Assignment #11 Solutions

§11.1 Problem 4c. The first thing to do is to write a file for the system of equations to be solved by the Runge-Kutta method. For this problem the system is:

$$
\begin{aligned}
u1' &= u2, \qquad u1(1) = 1/2;\\
u2' &= -\frac{4}{x}u2 - \frac{2}{x^2}u1 + \frac{2}{x^2}log(x), \qquad u2(1) = 0;\\
u3' &= u4, \qquad u4(1) = 0;\\
u4' &= -\frac{4}{x}u2 - \frac{2}{x^2}u1, \qquad u4(1) = 1;
\end{aligned}
$$

Therefore, an m-file for use in the implementation shoot111.m for linear shooting method is

```
function V=fn11p4c(x,U);
%% system for problem 4c of Section 11.1
V=[U(2),-(4./x).*U(2)-(2./x.^2).*U(1)+(2./x.^2).*log(x), ...
U(4),-(4./x).*U(4)-(2./x.^2).*U(3)];
```

The actual solution to the boundary value problem is given in the m-file

```
function y=sol11p4c(x);
%% actual solution for problem 4c of Section 11.1
y=(4./x)-(2./x.^2)+log(x)-3/2;
```

With these m-files in hand we are ready to implement the algorithm:

```
I=[1,2];bv=[1/2,log(2)]; h=0.05; N=(2-1)/h;
yprime='fn11p4c';
[x,W1,W2]=shoot111(yprime,I,bv,N);
y=sol11p4c(x);
[x,W1,y,abs(y-W1)]
```

x	W1	y	Error
1.00	5.0000e-01	5.0000e-01	0
1.05	5.4425e-01	5.4426e-01	7.0095e-07
1.10	5.7878e-01	5.7878e-01	1.0763e-06
1.15	6.0573e-01	6.0574e-01	1.2508e-06
1.20	6.2676e-01	6.2677e-01	1.3019e-06
1.25	6.4314e-01	6.4314e-01	1.2781e-06
1.30	6.5585e-01	6.5586e-01	1.2101e-06
1.35	6.6567e-01	6.6567e-01	1.1174e-06
1.40	6.7321e-01	6.7321e-01	1.0123e-06
1.45	6.7893e-01	6.7894e-01	9.0283e-07
1.50	6.8324e-01	6.8324e-01	7.9370e-07
1.55	6.8643e-01	6.8643e-01	6.8790e-07
1.60	6.8875e-01	6.8875e-01	5.8717e-07

1.65	6.9040e-01	6.9040e-01	4.9240e-07
1.70	6.9153e-01	6.9153e-01	4.0400e-07
1.75	6.9227e-01	6.9227e-01	3.2201e-07
1.80	6.9272e-01	6.9272e-01	2.4630e-07
1.85	6.9298e-01	6.9298e-01	1.7659e-07
1.90	6.9310e-01	6.9310e-01	1.1256e-07
1.95	6.9314e-01	6.9314e-01	5.3818e-08
2.00	6.9315e-01	6.9315e-01	1.1102e-16

§**11.1 Problem 5.** The first thing to do is to write a m-file for the system of equations to be solved by the Runge-Kutta method. For this problem the system is:

$$\begin{aligned} u1' &= u2, \qquad u1(0) = 1; \\ u2' &= 100u1, \qquad u2(0) = 0; \\ u3' &= u4, \qquad u3(0) = 0; \\ u4' &= 100u3, \qquad u4(0) = 1; \end{aligned}$$

An m-file for use in the implementation shoot111.m for linear shooting method is

```
function V=fn11p5(x,U);
%% system for problem 5 of Section 11.1
V=[U(2),100*U(1), U(4),100*U(3)];
```

We are to apply the shooting method with step size $h = 0.1$ and $h = 0.05$ which corresponds to $N = 10$ and $N = 20$ respectively (since the length of the interval is one).

```
I=[0,1];bv=[1,exp(-10)]; N=10;
yprime='fn11p5';
[x,W1,W2]=shoot111(yprime,I,bv,N);
y=exp(-10*x);
[x,W1,y,abs(y-W1)]
```

x	W1	y	Error
0	1.0e+00	1.0000e+00	0
1.0e-01	3.7500e-01	3.6788e-01	7.1206e-03
2.0e-01	1.4062e-01	1.3534e-01	5.2897e-03
3.0e-01	5.2734e-02	4.9787e-02	2.9473e-03
4.0e-01	1.9775e-02	1.8316e-02	1.4597e-03
5.0e-01	7.4157e-03	6.7379e-03	6.7776e-04
6.0e-01	2.7807e-03	2.4788e-03	3.0198e-04
7.0e-01	1.0424e-03	9.1188e-04	1.3048e-04
8.0e-01	3.8976e-04	3.3546e-04	5.4296e-05
9.0e-01	1.4311e-04	1.2341e-04	1.9698e-05
1.0e+00	4.5400e-05	4.5400e-05	6.1306e-13

```
I=[0,1];bv=[1,exp(-10)]; N=20;
yprime='fn11p5';
[x,W1,W2]=shoot111(yprime,I,bv,N);
```

```
y=exp(-10*x);
[x,W1,y,abs(y-W1)]
```

x	W1	y	Error
0	1.0000e+00	1.0000e+00	0
5.0e-02	6.0677e-01	6.0653e-01	2.4017e-04
1.0e-01	3.6817e-01	3.6788e-01	2.9140e-04
1.5e-01	2.2340e-01	2.2313e-01	2.6517e-04
2.0e-01	1.3555e-01	1.3534e-01	2.1449e-04
2.5e-01	8.2248e-02	8.2085e-02	1.6265e-04
3.0e-01	4.9905e-02	4.9787e-02	1.1840e-04
3.5e-01	3.0281e-02	3.0197e-02	8.3802e-05
4.0e-01	1.8374e-02	1.8316e-02	5.8100e-05
4.5e-01	1.1149e-02	1.1109e-02	3.9652e-05
5.0e-01	6.7647e-03	6.7379e-03	2.6726e-05
5.5e-01	4.1046e-03	4.0868e-03	1.7832e-05
6.0e-01	2.4905e-03	2.4788e-03	1.1797e-05
6.5e-01	1.5112e-03	1.5034e-03	7.7468e-06
7.0e-01	9.1693e-04	9.1188e-04	5.0503e-06
7.5e-01	5.5635e-04	5.5308e-04	3.2646e-06
8.0e-01	3.3755e-04	3.3546e-04	2.0828e-06
8.5e-01	2.0476e-04	2.0347e-04	1.2935e-06
9.0e-01	1.2416e-04	1.2341e-04	7.4977e-07
9.5e-01	7.5198e-05	7.4852e-05	3.4623e-07
1.0e+00	4.5400e-05	4.5400e-05	6.1306e-13

§**11.2 Problem 3a.** The function f in the problem is

$$f(x, y, y') = y^3 - yy'.$$

We first need to set up the function files to be called by the shooting algorithm; one file for each of the function f, f112p3a.m, its partial derivative with respect to y, f112p3a2.m, and its partial derivative with respect to y', f112p3a3.m:

```
function z=f112p3a(x,y,yprime);
%% function f(x,y,y') for problem 3a of section 11.2
z=y.^3-y.*yprime;

function z=f112p3a2(x,y,yprime);
%% function f_y(x,y,y') for problem 3a of section 11.2
z=3*y.^2-yprime;

function z=f112p3a3(x,y,yprime);
%% function f_yprime(x,y,y') for problem 3a of section 11.2
z=-y;
```

The system of initial values problems that is required for the algorithm is (see `help shoot112.m`):

$$w1' = w2, \qquad w1(1) = 1/2;$$
$$w2' = w1^3 - w1w2; \qquad w2(1) = t;$$
$$u1' = u2; \qquad u1(1) = 0;$$
$$u2' = u1(3u1^2 - u2) + u2(-u1), \qquad u2(1) = 1;$$

An m-file to solve this system via the Runge-Kutta method is

```
function Z=rk112p3a(x,V);
%% system file for problem 3a of section 11.2
Z=[V(2),V(1).^3-V(1).*V(2),...
V(4), V(3).*(3*V(3).^2-V(4))-V(4).*V(3)];
  or more compactly as
function Z=rk112p3a(x,V);
%% system file for problem 3a of section 11.2
Z=[V(2),f112p3a(x,[V(1),V(2)]),V(4),...
V(3).*f112p3a2(x,[V(3),V(4)])+V(4).*f112p3a3(x,[V(3),V(4)])];
```

The actual solution of the boundary value problem, $y = (x+1)^{-1}$, is given in the mfile s112p3a.m:

```
function y=s112p3a(x);
%% actual solution of problem 3a of section 11.2
y=1./(x+1);
```

We are now in a position to run the algorithm and to compare the results with the actual solution:

```
yprime='rk112p3a';I=[1,2];bv=[1/2,1/3];
N=(2-1)/0.1;Nmax=50;TOL=10e-5;
[x,W1,W2]=shoot112(yprime,I,bv,N,Nmax,TOL);
y=s112p3a(x);
format short e
[x,W1,y,abs(y-W1)]
```

1.0	5.0000e-01	5.0000e-01	0
1.1	4.7619e-01	4.7619e-01	1.9982e-06
1.2	4.5454e-01	4.5455e-01	3.9324e-06
1.3	4.3478e-01	4.3478e-01	5.8204e-06
1.4	4.1666e-01	4.1667e-01	7.6770e-06
1.5	3.9999e-01	4.0000e-01	9.5143e-06
1.6	3.8460e-01	3.8462e-01	1.1343e-05
1.7	3.7036e-01	3.7037e-01	1.3170e-05
1.8	3.5713e-01	3.5714e-01	1.5004e-05
1.9	3.4481e-01	3.4483e-01	1.6850e-05
2.0	3.3331e-01	3.3333e-01	1.8714e-05

§**11.2 Problem 4a.** The change in the algorithm to use the secant method is given in the m-file shoo112s.m. To apply this to the problem 3a, we need to modify the m-file used for the system:

```
function Z=rk112p4a(x,V);
%% system file for problem 4/5 section 11.2
Z=[V(2),V(1).^3-V(1).*V(2)];
```

(Note, this just drops the last two components of the vectors in the m-file rk112p3a.m.)

Now we apply the algorithm:

```
yprime='rk112p4a';I=[1,2];bv=[1/2,1/3];
N=(2-1)/0.1;Nmax=50;TOL=10e-5;
[x,W1,W2]=shoo112s(yprime,I,bv,N,Nmax,TOL);
y=s112p3a(x);
format short e
[x,W1,y,abs(y-W1)]
```

1.0	5.0000e-01	5.0000e-01	0
1.1	4.7619e-01	4.7619e-01	3.6560e-06
1.2	4.5454e-01	4.5455e-01	7.1853e-06
1.3	4.3477e-01	4.3478e-01	1.0624e-05
1.4	4.1665e-01	4.1667e-01	1.4000e-05
1.5	3.9998e-01	4.0000e-01	1.7338e-05
1.6	3.8459e-01	3.8462e-01	2.0657e-05
1.7	3.7035e-01	3.7037e-01	2.3973e-05
1.8	3.5712e-01	3.5714e-01	2.7300e-05
1.9	3.4480e-01	3.4483e-01	3.0648e-05
2.0	3.3330e-01	3.3333e-01	3.4027e-05

§**11.3 Problem 3c.** The linear finite difference algorithm as implemented in `lfd113.m` requires that a function file be set up to give the coefficients $[p, q, r]$ from the linear equation

$$y'' = p(x)y' + q(x)y + r(x)$$

for the boundary value problem. In this case, the m-file may look like

```
function [p,q,r]=f113p3c(x);
%% function file for problem 3c of section 11.3
p=-4./x;q=2./x.^2;r=(-2./x.^2).*log(x);
```

The algorithm is then implemented via

```
I=[1,2];bv=[-1/2,log(2)];h=0.05;N=1/h-1;
Funs='f113p3c';
[X,W]=lfd113(Funs,I,bv,N);
[X,W]
```

X	W
1.0000	-0.5000

1.0500	-0.3111
1.1000	-0.1566
1.1500	-0.0288
1.2000	0.0780
1.2500	0.1680
1.3000	0.2445
1.3500	0.3100
1.4000	0.3665
1.4500	0.4156
1.5000	0.4584
1.5500	0.4960
1.6000	0.5291
1.6500	0.5585
1.7000	0.5846
1.7500	0.6079
1.8000	0.6287
1.8500	0.6474
1.9000	0.6643
1.9500	0.6794
2.0000	0.6931

§**11.3 Problem 4a.** The linear finite difference algorithm as implemented in `lfd113.m` requires that a function file be set up to give the coefficients $[p, q, r]$ from the linear equation

$$y'' = p(x)y' + q(x)y + r(x)$$

for the boundary value problem. In this case, the m-file may look like

```
function [p,q,r]=f113p4a(x);
%% function for problem 4a of section 11.3
p=0;q=-1;r=0;
```

The function file forthe actual solution is

```
function y=s113p4a(x);
%% actual solution for problem 4a of section 11.3
y=cos(x)+(sqrt(2)-1)*sin(x);
```

We are now in a position to run the algorithm and to compare the results with the actual solution:

```
I=[0,pi/4];h=pi/20;N=pi/(4*h)-1;
bv=[1,1];Funs='f113p4a';
[X,W]=lfd113(Funs,I,bv,N);
Y=s113p4a(X);
format short e
[X,W,Y,abs(Y-W)]
```

0	1.0000e+00	1.0000e+00	0

1.5708e-01	1.0526e+00	1.0525e+00	1.1519e-04
3.1416e-01	1.0792e+00	1.0791e+00	1.7419e-04
4.7124e-01	1.0792e+00	1.0791e+00	1.7419e-04
6.2832e-01	1.0526e+00	1.0525e+00	1.1519e-04
7.8540e-01	1.0000e+00	1.0000e+00	0

§**11.4 Problem 3a.** This is the same problem as in Section 11.3, #3a so we can use the m-files made in that case when appropriate. We need to make a m-file to use in `nlfd114.m`:

```
function [f,fy,fyprime]=f114p3a(x,y,yprime);
%% function for problem 3a of section 11.4
f=f112p3a(x,y,yprime);fy=f112p3a2(x,y,yprime);
fyprime=f112p3a3(x,y,yprime);
```

We are now in a position to run the algorithm and to compare the results with the actual solution:

```
Funs='f114p3a';I=[1,2];bv=[1/2,1/3];
N=(2-1)/0.1;Nmax=50;TOL=10e-5;
[X,W]=nlfd114(Funs,I,bv,N,Nmax,TOL);
Y=s112p3a(X);
format short e
[X,W,Y,abs(Y-W)]
```

0	1.0000e+00	1.0000e+00	0
1.5708e-01	1.0526e+00	8.6424e-01	1.8836e-01
3.1416e-01	1.0792e+00	7.6094e-01	3.1829e-01
4.7124e-01	1.0792e+00	6.7970e-01	3.9953e-01
6.2832e-01	1.0526e+00	6.1413e-01	4.3847e-01
7.8540e-01	1.0000e+00	5.6010e-01	4.3990e-01

§**11.4 Problem 3f.** The function file needed in the nonlinear finite difference method for this problem might look like:

```
function [f,fy,fyprime]=f114p3f(x,y,yprime);
%% function for problem 3f in section 11.4
f=((x.^2).*yprime.^2-9*y.^2+4*x.^4)./x.^5;
fy=(-18*y)/x.^5;
fyprime=(2*x.^2).*yprime./x.^5;
```

A function file for the actual solution is

```
function y=s114p3f(x);
%% actual solution for problem 3f in section 11.4
y=(x.^3).*log(x);
```

We are now in a position to run the algorithm and to compare the results with the actual solution:

```
Funs='f114p3f';I=[1,2];bv=[0,log(256)];
N=(2-1)/0.05-1;Nmax=50;TOL=10e-5;
```

```
[X,W]=nlfd114(Funs,I,bv,N,Nmax,TOL);
Y=s114p3f(X);
format short e
[X,W,Y,abs(Y-W)]
```

1.00e+00	0	0	0
1.05e+00	8.9843e-02	5.6481e-02	3.3362e-02
1.10e+00	1.9749e-01	1.2686e-01	7.0632e-02
1.15e+00	3.2398e-01	2.1256e-01	1.1142e-01
1.20e+00	4.7022e-01	3.1505e-01	1.5517e-01
1.25e+00	6.3696e-01	4.3583e-01	2.0113e-01
1.30e+00	8.2481e-01	5.7641e-01	2.4839e-01
1.35e+00	1.0342e+00	7.3837e-01	2.9581e-01
1.40e+00	1.2653e+00	9.2328e-01	3.4203e-01
1.45e+00	1.5182e+00	1.1328e+00	3.8547e-01
1.50e+00	1.7928e+00	1.3684e+00	4.2432e-01
1.55e+00	2.0885e+00	1.6320e+00	4.5654e-01
1.60e+00	2.4050e+00	1.9251e+00	4.7983e-01
1.65e+00	2.7412e+00	2.2495e+00	4.9167e-01
1.70e+00	3.0963e+00	2.6070e+00	4.8934e-01
1.75e+00	3.4691e+00	2.9992e+00	4.6986e-01
1.80e+00	3.8581e+00	3.4280e+00	4.3011e-01
1.85e+00	4.2619e+00	3.8951e+00	3.6676e-01
1.90e+00	4.6788e+00	4.4025e+00	2.7636e-01
1.95e+00	5.1072e+00	4.9519e+00	1.5532e-01
2.00e+00	5.5452e+00	5.5452e+00	0

§**11.4 Problem 4a.** We only need to apply extrapolation to the problem given in #3a of Section 11.4. This is implemented through the algorithm `nlfd114r`

```
Funs='f114p3a';I=[1,2];bv=[1/2,1/3];
N=(2-1)/0.1-1;Nmax=50;TOL=10e-5;
[X,Y1,W2,W3,W4]=nlfd114r(Funs,I,bv,N,Nmax,TOL);
Y=s112p3a(X);
format short e
[X,W4,Y,abs(Y-W4)]
```

1.0e+00	5.0000e-01	5.0000e-01	0
1.1e+00	4.7619e-01	4.7619e-01	3.5780e-12
1.2e+00	4.5455e-01	4.5455e-01	8.1932e-12
1.3e+00	4.3478e-01	4.3478e-01	1.9403e-11
1.4e+00	4.1667e-01	4.1667e-01	4.3062e-11
1.5e+00	4.0000e-01	4.0000e-01	7.8234e-11
1.6e+00	3.8462e-01	3.8462e-01	1.1329e-10
1.7e+00	3.7037e-01	3.7037e-01	1.3028e-10
1.8e+00	3.5714e-01	3.5714e-01	1.1521e-10

1.9e+00	3.4483e-01	3.4483e-01	6.7161e-11
2.0e+00	3.3333e-01	3.3333e-01	0