Assignment #5 Solutions

§7.1 Problem 3a. Since $\lim_{k\to\infty} 1/k = 0$, $\lim_{k\to\infty} e^{-k} = 0$, and $\lim_{k\to\infty} 1/k^2 = 0$, we have

$$x^{(k)} = (1/k, e^{1-k}, -2/k^2)^T \to (0, e \cdot 0, -2 \cdot 0)^T = (0,0,0)^T, \quad \text{as } k \to \infty.$$

§7.1 Problem 4a. Let $A = \begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}$. Find $||A||_\infty$:

$$||A||_\infty = \max\{|10| + |15|, |0| + |1|\} = 25.$$

§7.1 Problem 4c. Find $||A||_\infty$ for

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$\begin{aligned} ||A||_\infty &= \max\{|2| + |-1| + |0|, |-1| + |2| + |-1|, |0| + |-1| + |2|\} \\ &= \max\{3, 4, 3\} = 4. \end{aligned}$$

§7.1 Problem 12. We know that $||\cdot||$ is a norm, so that it satisfies (i) $||x|| \geq 0$ with equality if and only if $x = 0$; (ii) $||\alpha x|| = |\alpha|\,||x||$, for all n-vectors x and scalars α; and (iii) $||x+y|| \leq ||x|| + ||y||$ for any two n-vectors x and y. We must show that the norm defined by $||x||' := ||Sx||$ for a nonsingular matrix S also satifies the norm properties.

The matrix S is nonsingular means in particular that $Sx = 0$ if and only if $x = 0$ (the zero solution, $x = 0$, is the unique solution). Therefore, by (i) for $||\cdot||$, we have $||x||' = ||Sx|| \geq 0$, with equality if and only if $x = 0$.

Since $S(\alpha x) = \alpha Sx$ for scalars α and n-vectors x, by (ii) for $||\cdot||$, we have $||\alpha x||' = ||S(\alpha x)|| = ||\alpha Sx|| = |\alpha|\,||Sx|| = |\alpha|\,||x||'$.

Finally, since $S(x+y) = Sx + Sy$ for any n-vectors x and y, by (iii) for $||\cdot||$, we have

$$||x+y||' = ||S(x+y)|| = ||Sx + Sy|| \leq ||Sx|| + ||Sy|| = ||x||' + ||y||'.$$

§7.2 Problem 1a. To find the eigenvalues, consider

$$\begin{aligned} det(\lambda I - A) &= det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1). \end{aligned}$$

Thus, the eigenvalues are 1 and 3.

§**7.2 Problem 4.** To see if the matrices are convergent we check the spectral radius of each one.

$$
\begin{aligned}
det(\lambda I - A_1) &= det \begin{bmatrix} \lambda - 1 & 0 \\ -1/4 & \lambda - 1/2 \end{bmatrix} \\
&= (\lambda - 1)(\lambda - 1/2)
\end{aligned}
$$

which means that the eigenvalues of A_1 are 1 and 1/2. Thus, the spectral radius of A_1 is 1, and A_1 does not converge by Theorem 7.16.

On the other hand,

$$
\begin{aligned}
det(\lambda I - A_2) &= det \begin{bmatrix} \lambda - 1/2 & 0 \\ -16 & \lambda - 1/2 \end{bmatrix} \\
&= (\lambda - 1/2)^2
\end{aligned}
$$

which means that the eigenvalues of A_2 are 1/2 with multiplicity two. Hence, the spectral radius of A_2 is 1/2 and A_2 does converge by Theorem 7.16.

§**7.2 Problem 10a.** λ is an eigenvalue of A with eigenvector $x \neq 0$. Then, since the following properties of transpose hold $det(A) = det(A^t)$, $(A+B)^t = A^t + B^t$, and $(\lambda I)^t = \lambda I$, we have

$$
\begin{aligned}
det(\lambda I - A^t) &= det(\lambda I^t - A^t) \\
&= det((\lambda I - A)^t) = det(\lambda I - A) = 0.
\end{aligned}
$$

Hence, λ is an eigenvalue of A^t.

§**7.2 Problem 10b.** Since λ is an eigenvalue of A with eigenvector x, we have $Ax = \lambda x$. Applying A to both sides of the last equation, we have $A^2 x = A(\lambda x) = \lambda A x = \lambda \lambda x = \lambda^2 x$. Hence, λ^2 is an eigenvalue of A^2. The proof is completed by induction. We have just seen that the result is true for $k = 1$ and $k = 2$. Now, assume it is true for $k - 1$, i.e. that $A^{k-1} x = \lambda^{k-1} x$. Applying A to both sides of the last equation yields $A^k x = A(\lambda^{k-1} x) = \lambda^{k-1} A x = \lambda^{k-1} \lambda x = \lambda^k x$. Hence, it is also true for k and the proof follows by mathematical induction.

§**7.2 Problem 10c.** If A^{-1} exits, then apply A^{-1} to both sides of $Ax = \lambda x$ to get

$$
\begin{aligned}
A^{-1} A x = A^{-1}(\lambda x) = \lambda A^{-1} x &\Longrightarrow x = \lambda A^{-1} x \\
&\Longrightarrow \lambda^{-1} x = A^{-1} x.
\end{aligned}
$$

Hence, λ^{-1} is an eigenvalue of A^{-1} with eigenvector x.

§**7.3 Problem 1d.** Rewriting the system, we have

$$
\begin{aligned}
x(1) &= (6 - x(2) + 2 * x(3))/10 \\
x(2) &= (25 - x(1) + x(3) - 3 * x(4))/10 \\
x(3) &= (-11 + 2 * x(1) + x(2) + x(4))/8 \\
x(4) &= (-11 - 3 * x(2) + x(4))/5
\end{aligned}
$$

Substituting $x = [0, 0, 0, 0]'$ into these equations gives the new value of

$$x = [0.6000, 2.5000, -1.3750, -2.2000]'.$$

Repeating the process by substituting the new x into the equations gives

$$x = [0.0750, 2.9625, -1.1875, -4.1400]'.$$

§**7.3 Problem 1g.** Rewriting the system, we have

$$\begin{aligned} x(1) &= (-2 - x(2) + x(3) - x(4))/4 \\ x(2) &= (-1 - x(1) + x(3) + x(4))/4 \\ x(3) &= (0 + x(1) + x(2) - x(4))/5 \\ x(4) &= (1 - x(1) + x(2) - x(3))/3 \end{aligned}$$

Substituting $x = [0, 0, 0, 0]'$ into these equations gives the new value of

$$x = [-0.5000, -0.2500, -0.0000, 0.3333]'.$$

Repeating the process by substituting the new x into the equations gives

$$x = [-0.5208, -0.0417, -0.2167, 0.4167]'.$$

§**7.3 Problem 2d.** Referring to the equations of Problem 1d, we use the Gauss-Seidel Method. The first pass gives

$$x = [0.6000, 2.4400, -0.9200, -3.6640]',$$

while the second pass through gives

$$x = [0.1720, 3.4900, -1.3538, -5.0268]'.$$

§**7.3 Problem 2g.** Referring to the equations of Problem 1g, we use the Gauss-Seidel Method. The first pass gives $x = [-0.5000, -0.1250, -0.1250, 0.5000]'$, while the second pass through gives $x = [-0.6250, 0, -0.2250, 0.6167]'$.

§**7.3 Problem 5a.** We use the program sor73.m with the form [x,X]=sor73(A,b,x0,omega,N,TOL); Here we take $A = [3, -1, 1; 3, 6, 2; 3, 3, 7]$, $b = [1, 0, 4]'$, $x0 = [0, 0, 0]'$, $omega = 1.1$, $N = 2$ and $TOL = .001$ (just to take something since the iterations being so small means it should not come into play). Applying the algorithm, we get $x = [0.3667, -0.2017, 0.5508]'$ for the first iteration and $x = [0.0541, -0.2115, 0.6477]'$ as the second iteration.

§**7.3 Problem 6a.** We use the program sor73.m with the form [x,X]=sor73(A,b,x0,omega,N,TOL); Here we take $A = [3,-1,1;3,6,2;3,3,7]$, $b = [1,0,4]'$, $x0 = [0,0,0]'$, $omega = 1.3$, $N = 2$ and $TOL = .001$ (just to take something since the iterations being so small means it should not come into play). Applying the algorithm, we get $x = [0.4333, -0.2817, 0.6584]'$ for the first iteration and $x = [-0.1040, -0.1332, 0.6775]'$ as the second iteration.

The true solution of the example in Problems 5a and 6a is $x = [0.0351, -0.2368, 0.6579]'$.

§**7.4 Problem 1e.** We are to compute the condition number relative to the infinity norm for the matrix

$$A = [1.003, 58.09; 5.550, 321.8].$$

We first compute $||A||_\infty = \max\{1.003 + 58.09,\ 5.550 + 321.8\} = 327.350$. Next, we need to find A^{-1}, which is easy for a 2 by 2 matrix

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} 321.8 & -58.09 \\ -5.550 & 1.003 \end{bmatrix}.$$

Here, $det(A) = 1.003 * 321.8 - 58.09 * 5.550 = 0.36590$. Hence,

$$\begin{aligned} ||A^{-1}||_\infty &= \max\{(321.8 + 58.09)/det(A),\ (5.55 + 1.003)/det(A)\} \\ &= 379.89/det(A) = 1038.23449... \end{aligned}$$

Finally, we find that

$$K(A) = ||A||_\infty * ||A^{-1}||_\infty = 327.35 * 1038.23449 = 339866.06...$$

§**7.4 Problem 6a.** The Hilbert matrix $H^{(4)}$ is

$$H^{(4)} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

To show that the given matrix is the inverse of $H^{(4)}$ simply carry out the multiplication $H^{(4)}[H^{(4)}]^{-1}$. Using MATLAB, one does

```
A=[1,1/2,1/3,1/4;
1/2,1/3,1/4,1/5;
1/3,1/4,1/5,1/6;
1/4,1/5,1/6,1/7];
invA=[16,-120,240,-140;
-120,1200,-2700,1680;
240,-2700,6480,-4200;
-140,1680,-4200,2800];
A*invA;
```

to see this (although you may notice the multiplication is not perfect if you put it in the long form).

§7.4 **Problem 6c.** In three digit arithmetic, the augmented matrix for $H^{(4)}x = [1,0,0,1]'$ is

$$\begin{bmatrix} 1.00 & .500 & .333 & .250 & 1 \\ .500 & .333 & .250 & .200 & 0 \\ .333 & .250 & .200 & .167 & 0 \\ .250 & .200 & .167 & .143 & 1 \end{bmatrix}.$$

Using straight Gaussian elimination on the first column with 3-digit rounding arithmetic, we get

$$\begin{bmatrix} 1.00e+00 & 5.00e-01 & 3.33e-01 & 2.50e-01 & 1.00e+00 \\ 0 & 8.30e-02 & 8.30e-02 & 7.50e-02 & -5.00e-01 \\ 0 & 8.40e-02 & 8.90e-02 & 8.40e-02 & -3.33e-01 \\ 0 & 7.50e-02 & 8.37e-02 & 8.05e-02 & 7.50e-01 \end{bmatrix}$$

Moving to the next column

$$\begin{bmatrix} 1.e+00 & 5.e-01 & 3.33e-01 & 2.50e-01 & 1.00e+00 \\ 0 & 8.30e-02 & 8.30e-02 & 7.50e-02 & -5.00e-01 \\ 0 & 0 & 5.00e-03 & 8.10e-03 & 1.73e-01 \\ 0 & 0 & 8.70e-03 & 1.27e-02 & 1.20e+00 \end{bmatrix}$$

Doing the third column

$$\begin{bmatrix} 1.e+00 & 5.e-01 & 3.33e-01 & 2.50e-01 & 1.00e+00 \\ 0 & 8.30e-02 & 8.30e-02 & 7.50e-02 & -5.00e-01 \\ 0 & 0 & 5.00e-03 & 8.10e-03 & 1.73e-01 \\ 0 & 0 & 0 & -1.40e-03 & 8.99e-01 \end{bmatrix}$$

Backward substitution gives

$$\begin{aligned} x4 &= (8.99e-01)/(-1.40e-03) = -6.42e+02 \\ x3 &= ((1.73e-01) - x4 * (8.10e-03))/(5.00e-03) \\ &= ((1.73e-01) - 5.20)/(5.00e-03) = -1.01e+03 \\ x2 &= \frac{(-5.00e-01) - x4 * (7.50e-02) - x3 * (8.30e-02)}{(8.30e-02)} \\ &= \frac{(-5.00e-01) + (4.82e+01) + (8.38e+01)}{(8.30e-02)} = 1.58e+03 \\ x1 &= 1.00 - x4 * (2.50e-01) - x3 * (3.33e-01) - x2 * (5.e-01) \\ &= 1 + (1.61e+02) + (3.36e+02) - 790 = 292 \end{aligned}$$

The actual solution is $x = [-124, 1560, -3960, 2660]'$. In (7.20), $\delta b = 0$ since b is represented exactly in 3-digit arithmetic. However,

$$\delta A = 1.0e-03*$$

$$\begin{bmatrix} 0 & 0 & -0.333333333 & 0 \\ 0 & -0.333333333 & 0 & 0 \\ -0.333333333 & 0 & 0 & 0.333333333 \\ 0 & 0 & 0.333333333 & 0.142857142 \end{bmatrix}$$

which has $||\delta A||_\infty = (1.0e-03) * 0.66666666$. Therefore,

$$||\delta A||_\infty / ||A||_\infty = 2.0365561834e-06,$$

so that the bound on the right in (7.20) is

$$\begin{aligned}
&\frac{K(A)}{1-K(A)(||\delta A||/||A||)}\left(\frac{||\delta b||}{||b||}+\frac{||\delta A||}{||A||}\right) \\
&= \frac{339866.06}{1-339866.06 * 2.0365561834e-06}(0+2.0365561834e-06) \\
&= 2.248401980
\end{aligned}$$

The relative error is

$$\frac{||x-\tilde{x}||_\infty}{||x||_\infty} = \frac{3302}{3960} = 0.83383838383838$$