Assignment #1 Solutions

§5.1 Problem 1a. First, we are required to show that the initial value problem $y' = y\cos(t)$, $0 \le t \le 1$, $y(0) = 1$, has a unique solution by applying Theorem 5.4. First observe that the function $f(t, y) = y\cos(t)$ is continuous since the trigonometric and polynomial functions are continuous. We therefore have to show that the function f satisfies a Lipschitz condition on the region $D := \{(t, y) : 0 \le t \le 1, \quad -\infty < y < \infty\}$ in the variable y:

$$
\begin{aligned}
|f(t, y_1) - f(t, y_2)| &= |y_1\cos(t) - y_2\cos(t)| \\
&|\cos(t)(y_1 - y_2)| = |\cos(t)||y_1 - y_2| \\
&\le 1|y_1 - y_2|
\end{aligned}
$$

Thus, it satisfies a Lipschitz condition in the variable y with constant $L = 1$.

Next, we find the actual solution: The equation $dy/dt = y\cos(t)$ is separable, so we carry out the separation of variables and then integrate both sides:

$$
\begin{aligned}
&dy/dt = y\cos(t) \Longrightarrow \frac{dy}{y} = \cos(t)dt \\
&\Longrightarrow \int \frac{dy}{y} = \int \cos(t)dt \Longrightarrow \ln y = \sin(t) + C \\
&\Longrightarrow y = e^{(\sin(t)+C)} = e^C e^{\sin(t)} = ke^{\sin(t)}
\end{aligned}
$$

The initial condition $y(0) = 1$ implies $1 = ke^{\sin(0)} = ke^0 = k$; hence the solution is $y = e^{\sin(t)}$.

§5.1 Problem 2a. The function is $f(t, y) = t^2y + 1$. Does it satisfy a Lipschitz condition on $D := \{(t, y) : 0 \le t \le 1, \quad -\infty < y < \infty\}$ in the variable y)? Yes:

$$
\begin{aligned}
|f(t, y_1) - f(t, y_2)| &= |t^2y_1 + 1 - t^2y_2 - 1| \\
&= |t^2(y_1 - y_2)| = |t^2||y_1 - y_2| \\
&\le 1|y_1 - y_2|
\end{aligned}
$$

since $|t^2| \le 1$ when $0 \le t \le 1$.

Theorem 5.6 can be used to show that the initial value problem is well-posed because f is also continuous on D (as well as satisfying a Lipschitz condition in the variable y). The Lipschitz constant is $L = 1$.

§5.1 Problem 2b. The function is $f(t, y) = ty$. Does it satisfy a Lipschitz condition on $D := \{(t, y) : 0 \le t \le 1, \quad -\infty < y < \infty\}$ in the variable y? Yes:

$$|f(t, y_1) - f(t, y_2)| = |ty_1 - ty_2| \le 1|y_1 - y_2|.$$

Again Theorem 5.6 can be used to show that the initial value problem is well-posed because f is also continuous on D (as well as satisfying a Lipschitz condition in the variable y. The

Lipschitz constant is $L = 1$. §**5.1 Problem 6a.** Suppose $\delta(t) = \delta t$ for some constant δ. The original problem $y' = 1 - y$, $0 \le t \le 2$, $y(0) = 0$ has the solution:

$$\begin{aligned}
&dy/dt = 1 - y \implies \frac{dy}{1-y} = dt \implies \int \frac{dy}{1-y} = \int dt \\
&\implies -\ln(1-y) = t + C \\
&\implies 1 - y = e^{-t-C} \implies y = 1 - e^{-C}e^{-t} \implies y = 1 - ke^{-t}
\end{aligned}$$

and the initial value $y(0) = 0$ implies $0 = 1 - ke^0 = 1 - k$, or $k = 1$. Hence, $y = 1 - e^{-t}$ is the solution.

The perturbed problem $z' = 1 - z + \delta t$ has the solution:

$$\begin{aligned}
&dz/dt = 1 - z + \delta t \implies z' + z = 1 + \delta t \\
&\implies e^t z' + e^t z = e^t(1 + \delta t) \implies (e^t z)' = e^t + \delta t e^t \\
&\implies e^t z = \int (e^t + \delta t e^t) dt + C \\
&\implies e^t z = e^t + \delta(te^t - e^t) + C \implies z = 1 + \delta(t-1) + Ce^{-t}
\end{aligned}$$

and the initial condition $z(0) = 0 + \epsilon_0$ implies that $0 + \epsilon_0 = 1 - \delta + C$ or $C = \epsilon_0 + \delta - 1 =: \kappa - 1$. Hence, $z = 1 + \delta(t-1) + (\kappa - 1)e^{-t}$.

Given $\epsilon > 0$, in order to have $|\delta(t)| = |\delta t| < \epsilon$ on $0 \le t \le 2$, we must have $|\delta| < \epsilon/2$. If $|\epsilon_0| < \epsilon$, then $|\kappa| = |\epsilon_0 + \delta| < \epsilon + \epsilon/2 = 3\epsilon/2$. Then

$$\begin{aligned}
|y - z| &= |1 - e^{-t} - (1 + \delta(t-1) + (\kappa - 1)e^{-t})| \\
&= |\delta(t-1) - \kappa e^{-t}| \\
&= |\delta||t-1| + |\kappa|e^{-t} \\
&= \epsilon/2 \cdot 1 + \frac{3}{2}\epsilon \cdot 1 \\
&= 2 \cdot \epsilon
\end{aligned}$$

§**5.2 Problem 1c.** We are given $f(t, y) = 1 + y/t$ on the interval $1 \le t \le 2$ with $y(1) = 2$ and $h = 0.25$. So

t	w
1	2
1.25	$2 + (0.25) * (1 + 2/1) = 2.75$
1.50	$2.75 + (0.25) * (1 + 2.75/1.25) = 3.55$
1.75	$3.55 + (0.25) * (1 + 3.55/1.50) = 4.39167$
2.00	$4.39167 + (0.25) * (1 + 4.39167/1.75) = 5.26905$

§**5.2 Problem 2c.** The actual solution to the above problem is $y(t) = t\ln t + 2t$. To compare the actual error to the error bound, we need to find L, the Lipschitz constant of $f(t, y)$ in the variable y, and M, the maximum of y''. First,

$$|f(t, y_1) - f(t, y_2)| = |1 + y_1/t - 1 - y_2/t| = |y_1 - y_2|/|t| \le |y_1 - y_2|,$$

since $1/|t| \leq 1$ has maximum equal 1 on $1 \leq t \leq 2$. Hence, $L = 1$. Next, $y'(t) = \ln t + 1 + 2 = \ln t + 3$, and $y''(t) = 1/t$. Hence, $M = \max_{1 \leq t \leq 2} |y''(t)| = \max_{1 \leq t \leq 2} 1/t = 1$. Therefore, the error bound is

$$\begin{aligned} |y(t_i) - w_i| &\leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] = \frac{0.25 \cdot 1}{2 \cdot 1}\left[e^{i \cdot h} - 1\right] \\ &= 0.125\left[e^{0.25 i} - 1\right]. \end{aligned}$$

Hence,

t_i	Actual Error	Error Bound
1.0000	0	0
1.2500	0.0289	0.0355
1.5000	0.0582	0.0811
1.7500	0.0877	0.1396
2.0000	0.1172	0.2148

§5.2 Problem 6a. Here $y' = f(t, y) = \frac{1}{t^2} - \frac{y}{t} - y^2$ on $1 \leq t \leq 2$, with $y(1) = 1$ and $h = 0.05$. The actual solution is $y = -1/t$. Apply Euler's method and compare it.

t_i	w_i	$y(t_i)$	$\lvert w_i - y(t_i)\rvert$
1.0000	−1.0000	−1.0000	0
1.0500	−0.9500	−0.9524	0.0024
1.1000	−0.9045	−0.9091	0.0046
1.1500	−0.8630	−0.8696	0.0066
1.2000	−0.8249	−0.8333	0.0084
1.2500	−0.7898	−0.8000	0.0102
1.3000	−0.7574	−0.7692	0.0118
1.3500	−0.7274	−0.7407	0.0133
1.4000	−0.6995	−0.7143	0.0148
1.4500	−0.6735	−0.6897	0.0162
1.5000	−0.6491	−0.6667	0.0175
1.5500	−0.6264	−0.6452	0.0188
1.6000	−0.6049	−0.6250	0.0201
1.6500	−0.5848	−0.6061	0.0212
1.7000	−0.5658	−0.5882	0.0224
1.7500	−0.5479	−0.5714	0.0235
1.8000	−0.5309	−0.5556	0.0246
1.8500	−0.5148	−0.5405	0.0257
1.9000	−0.4996	−0.5263	0.0268
1.9500	−0.4850	−0.5128	0.0278
2.0000	−0.4712	−0.5000	0.0288

§5.2 Problem 6b. Use linear interpolation on the previous points to determine the values of $y(1.052)$, $y(1.555)$, and $y(1.978)$. Now $t = 1.052$ is between $t_1 = 1.05$ and $t_2 = 1.1$,

$t = 1.555$ is between $t_{11} = 1.55$ and $t_{12} = 1.6$, and $t = 1.978$ is between $t_{19} = 1.95$ and $t_{20} = 2.0$. We use the formula

$$w = \frac{t_{i+1} - t}{h} w_i + \frac{t - t_i}{h} w_{i+1}, \qquad t_i < t < t_{i+1}$$

to linearly interpolate. Thus,

t	w
1.0520	−0.9482
1.5550	−0.6242
1.9780	−0.4773

§5.2 Problem 6c. Compute the h necessary to give accuracy within 0.05 using the error formula

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right].$$

For the Lipschitz constant L, observe

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &\leq |y_1 - y_2|/|t| + |y_2^2 - y_1^2| \\ &\leq |y_1 - y_2|(1 + |y_1 + y_2|) \leq 3|y_1 - y_2| \end{aligned}$$

(Here, we used that $1/|t| \leq 1$ on $1 \leq t \leq 2$, and that $y = -1/t$ has absolute value at most 1.) Hence $L = 3$ works as a bound for the Lipschitz constant. Also, $y' = 1/t^2$ and $y'' = -2/t^3$ so that $M = \max_{1 \leq t \leq 2} |y''(t)| = 2$. Hence,

$$\frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right] = \frac{3h}{4}\left[e^{3(t_i-1)} - 1\right] \leq \frac{h}{3}\left[e^3 - 1\right] = 6.3618h$$

when $t_i = 2$. Thus, the error will be always less than 0.05 when $6.3618h < 0.05$, or $h < 0.05/14.3142 \approx= 0.0079$.

§5.3 Problem 4. Here $f(t, y) = 1 + t\sin(ty)$. To use Taylor's method of order 2 with $h = 0.1$, we need $f'(t, y(t))$ (where implicit differentiation and the substitution $y' = f(t, y)$ are used).

$$f'(t, y) = \sin(ty) + ty\cos(ty) + t^2\cos(ty)(1 + t\sin(ty)).$$

Applying Taylor's method of order 2 with $h = 0.1$, we find

t_i	w_i
0.00000000000000	0.00000000000000
0.10000000000000	0.10000000000000
0.20000000000000	0.20025004249672
0.30000000000000	0.30165255443463
0.40000000000000	0.40572652113584
0.50000000000000	0.51463888640165
0.60000000000000	0.63124269090628
0.70000000000000	0.75907976874000
0.80000000000000	0.90223431113663
0.90000000000000	1.06477676298440
1.00000000000000	1.24930487794059
1.10000000000000	1.45397788625285
1.20000000000000	1.66833943665646
1.30000000000000	1.87142297112678
1.40000000000000	2.03821372952049
1.50000000000000	2.15259867145876
1.60000000000000	2.21325917613240
1.70000000000000	2.22830048523407
1.80000000000000	2.20803843520317
1.90000000000000	2.16140919219282
2.00000000000000	2.09518496841462

§**5.3 Problem 6a.** Here $f(t, y) = \frac{1}{t^2} - \frac{y}{t} - y^2$ on $1 \le t \le 2$, and $y(1) = -1$. For Taylor's Method of order 2, we need $f'(t, y(t))$:

$$\begin{aligned} f'(t, y(t)) &= -\frac{2}{t^3} + \frac{y}{t^2} - (\frac{1}{t} + 2y)y' \\ &= -\frac{2}{t^3} + \frac{y}{t^2} - (\frac{1}{t} + 2y)(\frac{1}{t^2} - \frac{y}{t} - y^2). \end{aligned}$$

Applying Taylor's method of order 2 with $h = 0.05$, we find

t_i	w_i
1.00000000000000	−1.00000000000000
1.05000000000000	−0.95250000000000
1.10000000000000	−0.90931378976690
1.15000000000000	−0.86987989631255
1.20000000000000	−0.83373019049514
1.25000000005215	−0.80047127161437
1.30000000000000	−0.76977013425034
1.35000000000000	−0.74134301096662
1.40000000000000	−0.71494659943473
1.45000000000000	−0.69037110012043
1.50000000029802	−0.66743464321711
1.55000000000000	−0.64597879191898
1.60000000000000	−0.62586488713977
1.65000000000000	−0.60697105558169
1.70000000000000	−0.58918974486405
1.75000000006706	−0.57242568050369
1.80000000000000	−0.55659416286966
1.85000000000000	−0.54161963990291
1.90000000000000	−0.52743450488434
1.95000000000000	−0.51397807891968
2.00000000000000	−0.50119574586239

§5.3 Problem 6c. Here $f(t,y) = \frac{1}{t^2} - \frac{y}{t} - y^2$ on $1 \le t \le 2$, and $y(1) = -1$. For Taylor's Method of order 4, we need $f''(t,y(t))$ and $f'''(t,y(t))$ in addition to what we have from part a):

$$\begin{aligned}
f'(t,y(t)) &== \frac{-2}{t^3} + \frac{y}{t^2} - (\frac{1}{t} + 2y)y' \\
&= \frac{-2}{t^3} + \frac{y}{t^2} - (\frac{1}{t} + 2y)f(t,y) \\
f''(t,y(t)) &== \frac{6}{t^4} - \frac{2y}{t^3} + 2\frac{1}{t^2}y' - 2(y')^2 - (\frac{1}{t} + 2y)y'' \\
&= \frac{6}{t^4} - \frac{2y}{t^3} + 2\frac{1}{t^2}f(t,y) - 2f(t,y)^2 - (\frac{1}{t} + 2y)f'(t,y) \\
f'''(t,y(t)) &== (\frac{-24}{t^5} + \frac{6y}{t^4}) + \frac{-6}{t^3}y' + 3(-2)y'y'' - (\frac{1}{t} + 2y)y'''. \\
&= (\frac{-24}{t^5} + \frac{6y}{t^4}) + \frac{-6}{t^3}f(t,y) - 6f(t,y)f'(t,y) \\
&\quad - (\frac{1}{t} + 2y)f''(t,y).
\end{aligned}$$

Note: It is convenient to keep $f(t,y)$, $f'(t,y)$ and $f''(t,y)$ because these values have already been computed so there is no need to unravel those functions again. Applying Taylor's method of order 4 with $h = 0.05$, we find

t_i	w_i
1.00000000000000	−1.00000000000000
1.05000000000000	−0.95237979647911
1.10000000000000	−0.90908877930679
1.15000000000000	−0.86956225304012
1.20000000000000	−0.83332964226114
1.25000000000000	−0.79999566656697
1.30000000000000	−0.76922586006325
1.35000000000000	−0.74073530892927
1.40000000000000	−0.71427980248546
1.45000000000000	−0.68964881515976
1.50000000000000	−0.66665989212038
1.55000000000000	−0.64515412160243
1.60000000000000	−0.62499245620369
1.65000000000000	−0.60605270305749
1.70000000000000	−0.58822704516761
1.75000000000000	−0.57141998766902
1.80000000000000	−0.55554664638730
1.85000000000000	−0.54053131393553
1.90000000000000	−0.52630625222205
1.95000000000000	−0.51281067072993
2.00000000000000	−0.49998985805760

§5.3 Problem 6b. Use linear interpolation on the previous points to determine the values of $y(1.052)$, $y(1.555)$, and $y(1.978)$. Now $t = 1.052$ is between $t_1 = 1.05$ and $t_2 = 1.1$, $t = 1.555$ is between $t_{11} = 1.55$ and $t_{12} = 1.6$, and $t = 1.978$ is between $t_{19} = 1.95$ and $t_{20} = 2.0$. We use the formula

$$w = \frac{t_{i+1} - t}{h} w_i + \frac{t - t_i}{h} w_{i+1}, \qquad t_i < t < t_{i+1}$$

to linearly interpolate. Thus,

t	w
1.05200000000000	−0.95077255159068
1.55500000000000	−0.64396740144106
1.97800000000000	−0.50681997240760

§5.3 Problem 6d. We use Hermite cubic spline interpolation from the program hcubspl.m:

t	w
1.05200000000000	−0.95056917330616
1.55500000000000	−0.64307961559532
1.97800000000000	−0.50555114904112