Homework #7 Solutions

<u>ection 7.8 Exercise 20</u>

$$y'' + 5y' + 6y = e^{-t}\,\delta(t-2)$$
$$y(0) = 2, \quad y'(0) = -5$$

We apply the Laplace transform to get

$$s^2 Y(s) - s y(0) - y'(0) + 5sY(s) - 5y(0) + 6Y(s) = \int_0^\infty e^{-st} e^{-t}\,\delta(t-2)\,dt$$
$$(s^2 + 5s + 6)\,Y(s) - 2s + 5 - 10 = e^{-2s} e^{-2}$$

Isolating $Y(s)$ gives $\quad Y(s) = \dfrac{2s+5}{(s+2)(s+3)} + e^{-2} e^{-2s} \dfrac{1}{(s+2)(s+3)}$

Partial fractions yields $\quad Y(s) = \dfrac{1}{s+2} + \dfrac{1}{s+3} + e^{-2} e^{-2s} \left[\dfrac{1}{s+2} - \dfrac{1}{s+3} \right]$

Apply the inverse Laplace transform to get

$$\begin{aligned} y(t) &= e^{-2t} + e^{-3t} + e^{-2}\, u(t-2)\, \mathcal{L}^{-1}\left\{ \frac{1}{s+2} - \frac{1}{s+3} \right\}(t-2) \\ &= e^{-2t} + e^{-3t} + e^{-2}\left[e^{-2(t-2)} - e^{-3(t-2)} \right] u(t-2) \\ &= e^{-2t} + e^{-3t} + \left[e^{-2(t-1)} - e^{-3(t-4/3)} \right] u(t-2) \end{aligned}$$

<u>Section 7.8 Exercise 24</u>

$$y'' + y = \delta(t-\pi) - \delta(t-2\pi)$$
$$y(0) = 0, \quad y'(0) = 1$$

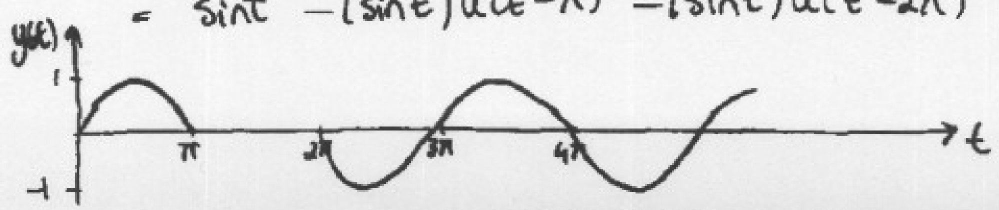
We apply the Laplace transform to get

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = e^{-\pi s} - e^{-2\pi s}$$
$$(s^2+1)\,Y(s) - 1 = e^{-\pi s} - e^{-2\pi s}$$

Isolating $Y(s)$ gives $\quad Y(s) = \dfrac{1}{s^2+1} + e^{-\pi s} \dfrac{1}{s^2+1} - e^{-2\pi s} \dfrac{1}{s^2+1}$

Apply the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \sin t + u(t-\pi)\, \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\}(t-\pi) - u(t-2\pi)\, \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\}(t-2\pi) \\ &= \sin t + \sin(t-\pi)\, u(t-\pi) - \sin(t-2\pi)\, u(t-2\pi) \\ &= \sin t - (\sin t)\, u(t-\pi) - (\sin t)\, u(t-2\pi) \end{aligned}$$



Section 7.9 Exercise 10 $\quad x'' + y = 1 \quad x(0) = 1 \quad x'(0) = 1$

$\quad x + y'' = -1 \quad y(0) = 1 \quad y'(0) = -1$

2/5

We apply the Laplace transform to both equations to get

$$\begin{cases} s^2 X(s) - s x(0) - x'(0) + Y(s) = \frac{1}{s} \\ X(s) + s^2 Y(s) - s y(0) - y'(0) = -\frac{1}{s} \end{cases}$$

Substitute initial conditions to get

$$\begin{cases} s^2 X(s) + Y(s) = \frac{1}{s} + s + 1 = \frac{s^2+s+1}{s} \\ X(s) + s^2 Y(s) = -\frac{1}{s} + s - 1 = \frac{s^2-s-1}{s} \end{cases}$$

From 2nd equation, write $X(s)$ in terms of $Y(s)$: $X(s) = \frac{s^2-s-1}{s} - s^2 Y(s)$

Substitute into 1st equation to get

$$s^2\left[\frac{s^2-s-1}{s} - s^2 Y(s)\right] + Y(s) = \frac{s^2+s+1}{s}$$

$$(1-s^4)Y(s) = \frac{s^2+s+1}{s} - s^3 + s^2 + s = \frac{-s^4+s^3+2s^2+s+1}{s}$$

$$Y(s) = \frac{-s^4+s^3+2s^2+s+1}{s(1-s)(1+s)(1+s^2)} = \frac{1}{s} + \frac{1}{1-s} + \frac{0}{1+s} + \frac{s+0}{1+s^2}$$

Apply the inverse Laplace transform to get $y(t) = 1 - e^t + \cos t$

To find $x(t)$, substitute
$$\begin{aligned} y(t) &= 1 - e^t + \cos t \\ y'(t) &= -e^t - \sin t \\ y''(t) &= -e^t - \cos t \end{aligned}$$
into the second differential equation:

$$\begin{aligned} x(t) &= -1 - y''(t) \\ &= -1 + e^t + \cos t \end{aligned}$$

So:
$$\begin{aligned} x(t) &= e^t + \cos t - 1 \\ y(t) &= -e^t + \cos t + 1 \end{aligned}$$

Section 7.9 Exercise 18

$$x' - 2y = 0 \qquad x(0) = 0 \qquad ①$$
$$x' - z' = 0 \qquad y(0) = 0 \qquad ②$$
$$x + y' - z = 3 \qquad z(0) = -2 \qquad ③$$

We apply the Laplace transform to all three equations to get

$$sX(s) - x(0) - 2Y(s) = 0 \Rightarrow sX(s) - 2Y(s) = 0 \qquad ④$$
$$sX(s) - x(0) - sZ(s) + z(0) = 0 \Rightarrow sX(s) - sZ(s) = 2 \qquad ⑤$$
$$X(s) + sY(s) - y(0) - Z(s) = \frac{3}{s} \Rightarrow X(s) + sY(s) - Z(s) = \frac{3}{s} \qquad ⑥$$

Rewrite ④ : $Y(s) = \frac{s}{2} X(s)$
Rewrite ⑤ ; $Z(s) = X(s) - \frac{2}{s}$
and substitute into ⑥ to get

$$X(s) + \frac{s^2}{2} X(s) - X(s) + \frac{2}{s} = \frac{3}{s}$$

or

$$X(s) = \frac{2}{s^3}$$

Apply the inverse Laplace transform to get $x(t) = t^2$

Then ① gives $y(t) = \frac{1}{2} x'(t) = \frac{1}{2}(2t) = t$
and ③ gives $z(t) = x(t) + y'(t) - 3 = t^2 + 1 - 3 = t^2 - 2$

So:

$$x(t) = t^2$$
$$y(t) = t$$
$$z(t) = t^2 - 2$$

Section 8.2 Exercise 4 Determine the convergence set of the power series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}(x-3)^n$

Let $a_n = \frac{4}{n^2+2n}$ and apply the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{4}{(n+1)^2 + 2(n+1)} \cdot \frac{n^2+2n}{4} \right] = \lim_{n\to\infty} \frac{n^2+2n}{n^2+4n+3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{4}{n}+\frac{3}{n^2}} = 1$$

Therefore the radius of convergence is 1
Since the center of the power series is at $x=3$, we have convergence on
$x \in (3-1, 3+1)$ or $x \in (2,4)$.
But we still need to check the endpoints.

At $x=2$, the power series reads $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}(2-3)^n = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2+2n}$
We use the Alternating Series Test from MATH 101.
Since (a) $\frac{4}{(n+1)^2+2(n+1)} < \frac{4}{n^2+2n}$ for all n
and (b) $\lim_{n\to\infty} \frac{4}{n^2+2n} = 0$, this series converges

At $x=4$, the power series reads $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}(4-3)^n = \sum_{n=1}^{\infty} \frac{4}{n^2+2n}$
We use the Comparison Test from MATH 101 and compare
$\frac{4}{n^2+2n} < \frac{4}{n^2}$
Since ~~the~~ $\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges [it is a p-series with $p = 2 > 1$],
therefore our series $\sum_{n=1}^{\infty} \frac{4}{n^2+2n}$ also converges.

We conclude that the given power series converges for $x \in [2,4]$.

<u>Section 8.2 Exercise 12</u>

$$\begin{cases} f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ g(x) = \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \end{cases}$$

Find the first three nonzero terms in the power series expansion for the product $f(x)g(x)$.

$$\begin{aligned} f(x)g(x) &= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\ &\qquad - \frac{x^3}{3!} + \frac{x^5}{2!3!} - \cdots \\ &\qquad\qquad + \frac{x^5}{5!} - \cdots \\ &= x - \left(\tfrac{1}{2}+\tfrac{1}{6}\right)x^3 + \left(\tfrac{1}{24}+\tfrac{1}{12}+\tfrac{1}{120}\right)x^5 - \cdots \\ &= x - \tfrac{2}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \end{aligned}$$

<u>Section 8.2 Exercise 26</u> Express $\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3}$ as a series with generic term x^k

$$\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3} = \sum_{k=4}^{\infty} \frac{a_{k-3}}{k} x^k$$

<u>Section 8.2 Exercise 30</u> Determine the Taylor series about the point $x_0 = 1$ for $f(x) = x^{-1}$

$$\begin{aligned}
f(x) &= x^{-1} &&= 0!\, x^{-1} & f(1) &= 0! \\
f'(x) &= -x^{-2} &&= -1!\, x^{-2} & f'(1) &= -1! \\
f''(x) &= 2x^{-3} &&= 2!\, x^{-3} & f''(1) &= 2! \\
f'''(x) &= -6x^{-4} &&= -3!\, x^{-4} & f'''(1) &= -3! \\
&&&& &\vdots \\
&&&& f^{(n)}(1) &= (-1)^n n!
\end{aligned}$$

Then the Taylor series is

$$\begin{aligned}
f(x) &= f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n \\
&= \sum_{n=0}^{\infty} (-1)^n (x-1)^n
\end{aligned}$$

HW #8

Section 8.3 Exercise 18

Find the first 4 nonzero terms in a power series solution about $x=0$ to $(2x-3)y'' - xy' + y = 0$

Let $y(x) = \sum_{n=0} a_n x^n$

Then $y'(x) = \sum_{n=1} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2} n(n-1) a_n x^{n-2}$

Substitute into the DE to get

$$(2x-3)\sum_{n=2} n(n-1)a_n x^{n-2} - x\sum_{n=1} n a_n x^{n-1} + \sum_{n=0} a_n x^n = 0$$

Multiply:

$$\sum_{n=2} 2n(n-1)a_n x^{n-1} - \sum_{n=2} 3n(n-1)a_n x^{n-2} - \sum_{n=1} n a_n x^n + \sum_{n=0} a_n x^n = 0$$

Rewrite using x^n in each series:

$$\sum_{n=1} 2(n+1)(n)a_{n+1} x^n - \sum_{n=0} 3(n+2)(n+1)a_{n+2} x^n - \sum_{n=1} n a_n x^n + \sum_{n=0} a_n x^n = 0$$

Start each series at $n=1$:

$$-6a_2 + a_0 + \sum_{n=1}\left[2(n+1)(n)a_{n+1} - 3(n+2)(n+1)a_{n+2} - n a_n + a_n\right]x^n = 0$$

Set all coefficients in LHS equal to zero, that is

$$a_2 = \frac{1}{6}a_0$$

and

$$a_{n+2} = \frac{2(n+1)(n)a_{n+1} - (n-1)a_n}{3(n+2)(n+1)} \quad \text{for } n = 1, 2, 3, \cdots$$

$$n=1 \Rightarrow \quad a_3 = \frac{2(2)(1)a_2 - 0}{3(3)(2)} = \frac{4}{18}a_2 = \frac{2}{9}a_2 = \frac{2}{9}\left(\frac{1}{6}a_0\right) = \frac{1}{27}a_0$$

Note that since both a_2 and a_3 are in terms of a_0 only, all remaining terms will be in terms of a_0 only.

So, we have

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + a_1 x + \tfrac{1}{6}a_0 x^2 + \tfrac{1}{27}a_0 x^3 + \cdots \\ &= a_0\left[1 + \tfrac{1}{6}x^2 + \tfrac{1}{27}x^3 + \cdots\right] + a_1 x. \end{aligned}$$

Section 8.3 Exercise 24

Find a power series solution about $x=0$ to

$$(x^2+1)y'' - xy' + y = 0$$

Your sol$^{\underline{n}}$ should include a general formula for the coefficients.

Let $y(x) = \sum_{n=0} a_n x^n$

Then $y'(x) = \sum_{n=1} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2} n(n-1) a_n x^{n-2}$

Substitute into the DE to get

$$(x^2+1)\sum_{n=2} n(n-1) a_n x^{n-2} - x\sum_{n=1} n a_n x^{n-1} + \sum_{n=0} a_n x^n = 0$$

Multiply:

$$\sum_{n=2} n(n-1) a_n x^n + \sum_{n=2} n(n-1) a_n x^{n-2} - \sum_{n=1} n a_n x^n + \sum_{n=0} a_n x^n = 0$$

Rewrite using x^n in each series:

$$\sum_{n=2} n(n-1) a_n x^n + \sum_{n=0} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1} n a_n x^n + \sum_{n=0} a_n x^n = 0$$

Start each series at $n=2$:

$$2a_2 + 6a_3 x - a_1 x + a_0 + a_1 x + \sum_{n=2} \left[n(n-1)a_n + (n+2)(n+1)a_{n+2} - na_n + a_n \right] x^n = 0$$

Set all coefficients in LHS equal to zero:

x^0 terms $\Rightarrow$ $2a_2 + a_0 = 0$ $\Rightarrow$ $a_2 = -\frac{1}{2}a_0$

x^1 terms $\Rightarrow$ $6a_3 - a_1 + a_1 = 0$ $\Rightarrow$ $a_3 = 0$

x^n terms $\Rightarrow$ $a_{n+2} = \dfrac{-(n-1)^2}{(n+2)(n+1)} a_n$ for $n = 2, 3, 4, \cdots$

$n=2$ gives $a_4 = \dfrac{a_0}{4!} = \dfrac{1}{24}a_0$

$n=3$ gives $a_5 = 0$

$n=4$ gives $a_6 = -\dfrac{(3)^2 a_4}{6\cdot 5} = -\dfrac{(3)^2}{6!}a_0$

$n=5$ gives $a_7 = 0$

$n=6$ gives $a_8 = -\dfrac{(5)^2 a_6}{8\cdot 7} = \dfrac{(5)^2(3)^2}{8!}a_0$

in general,

$$a_{2k} = \frac{(-1)^k (3)^2(5)^2 \cdots (2k-5)^2(2k-3)^2}{(2k)!} a_0$$

for $k = 3, 4, \cdots$

Then

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_1 x + a_0 \left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \sum_{k=3} \frac{(-1)^k (1)^2(3)^2(5)^2 \cdots (2k-5)(2k-3)^2}{(2k)!} \right]$$

Section 8.3 Exercise 26

Find the first 4 nonzero terms in a power series solution about $x=0$ to the following IVP problem: $(x^2-x+1)y''-y'-y=0$

$y(0)=0$ and $y'(0)=1$

Let $y(x)=\sum_{n=0} a_n x^n$

Then $y'(x)=\sum_{n=1} n a_n x^{n-1}$ and $y''(x)=\sum_{n=2} n(n-1)a_n x^{n-2}$

Substitute into the DE to get

$$(x^2-x+1)\sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=1} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0$$

Multiply:

$$\sum_{n=2} n(n-1)a_n x^n - \sum_{n=2} n(n-1)a_n x^{n-1} + \sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=1} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0$$

Rewrite using x^n in each series:

$$\sum_{n=2} n(n-1)a_n x^n - \sum_{n=1}(n+1)(n)a_{n+1} x^n + \sum_{n=0}(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}(n+1)a_{n+1} x^n - \sum_{n=0} a_n x^n = 0$$

Start each series at $n=2$:

$$-2a_2x + 2a_2 + 6a_3x - a_1 - 2a_2x - a_0 - a_1x + \sum_{n=2}\left[n(n-1)a_n - (n+1)(n)a_{n+1} + (n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n \right]x^n = 0$$

Set all coefficients in LHS equal to zero:

x^0 terms $\Rightarrow$ $2a_2 - a_1 - a_0 = 0$ $\Rightarrow$ $a_2 = \frac{1}{2}a_1 + \frac{1}{2}a_0$

x^1 terms $\Rightarrow$ $-2a_2 + 6a_3 - 2a_2 - a_1 = 0$ $\Rightarrow$ $a_3 = \frac{1}{2}a_1 + \frac{1}{3}a_0$

x^2 terms $\Rightarrow$ $2a_2 - 6a_3 + 12a_4 - 3a_3 - a_2 = 0$ $\Rightarrow$ $a_4 = \frac{1}{3}a_1 - \frac{5}{24}a_0$

So $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$= a_0 + a_1x + (\frac{1}{2}a_1 + \frac{1}{2}a_0)x^2 + (\frac{1}{2}a_1 + \frac{1}{3}a_0)x^3 + (4a_1 - \frac{5}{2}a_0)x^4 + \cdots$

$= a_0[1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{5}{2}x^4 + \cdots]$

$+ a_1[x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \cdots]$

Apply initial conditions: $y(0)=0 \Rightarrow a_0=0$ and $y'(0)=1 \Rightarrow a_1=1$

Then $y(x) = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \cdots$

Section 8.4 Exercise 8

Find the first 4 nonzero terms in a power series solution about $x=-1$ to $y' - 2xy = 0$

Let $x = t-1$ and $Y(t) = y(x(t)) = y(t-1)$

Then $Y'(t) = y'(x)$

and the DE for $Y(t)$ is $Y' - 2(t-1)Y = 0$

Let $Y(t) = \sum_{n=0} a_n t^n$ Then $Y'(t) = \sum_{n=1} n a_n t^{n-1}$

Substitute into DE to get

$$\sum_{n=1} n a_n t^{n-1} - 2(t-1)\sum_{n=0} a_n t^n = 0$$

Multiply:

$$\sum_{n=1} n a_n t^{n-1} - \sum_{n=0} 2a_n t^{n+1} + \sum_{n=0} 2a_n t^n = 0$$

Rewrite using t^n in each series:

$$\sum_{n=0} (n+1) a_{n+1} t^n - \sum_{n=1} 2a_{n-1} t^n + \sum_{n=0} 2a_n t^n = 0$$

Start each series at $n=1$:

$$a_1 + 2a_0 + \sum_{n=1} [(n+1)a_{n+1} - 2a_{n-1} + 2a_n] t^n = 0$$

Set all coefficients in LHS equal to zero:

$$a_1 + 2a_0 = 0 \implies a_1 = -2a_0$$

and

$$(n+1)a_{n+1} - 2a_{n-1} + 2a_n = 0 \implies a_{n+1} = \frac{2a_{n-1} - 2a_n}{n+1} \quad \text{for } n = 1, 2, 3, \cdots$$

$n=1$ gives $a_2 = 3a_0$

$n=2$ gives $a_3 = -\frac{10}{3}a_0$

$$\begin{aligned}
\text{So } Y(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \\
&= a_0 - 2a_0 t + 3a_0 t^2 - \tfrac{10}{3} a_0 t^3 + \cdots \\
&= a_0 [1 - 2t + 3t^2 - \tfrac{10}{3} t^3 + \cdots] \\
y(x) &= a_0 [1 - 2(x+1) + 3(x+1)^2 - \tfrac{10}{3}(x+1)^3 + \cdots]
\end{aligned}$$

Section 8.4 Exercise 16 5/8

Find the first 3 nonzero terms in a power series solution to

$y'' + ty' + e^t y = 0, \quad y(0) = 0, \quad y'(0) = -1$

Let $y(t) = \sum_{n=0} a_n t^n$

Then $y'(t) = \sum_{n=1} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2} n(n-1) a_n t^{n-2}$

Substitute into the DE, and expand e^t into its Taylor series about $t=0$:

$$\sum_{n=2} n(n-1) a_n t^{n-2} + \sum_{n=1} n a_n t^n + \left[1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 + \cdots\right] \sum_{n=0} a_n t^n = 0$$

$$\sum_{n=2} n(n-1) a_n t^{n-2} + \sum n a_n t^n + \left[1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^3 + \cdots\right]\left[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\right] = 0$$

Expand first two series, multiply terms in square brackets, group like terms in columns:

$$\begin{array}{rrrrl}
(2)(1)a_2 & + (3)(2)a_3 t & + (4)(3)a_4 t^2 & + (5)(4)a_5 t^3 & + \cdots\cdots \\
 & + a_1 t & + 2a_2 t^2 & + 3a_3 t^3 & + \cdots \\
+ a_0 & + a_1 t & + a_2 t^2 & + a_3 t^3 & + \cdots \\
 & + a_0 t & + a_1 t^2 & + a_2 t^3 & + \cdots \\
 & & + \tfrac{1}{2}a_0 t^2 & + \tfrac{1}{2}a_1 t^3 & + \cdots \\
 & & & + \tfrac{1}{6}a_0 t^3 & + \cdots\cdots \quad = 0
\end{array}$$

t^0 terms: $2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{1}{2}a_0$

t^1 terms: $6a_3 + 2a_1 + a_0 = 0 \Rightarrow a_3 = -\frac{1}{3}a_1 - \frac{1}{6}a_0$

t^2 terms: $12a_4 + 3a_2 + a_1 + \frac{1}{2}a_0 = 0 \Rightarrow a_4 = -\frac{1}{12}a_1 + \frac{1}{12}a_0$

t^3 terms: $20a_5 + 4a_3 + a_2 + \frac{1}{2}a_1 + \frac{1}{6}a_0 = 0 \Rightarrow a_5 = \frac{1}{24}a_1 + \frac{1}{20}a_0$

$$\begin{aligned}
\text{So } y(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots \\
&= a_0 + a_1 t - \tfrac{1}{2}a_0 t^2 + (-\tfrac{1}{3}a_1 - \tfrac{1}{6}a_0)t^3 + (-\tfrac{1}{12}a_1 + \tfrac{1}{12}a_0)t^4 + (\tfrac{1}{24}a_1 + \tfrac{1}{20}a_0)t^5 + \cdots \\
&= a_0\left[1 - \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{12}t^4 + \tfrac{1}{20}t^5 + \cdots\right] \\
&\quad + a_1\left[t - \tfrac{1}{3}t^3 - \tfrac{1}{12}t^4 + \tfrac{1}{24}t^5 + \cdots\right]
\end{aligned}$$

Apply initial conditions: $y(0) = 0 \Rightarrow a_0 = 0$, $y'(0) = -1 \Rightarrow a_1 = -1$

So $y(t) = \underbrace{-t + \tfrac{1}{3}t^3 + \tfrac{1}{12}t^4}_{\text{these are the terms asked for}} - \tfrac{1}{24}t^5 + \cdots$

Section 8.4 Exercise 22

Find the first 4 nonzero terms in a power series solution, about $x=0$ to
$$w' + xw = e^x$$

Let $w(x) = \sum_{n=0} a_n x^n$

Then $w'(x) = \sum_{n=1} n a_n x^{n-1}$

Substitute into the DE, and expand e^x about $x=0$, to get
$$\sum_{n=1} n a_n x^{n-1} + x \sum_{n=0} a_n x^n = \sum_{n=0} \frac{x^n}{n!}$$

Multiply:
$$\sum_{n=1} n a_n x^{n-1} + \sum_{n=0} a_n x^{n+1} = \sum_{n=0} \frac{x^n}{n!}$$

Rewrite using x^n in each series:
$$\sum_{n=0} (n+1) a_{n+1} x^n + \sum_{n=1} a_{n-1} x^n = \sum_{n=0} \frac{x^n}{n!}$$

Start each series at $n=1$:
$$a_1 + \sum_{n=1} [(n+1)a_{n+1} + a_{n-1}] x^n = 1 + \sum_{n=1} \frac{x^n}{n!}$$

Equate coefficients of like terms on LHS and RHS:

x^0 terms $\Rightarrow$ $a_1 = 1$

x^n terms $\Rightarrow$ $(n+1)a_{n+1} + a_{n-1} = \frac{1}{n!}$ for $n = 1, 2, 3, \cdots$

$n=1$ gives $2a_2 + a_0 = 1 \Rightarrow a_2 = \frac{1}{2} - \frac{1}{2}a_0$

$n=2$ gives $3a_3 + a_1 = \frac{1}{2} \Rightarrow a_3 = -\frac{1}{6}$

Then the solution is
$$\begin{aligned} w(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + x + \left(\tfrac{1}{2} - \tfrac{1}{2}a_0\right)x^2 - \tfrac{1}{6}x^3 + \cdots \\ &= a_0\left[1 - \tfrac{1}{2}x^2 + \cdots\right] + \left[x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots\right] \end{aligned}$$

Section 8.5 Exercise 14

Use variation of parameters to find a general solution to

$$x^2 y'' + 2xy' - 2y = \frac{6}{x^2} + 3x, \qquad x > 0$$

We first solve the homogeneous DE $x^2y'' + 2xy' - 2y = 0$

Let $y(x) = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$

Substitute into DE to get $r(r-1) + 2r - 2 = 0 \Rightarrow r^2 + r - 2 = 0$

$\Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2$ or $r = 1$.

So $y_H(x) = C_1 x^{-2} + C_2 x$

Now we look for a particular solution of the form

$y_p(x) = u(x) \cdot \frac{1}{x^2} + v(x) \cdot x$

By the method of variation of parameters, $u'(x)$ and $v'(x)$ are given by the following system of equations:

$$\begin{bmatrix} \frac{1}{x^2} & x \\ -2/x^3 & 1 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ 6/x^4 + \frac{3}{x} \end{bmatrix}$$

Cramer's Rule gives the solution: $u'(x) = \dfrac{-\left(\frac{6}{x^3} + 3\right)}{3/x^2} = -\frac{2}{x} - x^2$

$$v'(x) = \frac{\frac{6}{x^6} + \frac{3}{x^3}}{3/x^2} = \frac{2}{x^4} + \frac{1}{x}$$

Integrate to get $u(x) = -2\ln x - \frac{1}{3}x^3$

$v(x) = -\frac{2}{3}\frac{1}{x^3} + \ln x$

The solution is
$$\begin{aligned} y(x) &= y_H(x) + y_p(x) \\ &= C_1 x^{-2} + C_2 x + \frac{1}{x^2}\left[-2\ln x - \frac{1}{3}x^3\right] + x\left[-\frac{2}{3}\frac{1}{x^3} + \ln x\right] \\ &= C_1 x^{-2} + C_2 x - 2x^{-2}\ln x - \frac{1}{3}x - \frac{2}{3}x^{-2} + x\ln x \end{aligned}$$

We absorb $-\frac{2}{3}x^{-2}$ into $C_1 x^{-2}$ and $-\frac{1}{3}x$ into $C_2 x$.

Then the solution is

$$y(x) = \frac{C_1}{x^2} + C_2 x - \frac{2}{x^2}\ln x + x\ln x$$

Section 8.5 Exercise 16

Solve $x^2 y'' + 5xy' + 4y = 0, \quad y(1) = 3, \quad y'(1) = 7$

Let $y(x) = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$

Substitute into the DE to get

$r(r-1) + 5r + 4 = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \qquad \Rightarrow r = -2$ is a double root.

Therefore the general sol$^{\underline{n}}$ is

$y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x$

Apply the initial conditions to determine C_1 and C_2

$y(1) = 3 \quad \Rightarrow \quad C_1 = 3$

$y'(1) = 7 \quad \Rightarrow \quad C_2 = 13$

So the solution is $y(x) = \dfrac{3}{x^2} + \dfrac{13 \ln x}{x^2}$

Math 201 Assignment #9

§ 10.2 #6

$$y'' + y = 0 \ , \quad 0 < x < 2\pi$$
$$y(0) = 0 \ , \quad y(2\pi) = 1$$

General solution is

$$y(x) = a\cos x + b\sin x .$$
$$y(0) = 0 \implies a = 0$$
$$y(2\pi) = 1 \implies b\sin 2\pi = 1.$$

Contradiction! Therefore no solution.

§ 10.2 #10

$$y'' + \lambda y = 0 \ , \quad 0 < x < \pi,$$
$$y'(0) = 0 \ , \quad y(\pi) = 0$$

Only nontrivial solution is for $\lambda > 0$.

General solution is

$$y = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$$
$$y'(x) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,x + b\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$
$$y'(0) = 0 \implies b = 0$$
$$y(\pi) = a\cos\sqrt{\lambda}\,\pi = 0 \implies$$

$$\sqrt{\lambda}\,\pi = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, \cdots$$

$$\therefore \lambda_n = \frac{(2n-1)^2}{4}$$

$$y_n = C_n \cos\left[\frac{(2n-1)x}{2}\right], \quad n = 1, 2, \cdots.$$

§ 10.2 #14

$$y'' - 2y' + \lambda y = 0, \quad 0 < x < \pi$$

$$y(0) = 0, \quad y(\pi) = 0.$$

Characteristic eqn.

$$r^2 - 2r + \lambda = 0$$

$$r = 1 \pm i\sqrt{\lambda - 1}, \quad \lambda > 1$$ for nontrivial solutions.

General solution

$$y = a e^x \cos\sqrt{\lambda-1}\,x + b e^x \sin\sqrt{\lambda-1}\,x.$$

$$y(0) = 0 \Rightarrow a = 0.$$

$$y(\pi) = 0 \Rightarrow b e^{\pi} \sin\sqrt{\lambda-1}\,\pi = 0$$

$$\therefore \sqrt{\lambda-1}\,\pi = n\pi$$

$$\lambda = n^2 + 1$$

$$\therefore \lambda_n = n^2 + 1$$

$$y_n = C_n e^x \sin nx, \quad n = 1, 2, \cdots$$

§10.2 #16.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{3}\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0.$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$

$$u(x,0) = \sin 3x + 5\sin 7x - 2\sin 13x.$$

Boundary conditions homogeneous so separation of variables works.

$$u(x,t) = X(x)T(t).$$

$$\frac{X''}{X} = \frac{1}{3}\frac{T'}{T} = -\lambda^2 < 0 \text{ (separation cst.)}$$

$$\therefore \; X'' + \lambda^2 X = 0, \quad T' + 3\lambda^2 T = 0,$$

$$X(0) = X(\pi) = 0.$$

$$X = a\cos\lambda x + b\sin\lambda x.$$

$$X(0) = 0 \Rightarrow a = 0$$

$$X(\pi) = 0 \Rightarrow b\sin\lambda\pi = 0$$

$$\Rightarrow \lambda_n = n, \; n = 1, 2, \ldots$$

$$T = e^{-3n^2 t}.$$

Superposition principle gives

$$u = \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin nx.$$

Apply initial condition

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sin 3x + 5\sin 7x - 2\sin 13x$$

$$\Rightarrow b_3 = 1,\ b_7 = 5,\ b_{13} = -2.$$

$$b_n = 0,\quad n \neq 3, 7, 13$$

$$\therefore u(x,t) = e^{-27t}\sin 3x + 5e^{-147t}\sin 7x - 2e^{-507t}\sin 13x.$$

---

§10.3 #6

$$f(x) = x^{1/5}\cos x^2$$

$$f(-x) = (-x)^{1/5}\cos(-x)^2$$

$$= (-1)^{1/5}x^{1/5}\cos x^2$$

$$= -x^{1/5}\cos x^2 = -f(x)$$

$\therefore$ odd.

§10.3 #10

$$f(x) = |x|,\quad -\pi < x < \pi.$$

$$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Since $f(x)$ is even we can conclude that in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

$b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

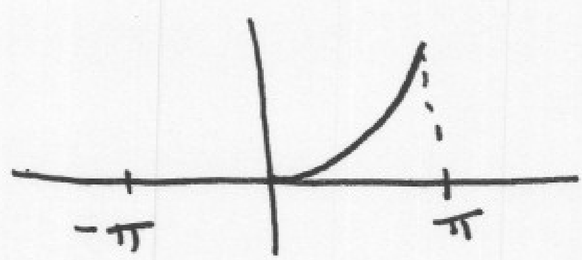
$$= \frac{2}{\pi n^2}[\cos n\pi - 1]$$

$$= \begin{cases} 0, & n \text{ even}, \\ -\frac{4}{\pi n^2}, & n \text{ odd}. \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \pi.$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)x].$$

<u>§ 10.3 # 12</u>



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_0^\pi x^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n}\Big|_0^\pi - 2\int_0^\pi x \frac{\sin nx}{n}\, dx\right.$$

$$= -\frac{2}{n\pi}\left[-x\frac{\cos nx}{n}\Big|_0^\pi + \underbrace{\int_0^\pi \frac{\cos nx}{n}\, dx}_{=0}\right]$$

$$= -\frac{2}{n\pi}\left[-\pi\frac{\cos n\pi}{n}\right] = \frac{2}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n}\Big|_0^\pi - \int_0^\pi\left(-\frac{\cos nx}{n}\right)2x\, dx\right]$$

$$= -\frac{\pi^2(-1)^n}{n\pi} + \frac{2}{n\pi}\int_0^\pi x\cos nx\, dx$$

$$= -\pi\frac{(-1)^n}{n} + \frac{2}{n^3\pi}\left((-1)^n - 1\right) = \frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos nx + \left[\frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi}\right]\sin nx\right\}$$

§ 10.3 #28.

$$f(x) = x^2, \quad -\pi < x < \pi$$

(a) f even $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{4(-1)^n}{n^2}.$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

(b) At $x = 0$, $f(0) = 0$.

$$\therefore \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0 \quad \text{or}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(c) By Th$^{\underline{m}}$ 2 pg. 600

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4}\left(1-\frac{1}{3}\right) = \frac{\pi^2}{6}.$$

Math 201 Assignment #10

§10.4 #6: $f(x) = \cos x$, FSS Odd extension

$a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx = \frac{2}{\pi}\left[-\frac{\cos(n-1)x}{2(n-1)} - \frac{\cos(n+1)x}{2(n+1)}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos(n-1)\pi}{2(n-1)} - \frac{\cos(n+1)\pi}{2(n+1)} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1}\right] = \frac{2n}{\pi}\,\frac{1+(-1)^n}{n^2-1}$$

$$\therefore \quad b_n = \begin{cases} 0, & n \text{ odd} \\ \dfrac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

$$\therefore f(x) = \sum_{n \text{ even}} \frac{4n}{\pi(n^2-1)} \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \sin 2nx .$$

§10.4 #10: $f(x) = e^x$, FSS odd extension

$a_n = 0$

$$b_n = 2\int_0^1 e^x \sin n\pi x \, dx$$

$$= \frac{2e^x}{1+n^2\pi^2}\left(\sin n\pi x - n\pi \cos n\pi x\right)\Big|_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left[e(-n\pi\cos n\pi) + n\pi\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[1 + e(-1)^{n+1}\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2n\pi\left[1+e(-1)^{n+1}\right]}{1+n^2\pi^2} \sin n\pi x.$$

§10.4 #14: $f(x) = e^{-x}$ FCS, even extension

$b_n = 0.$

$$a_n = 2\int_0^1 e^{-x}\cos n\pi x\,dx, \quad n = 0, 1, \ldots$$

$$= 2\,\frac{e^{-x}}{1+n^2\pi^2}\left(-\cos n\pi x + n\pi\sin n\pi x\right)\Big|_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left[e^{-1}(-\cos n\pi) - (-1)\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[1 + (-1)^{n+1}e^{-1}\right]$$

$$\therefore f(x) = 1 - e^{-1} + \sum_{n=1}^{\infty} \frac{2\left[1+(-1)^{n+1}e^{-1}\right]}{1+n^2\pi^2}\cos n\pi x.$$

§10.4 #16: $f(x) = x - x^2$, $0 < x < 1$ FCS even extension.

$$a_0 = 2\int_0^1 (x - x^2)\,dx = 2\left(\tfrac{1}{6}\right)$$

$$a_n = 2\int_0^1 (x - x^2)\cos n\pi x\,dx, \quad n = 1, 2, \cdots$$

$$= 2\int_0^1 x\cos n\pi x\,dx - 2\int_0^1 x^2 \cos n\pi x\,dx.$$

$$\int_0^1 x\cos n\pi x\,dx = x\frac{\sin n\pi x}{n\pi} - \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx$$

$$= \frac{\cos n\pi x}{n^2\pi^2}\Big|_0^1 = \frac{1}{n^2\pi^2}\left((-1)^n - 1\right).$$

$$\int_0^1 x^2\cos n\pi x\,dx = x^2\frac{\sin n\pi x}{n\pi} - 2\int_0^1 x\frac{\sin n\pi x}{n\pi}\,dx$$

$$= -\frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx$$

$$= -\frac{2}{n\pi}\left[-\frac{(-1)^n}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\Big|_0^1\right]$$

$$= 2(-1)^n / n^2\pi^2.$$

$$\therefore a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1 - 2(-1)^n\right] = \frac{-2}{n^2\pi^2}\left[1 + (-1)^n\right]$$

$$= \begin{cases} -4/n^2\pi^2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\therefore\ f(x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{4}{4n^2\pi^2} \cos 2n\pi x.$$

§ 10.5 #2: $u_{xx} = u_t$, $0 < x < \pi$, $t > 0$.

$u(0,t) = u(\pi,t) = 0$, $t > 0$,

$u(x,0) = x^2$, $0 < x < \pi$.

$$u = XT \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2.$$

$$\therefore\ X = a\cos\lambda x + b\sin\lambda x,\ T = e^{-\lambda^2 t}.$$

$X(0) = 0 \Rightarrow a \equiv 0$. $X(\pi) = 0 \Rightarrow \lambda = \lambda_n = n$.

Solution

$$u = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx.$$

$$u(x,0) = x^2 = \sum_{n=1}^{\infty} b_n \sin nx \quad \Rightarrow$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx.$$

$$= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left((-1)^n - 1\right), \quad n = 1, 2, \ldots$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left((-1)^n - 1\right) \right] e^{-n^2 t} \sin nx .$$

$u \to 0$ as $t \to \infty$

§ 10.5 #4. $u_{xx} = \frac{1}{2} u_t \,, 0 < x < 1 \,, t > 0$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \,, \quad t > 0.$$

$$u(x,0) = x - x^2 .$$

$$u = XT \Rightarrow \frac{X''}{X} = \frac{T'}{2T} = -\lambda^2 .$$

$$\Rightarrow X = a\cos\lambda x + b\sin\lambda x .$$

$$T = e^{-2\lambda^2 t} .$$

$X'(0) = 0 \Rightarrow b = 0 \,, \ X'(1) = 0 \Rightarrow$

$$\lambda_n = n\pi \,, \quad n = 0, 1, \ldots$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-2n^2\pi^2 t} \cos n\pi x .$$

$$u(x,0) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x .$$

Using §10.4 #16 solution then gives

$$a_0 = 1/3 \,, \quad a_n = -4/n^2\pi^2 ,\ n \text{ even}$$

$$\therefore u(x,t) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} e^{-8n^2\pi^2 t} \cos 2n\pi x.$$

§10.5 #10: $u_t = 3u_{xx} + x$, $0 < x < \pi$, $t > 0$

$u(0,t) = u(\pi,t) = 0$, $t > 0$

$u(x,0) = \sin x$, $0 < x < \pi$.

Introduce transient $w(x,t)$ and steady state $v(x)$, i.e.,

$$u = v(x) + w(x,t)$$

Steady state satisfies the heat eqn.

$$v'' = -x/3.$$

The boundary conditions

$u(0,t) = v(0) + w(0,t) = 0 \Rightarrow v(0) = 0,$

$u(\pi,t) = v(\pi) + w(\pi,t) = 0 \Rightarrow v(\pi) = 0.$

Now $\quad v(x) = -\dfrac{x^3}{18} + Ax + B$

and $\quad v(0) = 0 \Rightarrow B = 0$

$v(\pi) = 0 \Rightarrow A = \pi^2/18.$

$$\therefore v(x) = -\frac{x^3}{18} + \frac{\pi^2 x}{18}.$$

The transient $w(x,t)$ satisfies the IBVP

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{3}\frac{\partial w}{\partial t}, \quad 0 < x < \pi, \ t > 0$$

$$w(0,t) = w(\pi,t) = 0$$

$$w(x,0) = u(x,0) - v(x)$$

$$= \sin x + \frac{x^3}{18} - \frac{\pi^2}{18}x.$$

Homogeneous BC's and equation $\Rightarrow$ separation of variables approach.

$$w = XT.$$

This leads to

$$\frac{X''}{X} = \frac{1}{3}\frac{T'}{T} = -\lambda^2 \text{ (separation constant)}$$

$$\therefore X = a\cos\lambda x + b\sin\lambda x.$$

$$X(0) = 0 \Rightarrow a = 0$$

$$X(\pi) = 0 \Rightarrow \lambda = \lambda_n = n, \ n = 1, 2, \cdots$$

Thus

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t}\sin nx$$

The initial condition for $w(x,t)$ gives

$$w(x,0) = \sin x + \frac{x^3}{18} - \frac{\pi^2}{18}x = \sum_{n=1}^{\infty} c_n \sin nx.$$

Since '$\sin x$' is already in form of FSS we need only determine the FSS for

$\frac{x^3}{18} - \frac{\pi^2 x}{18}$ i.e.,

$$\frac{x^3}{18} - \frac{\pi^2 x}{18} = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x^3}{18} - \frac{\pi^2 x}{18}\right)\sin nx\,dx$$

Now

$$\frac{1}{9\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{1}{9\pi}\left[-\frac{\pi^3}{n}\cos n\pi + 3x^2\frac{\sin nx}{n^2} - \frac{3}{n}\int_0^{\pi}\frac{\sin nx}{n}\cdot 2x\,dx\right.$$

$$= \frac{1}{9\pi}\left[-\frac{\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n\right]$$

and

$$-\frac{\pi}{9}\int_0^{\pi} x \sin nx\,dx = \frac{\pi^2}{9n}(-1)^n$$

$$\therefore b_n = \frac{1}{9\pi}\left[-\frac{\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n\right] + \frac{\pi^2}{9n}(-1)^n$$

$$= \frac{2}{3}\frac{(-1)^n}{n^3}$$

$$u = -\frac{x^3}{18} + \frac{\pi^2 x}{18} + \sin x\, e^{-3t} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{3n^3} e^{-3n^2 t}\sin nx.$$