, \\ -\alpha\bar{g} & \text{if} \quad \bar{g} > 0. \end{cases} \tag{4.14}$$

Proof: The proof will be divided into several cases. Q.E.D.

Case 1: Assume that $0 < \alpha < 1$ and $\bar{g} < 0$.

In this case we find that $1 > \beta = \alpha/(1-\alpha) > 0$ and $w(x,t) \le 0$ by the maximum principle. Thus, it follows from the assumption $\bar{g} < 0$ that there exists $T_0 > 0$ such that $g(t) < 0$ for all $t \ge T_0$, and then $w(0,t) = w(1,t) = 0$ for $t \ge T_0$ in (4.10)-(4.11). Hence, (4.14) is valid.

Case 2: Assume that $0 < \alpha < 1$ and $\bar{g} = 0$.

In this case it is easy to see from the non-positiveness of w that $0 \ge w(0,t) = w(1,t) \ge -\beta|g(t)|$ for all $t > 0$, which in turn implies that $\lim_{t\to\infty} w(0,t) = w(1,t) = 0$ due to $\lim_{t\to\infty} g(t) = \bar{g} = 0$. Thus $\lim_{t\to\infty} w(x,t) = 0$.

Case 3: Assume that $\alpha < 0$ and $\bar{g} \le 0$.

In this case we find that $1 > \beta = -\alpha/(1-\alpha) > 0$ and $w(x,t) \ge 0$ by the maximum principle. From the assumption we find that for any ϵ, $0 < \epsilon < 1$, there exists $T_0 > 0$ such that $g(t) < [(1-\beta)/\beta]\epsilon$ for $t > T_0$, and consequently w satisfies the boundary condition:

$$\begin{aligned} w(0,t) = w(1,t) &= \beta \max\{\int_0^1 w(x,t)dx + g(t), 0\} \\ &\le \beta \int_0^1 w(x,t)dx + (1-\beta)\epsilon, \qquad t \ge T_0. \end{aligned} \tag{4.15}$$

Let $\lambda > 0$ and $\mu > 0$ be constants and let

$$V(x,t) = 2W_0(1 - \mu(x-1/2)^2)\exp\{-\lambda(t-T_0)\} \tag{4.16}$$

where

$$W_0 = \max_{0 \le x \le 1} |w(x,T_0)|.$$

It is easy to verify that there exists $\mu = \mu_0 > 0$ small, (e.g., $0 < \mu_0 < \min(1/2, 1-\beta)$), such that $V(0,t) = V(1,t) > \beta \int_0^1 V(x,t)dx$ and $V(x,T_0) > W_0$. For this μ_0 it is also easy to show that $V_t - V_{xx} > 0$ for $0 < x < 1$ and $t > T_0$ if $\lambda = \lambda_0, 0 < \lambda_0 < 2\mu_0$. Therefore it follows by letting $Z = \epsilon + V(x,t) - w(x,t)$ that

$$Z_t - Z_{xx} > 0, \quad 0 < x < 1, \quad t > T_0 \tag{4.17}$$

$$Z(0,t) = Z(1,t) > \beta \int_0^1 Z(x,t)dx \quad ,t > T_0 \tag{4.18}$$

$$Z(x,T_0) > 0, \quad 0 < x < 1. \tag{4.19}$$

An application of the maximum principle implies that $Z(x,t) \ge 0$ for all $t > T_0$. Thus, $w \le \epsilon + V(x,t)$ for $t > T_0$. Hence we have $0 \le \overline{\lim_{t\to\infty}} w(x,t) \le \epsilon$. Since ϵ can be chosen to be arbitrary small we find that $\lim_{t\to\infty} w(x,t) = 0$.

Case 4: Assume that $\alpha < 0$ and $\bar{g} > 0$.

In this case there exists $T_0 > 0$ such that $g(t) > 0$ for $t > T_0$, therefore we observe that for $t \ge T_0$,

$$w(0,t) = w(1,t) = \beta \int_0^1 w(x,t)dx + \beta g(t), \qquad 0 < \beta = \frac{-\alpha}{1-\alpha} < 1.$$

Let $w = \bar{w} - \alpha\bar{g}$. We obtain from a simple calculation that

$$\begin{aligned}
&\bar{w}_t = \bar{w}_{xx}, \quad 0 < x < 1, \quad t > T_0, \\
&\bar{w}(0,t) = \bar{w}(1,t) = \beta \int_0^1 \bar{w}(x,t)dx + \beta(g(t) - \bar{g}), \quad t > T_0 \\
&\bar{w}(x,T_0) = w(x,T_0) + \alpha\bar{g}, \quad 0 < x < 1.
\end{aligned}$$

From the proof of **Case 3**, it follows that $\lim_{t\to\infty} \bar{w}(x,t) = 0$ as a consequence of $\lim_{t\to\infty}(g(t) - \bar{g}) = 0$. Therefore $\lim_{t\to\infty} w(x,t) = -\alpha\bar{g}$.

Case 5: Assume that $0 < \alpha < 1/2$ and $\bar{g} > 0$.

In this case, the maximum principle implies that $w(x,t) \leq 0$. First we show that there exists $T_0 > 0$ such that $S(t) = \int_0^1 w(x,t)dx + g(t) > 0$ for $t \geq T_0$. Now, for $0 < \epsilon$ small, there exists $T_1 > 0$ such that $g(t) < \bar{g} + \epsilon$, for $t \geq T_1$. We therefore conclude that $0 \geq w(0,t) = w(1,t) \geq -\frac{\alpha}{1-\alpha}(\bar{g} + \epsilon)$ for $t > T_1$ which implies via comparison with the decay of a bounded solution of the heat equation with constant boundary conditions that

$$0 \geq \overline{\lim}_{t\to\infty} \max_{0<x<1} w(x,t) \geq \underline{\lim}_{t\to\infty} \min_{0<x<1} w(x,t) \geq -\frac{\alpha}{1-\alpha}(\bar{g} + \epsilon)$$

and then, we have

$$\begin{aligned}
\underline{\lim}_{t\to\infty}\left(\int_0^1 w(x,t)dx + g(t)\right) &\geq -\frac{\alpha}{1-\alpha}(\bar{g} + \epsilon) + \bar{g} \\
&\geq -\frac{\alpha}{1-\alpha}\epsilon + \bar{g}\left(1 - \frac{\alpha}{1-\alpha}\right),
\end{aligned}$$

which shows the existence of T_0 for ϵ small enough since $\bar{g} > 0$ and $\alpha < 1/2$.

Let $w = \bar{w} - \alpha\bar{g}$ and note that a simple calculation yields:

$$\bar{w}_t = \bar{w}_{xx}, \quad 0 < x < 1, \quad t > T_0, \tag{4.20}$$

$$\bar{w}(0,t) + \beta \int_0^1 \bar{w}(x,t)dx = \beta G(t), \quad t > T_0, \tag{4.21}$$

$$\bar{w}(1,t) + \beta \int_0^1 \bar{w}(x,t)dx = \beta G(t), \quad t > T_0, \tag{4.22}$$

$$\bar{w}(x,T_0) = w(x,T_0) + \alpha\bar{g}, \quad 0 < x < 1. \tag{4.23}$$

where $\beta = \alpha/(1-\alpha) > 0$ and $\lim_{t\to\infty} G(t) = (\bar{g} - g(t)) = 0$.

Now we show that $\bar{w} \to 0$ as $t \to \infty$ if $0 < \beta \le 1$. First, if $\bar{w}(0,t) = \bar{w}(1,t) \to 0$ as $t \to \infty$, then we are done. Second, if $\bar{w}(0,t) = \bar{w}(1,t)$ is eventually positive or negative, then $\int_0^1 \bar{w} dx$ will be eventually positive or negative, which is a contradiction to $\bar{w}(0,t) + \beta \int_0^1 \bar{w} dx \to 0$ as $t \to \infty$. Finally, suppose $\bar{w}(0,t) = \bar{w}(1,t)$ is oscillatory as $t \to \infty$. We know that $\bar{w}$ is bounded since u is bounded, which in turn indicates that there exist $\{T_j^+\}$ and $\{T_j^-\}$, and $w^+ \ge 0$ and $w^- \le 0$ such that $\bar{w}(0, T_j^+) \to w^+ = \overline{\lim}_{t\to\infty} \bar{w}(0,t)$ and $\bar{w}(0,T_j^-) \to w^- = \underline{\lim}_{t\to\infty} \bar{w}(0,t)$, and $w^+ - w^- > 0$. Also we find that $w^- \le \underline{\lim}_{t\to\infty} \min_{0<x<1} \bar{w} \le \overline{\lim}_{t\to\infty} \max_{0<x<1} \bar{w}(x,t) \le w^+$ due to the maximum principle. In addition, we see by taking limits that

$$w^+ + \beta w^- \le w^+ + \beta \lim_{j\to\infty} \int_0^1 \bar{w}(x, T_j^+) dx = 0$$

and

$$w^- + \beta w^+ \ge w^- + \beta \lim_{j\to\infty} \int_0^1 \bar{w}(x, T_j^-) dx = 0.$$

We conclude that $(1-\beta)(w^+ - w^-) \le 0$ so that $\beta \ge 1$, which contradicts the assumption $0 < \alpha < 1/2$. This completes the proof of Case 5. Q.E.D.

Case 6: $1/2 \le \alpha < 1$ (i.e. $\beta \ge 1$) and $\bar{g} > 0$.

Let the solution u of (2.1)-(2.3) be decomposed into $u = Z(x,t) + Z^1(x,t)$, where Z^1 satisfies

$$Z_t^1 - Z_{xx}^1 = f \quad \text{in} \quad Q_\infty, \tag{4.21}$$

$$Z^1(0,t) = g_1(t), \quad Z^1(1,t) = g_2(t), \quad t > 0, \tag{4.22}$$

$$Z^1(x,0) = u_0(x), \quad x \in (0,1), \tag{4.23}$$

and

$$Z_t - Z_{xx} = \alpha \frac{d}{dt} \max\{\int_0^1 Z dx + G(t), 0\} \quad \text{in} \quad Q_\infty, \tag{4.24}$$

$$Z(0,t) = Z(1,t) = 0, \quad t > 0, \tag{4.25}$$

$$Z(x,0) = 0, \quad 0 < x < 1, \tag{4.26}$$

where

$$G(t) = \int_0^1 Z^1(x,t) dx, \qquad t > 0, \tag{4.27}$$

and $G(t) \to \bar{g} > 0$ as $t \to \infty$ and $G_t \in L^2(0,\infty)$ by our assumptions on the data and (4.6)-(4.8) and (4.13) .

Now multiplying (4.24) by Z_t and integrating on $(0,1)$, it follows easily that

$$\begin{aligned}
\int_0^1 Z_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 Z_x^2 ds &= \alpha\frac{d}{dt}\max\{\int_0^1 Z dx + G(t), 0\}\int_0^1 Z_t dx \\
&\leq \alpha|\int_0^1 Z_t dx|^2 + \alpha|G_t|\,|\int_0^1 G_t dx| \\
&\leq \frac{1+\alpha^2}{1+\alpha}|\int_0^1 w_t dx|^2 + \frac{(1+\alpha)\alpha^2}{4(1-\alpha)}|G_t|^2,
\end{aligned}$$

Since

$$\int_0^1 Z_t^2 dx - \frac{1+\alpha^2}{1+\alpha}|\int_0^1 Z_t dx|^2$$

is equivalent to the $||Z_t||^2_{L^2(0,1)}$ norm due to $0 < \alpha < 1$, we have by integration that

$$\int_0^t\int_0^1 Z_t^2 dxds + \int_0^1 Z_x^2 dx \leq C\int_0^t |G_t|^2 ds \quad t \geq 0.$$

Thus,

$$F(t) = \alpha\frac{d}{dt}\max\{\int_0^1 Z dx + G(t), 0\} \in L^2(0,\infty, L^2(0,1)).$$

Now multiplying (4.24) by Z and integrating on $(0,1)$, it follows that

$$\frac{1}{2}\frac{d}{dt}||Z||^2 + ||Z_x||^2 \leq ||F(t)||\ ||w||,$$

where $||\cdot||$ denotes the norm on $L^2(0,1)$.

We find from Poincare's inequality that

$$\frac{d}{dt}||Z|| + \lambda_0||Z|| \leq ||F(t)||,$$

where $\lambda_0 = \pi^2$. Therefore we see that

$$\begin{aligned}
||Z(t)|| &\leq e^{-\lambda_0 t}\left\{||Z(T_0)|| + \int_{T_0}^t ||F(s)||e^{\lambda_0 s} ds\right\} \\
&\leq e^{-\lambda_0 t}||Z(T_0)|| + ||F||_{L^2(T_0,t)}\left(\frac{1-e^{-2\lambda_0(t-T_0)}}{2\lambda_0}\right)^{1/2},
\end{aligned}$$

and then

$$\overline{\lim_{t\to\infty}} ||Z(t)|| \leq ||F||_{L^2(T_0,\infty)} \to 0 \quad \text{as} \quad T_0 \to \infty.$$

Hence $||Z|| \to 0$ as $t \to \infty$, which in turn implies that $\int_0^1 u dx \to \bar{g}$ as $t \to \infty$.

But it follows from transformation (2.4) that

$$\int_0^1 v dx \to (1-\alpha)\bar{g} \ \text{ as } \ t \to \infty,$$

and then by the decomposition (4.6)-(4.13) that $\int_0^1 w dx \to -\alpha\bar{g}$ as $t \to \infty$. As a consequence we find from (4.10)-(4.11) that

$$w(0,t) = w(1,t) \to \frac{-\alpha}{1-\alpha}(1-\alpha)\bar{g} = -\alpha\bar{g} \ \text{ as } \ t \to \infty.$$

Hence, we have $w(x,t) \to -\alpha\bar{g}$ as $t \to \infty$. Therefore the proof of Case 6 is complete. Q.E.D.

Remark: In Theorem 4.1 it is assumed for our analysis that $g(t)$ in (4.13) has some extra properties when $1/2 \leq \alpha < 1$. Our conjecture is that this extra assumption is not necessary.

5. The case $\alpha > 1$

If we consider problem (2.1)-(2.3) for the case $\alpha > 1$, then as noted in Lemma 2.2, the earlier procedures fail since the map (2.5) is not invertible. Some results can nevertheless be obtained by considering problem (2.1)-(2.3) directly. Specifically suppose first that f, g_1, g_2 and u_0 are negative. In this case we note that any solution of

$$u_t - u_{xx} = f, \quad 0 < x < 1, \ \ t > 0, \tag{5.1}$$

$$u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \quad t > 0, \tag{5.2}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{5.3}$$

is negative by the maximum principle, and thus it is also a solution of (2.1)-(2.3). It is the only continuous solution since $u_0(x) < 0$ implies $\int_0^1 u dx < 0$ for all t small, and thus for all $t > 0$ by the maximum principle as f, g_1, $g_2 < 0$. These simple arguments also show that the asymptotic behavior as $t \to \infty$ of u is the same as for the local heat equations (5.1)-(5.3) in this case.

However, quite different behavior can also be exhibited for $\alpha > 1$. To see this, consider again (2.1)-(2.3) with now $f = g_1 = g_2 = 0$, and $u_0(x) > 0$ such that

$$\int_0^1 u_0^2 dx < \alpha (\int_0^1 u_0 dx)^2. \tag{5.4}$$

Notice that this is possible for some $u_0(x)$ and $\alpha > 1$. Consider now the auxilliary problem:

$$v_t - v_{xx} = 0, \quad 0 < x < 1, \ \ t > 0, \tag{5.5}$$

$$v(x,t) + \frac{\alpha}{1-\alpha} \int_0^1 v dx = 0, \quad t > 0, \quad x = 0, 1, \tag{5.6}$$

$$v(x,0) = u_0 - \alpha \int_0^1 u_0 dx, \quad 0 < x < 1. \tag{5.7}$$

By the earlier existence result in Section 3 this problem has a solution, and if we can show that $\int_0^1 v dx < 0$ for all $t > 0$, then

$$u(x,t) = v(x,t) + \frac{\alpha}{1-\alpha} \int_0^1 v dx \tag{5.8}$$

will yield a solution of (2.1)-(2.3). By arguments somewhat anologous to those given in Section 4, we have that

$$\begin{aligned} \left(\int_0^1 v^2 dx\right)_t &= v_x|_0^1, & (5.9)\\ \left(\frac{1}{2}\int_0^1 v dx\right)_t + \int_0^1 (v_x)^2 dx &= v v_x|_0^1. & (5.10) \end{aligned}$$

Hence we find using the boundary condition (5.6) and (5.9)-(5.10) that

$$\left(\frac{1}{2}\int_0^1 v^2 dx + \frac{\alpha}{2(1-\alpha)}(\int_0^1 v dx)^2\right)_t + \int_0^1 v_x^2 dx = 0. \tag{5.11}$$

We conclude by Lemma 5.1 below and (5.4), via an integration on t and simple calculations, that for some $\eta_0 > 0$,

$$\begin{aligned} \eta_0 \int_0^t \int_0^1 v^2 dx &+ \left(\frac{1}{2}\int_0^1 v^2 dx + \frac{\alpha}{2(1-\alpha)}(\int_0^1 v dx)^2\right) \\ &\leq \left(\frac{1}{2}\int_0^1 v^2(x,0) dx + \frac{\alpha}{2(1-\alpha)}(\int_0^1 v(x,0) dx)^2\right) \qquad (5.12)\\ &\leq \frac{1}{2}\left(\int_0^1 u_0^2 dx - \alpha(\int_0^1 u_0 dx)^2\right) = -\delta_0 < 0, \end{aligned}$$

and, in particular, that

$$\frac{1}{2}\int_0^1 v^2(x,t) dx + \delta_0 \leq \frac{\alpha}{2(\alpha-1)}(\int_0^1 v(x,t) dx)^2. \tag{5.13}$$

Since $\int_0^1 v(x,0) dx = (1-\alpha)\int_0^1 u_0 dx < 0$, we conclude by (5.13) that $\int_0^1 v dx \leq -\delta_1 < 0$ for all $t > 0$ and for some constant $\delta_1 > 0$. We have thus formulated a solution for (2.1)-(2.3) via the transformation given in (5.8). Note, however, that now we have $\delta_1^2 \leq (\int_0^1 v dx)^2 \leq \int_0^1 v^2 dx$ and thus,

$$(\frac{1}{2} + d_1 t)\delta_1 + \delta_0 \leq \frac{\alpha}{2(1-\alpha)}(\int_0^1 v dx)^2 \tag{5.14}$$

We conclude that $\int_0^1 v dx \to -\infty$ and consequently, that (2.1)-(2.3) has a solution with $\int_0^1 u dx \to \infty$ as $t \to \infty$, while the associated local problem (5.1)-(5.3) has a solution which tends to zero as $t \to \infty$. This indicates that the asymptotic behavior can be very different from the results obtained for $\alpha < 1$ in Section 4. We therefore have proved the following result:

Theorem 5.1: Assume that $u_0 > 0$ and $\alpha > 1$ satisfy (5.4) and $f = g_1 = g_2 = 0$. Then problem (2.1)-(2.3) has a solution u such that $u > 0$ for all $t > 0$ and $\int_0^1 u dx \to \infty$ as $t \to \infty$.

Proof: The proof of Theorem 5.1 will be complete once we show:

Lemma 5.1: There exists $\eta_0 > 0$ such that

$$\int_0^1 v_x^2 dx \geq \eta_0 \int_0^1 v^2 dx, \quad v \in H, \tag{5.15}$$

where the space H is defined by

$$H = \left\{ f \in H^1(0,1) | f(0) = f(1) = \beta \int_0^1 f(x) dx, \quad \beta \neq 1 \right\}.$$

Proof: Assume that (5.15) is not true, then there exists a sequence $\{f_n\} \subset H$ such that

$$\int_0^1 f_n^2 = 1, \qquad \int_0^1 (f_n)_x^2 dx \to 0, \ n \to \infty$$

which implies that there exists $M > 0$ such that

$$\int_0^1 f_n^2 + (f_n)_x^2 dx \leq M, \quad n = 1, 2, \cdots.$$

and $f \in H$ such that

$$f_n \to f, \quad \text{weakly in} H^1(0,1), \quad \text{strongly in} \quad L^2(0,1), \quad \text{a.e. in } (0,1).$$

Since $\int_0^1 f_n^2 dx = 1$, we have $\int_0^1 f^2 dx = 1$. But,

$$f_n(x) = f_n(0) + \int_0^x (f_n)_x dx = \beta \int_0^1 f_n dx + \int_0^x (f_n)_x dx,$$

and we have by taking the limit that $f(x) = \beta \int_0^1 f dx$ for all $x \in [0,1]$, which indicates that f is a constant and $(1 - \beta) f = 0$, and thus, $f = 0$ because $\beta \neq 1$. This contradicts $\int_0^1 f^2 dx = 1$. Q.E.D.

Finally, we observe that it is possible to give examples of problems with no solution at all. Consider the following problem with the second boundary data and $\alpha > 1$.

$$u_t - u_{xx} = \alpha \frac{d}{dt} \max\{\int_0^1 u dx,\ 0\} + f \ \text{ in } \ Q_T, \tag{5.16}$$

$$u_x(0,t) = g_1(t), \quad u_x(1,t) = g_2(t), \quad 0 < t < T, \tag{5.17}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{5.18}$$

Theorem 5.2: Let $S(t)$ be defined by

$$S(t) = \int_0^t (g_2 - g_1) ds + \int_0^t \int_0^1 f dx ds + \int_0^1 u_0 dx - \alpha \max\{\int_0^1 u_0 dx, 0\}. \tag{5.19}$$

(i) If $S(0) = 0$ or $\int_0^1 u_0 dx = 0$, and $S(t) > 0$ for $t > 0$, then problem (5.16)-(5.18) has no solution; (ii) If $S(t) > 0$ for $t \geq t_0 > 0$, then problem (5.16)-(5.18) has no global solution.

Proof: Assume that a solution u exists, then v by (2.4) must satisfy

$$v_t - v_{xx} = f \ \text{ in } \ Q_T, \tag{5.20}$$

$$v_x(0,t) = g_1(t), \quad v_x(1,t) = g_2(t), \quad 0 < t < T, \tag{5.21}$$

$$v(x,0) = u_0(x) - \alpha \max\{\int_0^1 u_0 dx, 0\}, \quad 0 < x < 1. \tag{5.22}$$

An integration of (5.20) on $(0,1) \times (0,t)$, then shows that for all $t \geq 0$,

$$\begin{aligned} S(t) = \int_0^1 v dx &= \int_0^1 u dx - \alpha \max\{\int_0^1 u dx,\ 0\} \\ &= \begin{cases} (1-\alpha) \int_0^1 u dx, & if \quad \int_0^1 u dx > 0, \\ \int_0^1 u dx, & if \quad \int_0^1 u dx \leq 0, \end{cases} \end{aligned}$$

is less or equal to zero since $\alpha > 1$. This contradiction proves the results. Q.E.D.

Theorem 5.3: Assume that the data in (5.19) is such that $S(t) \leq 0$. Then

(i) if $\int_0^1 u_0 dx > 0$, then $u = v + \frac{\alpha}{1-\alpha} \int_0^1 v dx$ is a solution for (5.16)-(5.18), where v is the solution of (5.20)-(5.22),

(ii) if $\int_0^1 u_0 dx < 0$, then $u = v$ is a solution for (5.16)-(5.18), where v is the solution of (5.20)-(5.22).

(iii) if $\int_0^1 u_0 dx = 0$, then both $u = v + \frac{\alpha}{1-\alpha}\int_0^1 v dx$ and $u = v$ are solutions for (5.16)-(5.18).

Proof: (i) If $\int_0^1 u_0 dx > 0$, then the solution v of (5.20)-(5.22) satisfies $\int_0^1 v(x,0)dx = (1-\alpha)\int_0^1 u_0 dx < 0$, so that $u = v + \frac{\alpha}{1-\alpha}\int_0^1 v dx$ is the inverse of (2.4) by Lemma 2.2, and satisfies (2.16)-(2.18) with $\int_0^1 u dx \geq 0$. (ii) If $\int_0^1 u_0 dx < 0$, then Lemma 2.2 and initial data matching implies that $u = v$ is a solution for (2.16)-(2.18). (iii) if $\int_0^1 u_0 dx = 0$, then Lemma 2.2 indicates that both $u = v$ and $u = v + \frac{\alpha}{1-\alpha}\int_0^1 v dx$ are solutions for (5.16)-(5.18). The former is such that $\int_0^1 u dx \leq 0$, while for the latter we conclude that $\int_0^1 u dx \geq 0$. Q.E.D.

Remark: (a) The solution in cases (i) and (ii) of Theorem 5.3 is unique as long as $S(t) < 0$; (b) If $\int_0^1 u_0 dx = 0$, then the two solutions in (iii) of Theorem 5.3 are the only two solutions if $S(t) < 0$ for $t > 0$; (c) If $S(t) \leq 0$, but the zeros of $S(t)$ are finite, say, $0 \leq t_0 < t_1 < \cdots < t_n$, then there are only 2^{n+1} solutions if $\int_0^1 u_0 dx = 0$ and $2^{n+1} - 1$ solutions if $\int_0^1 u_0 dx \neq 0$ since we can choose any one of the solutions in each sub-interval (t_i, t_{i+1}) as a consequence of the identity: $\int_0^1 v(x,t_i)dx = \int_0^1 u(x,t_i)dx = 0$ for any i. Notice that at t_i we must have $\int_0^1 v(x,t_i)dx = (\int_0^1 v(x_i,t_i)dx)_t = 0$ since $S(t) \leq 0$, and $S(t_i) = 0$ implies that t_i are maximum points; (d) if $S(t)$ has infinitely many discrete zeros, then problem (5.16)-(5.18) has infinitely many solutions.

Clearly existence and uniqueness for problem (2.1)-(2.3) with $\alpha > 1$ are more comlex questions than the discussions given above for problem (2.16)-(2.18). We conjecture that results similar to Theorem 5.3 hold, but these remain open questions.

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Email Addresses:

W. Allegretto: retl@retl.math.ualberta.ca

J. R. Cannon: jcannon@pegasus.cc.ucf.edu

Y. Lin: ylin@hilbert.math.ualberta.ca