Math 201—Assignment 1 Solutions

Section 2.2:

12.
$$x\frac{dv}{dx} = \frac{1-4v^2}{3v}$$
$$\Rightarrow \quad \frac{3v\,dv}{1-4v^2} = \frac{dx}{x}$$
$$\Rightarrow \quad \int \frac{3v\,dv}{1-4v^2} = \ln|x|$$

Substituting $1-4v^2 = w$ we easily get

$$-3/8 \ln|w| = \ln|x| + c_1$$

or taking the exponent of each side

$$1-4v^2 = \pm c x^{-8/3}$$

where $c = e^{c_1} > 0$. We can rewrite it as

$$1-4v^2 = kx^{-8/3}$$

where k is an arbitrary nonzero constant.

$$\ln|W| = \ln|x|^{-8/3} + C_2$$
$$\ln\frac{|W|}{|x|^{-8/3}} = C_2$$
$$|W| = e^{C_2}|x|^{-8/3}$$
$$1-4v^2 = \pm k x^{-8/3}$$

Section 2.3:

10. The standard form becomes

$$\frac{dy}{dx} + 2\frac{y}{x} = x^{-4}$$

$$\mu'(x) = \frac{2\mu(x)}{x}$$

or

$$\mu(x) = x^2$$

Then

$$\frac{d}{dx}(x^2y) = x^{-2}$$
$$x^2y = x^{-1} + C$$

or

$$y = -x^{-3} + Cx^{-2}$$

14 The integrating factor is given by:

$$\mu'(x) = \frac{3\mu}{x}$$

and after integration we find that $\mu(x) = x^3$. Multiply the standard form of the equation by μ to obtain

$$(x^3y)' = -2x^4 + x^5 + 4x^3$$

Integrating both sides we get

$$x^3y = x^6/6 - 2x^5/5 + x^4 + C$$

or

$$y = x^3/6 - 2x^2/5 + x + Cx^{-3}$$

Section 2.4:

10.

$$(2x + y)dx + (x - 2y)dy = M(x, y)dx + N(x, y)dy = 0$$

In order to check the exactness of the equation we compute

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

and it follows that the equation is exact.

$$F(x, y) = \int M(x, y)dx + g(y) = x^2 + xy + g(y)$$

$$\frac{\partial F}{\partial y} = x + g'(y) = N(x, y) = x - 2y$$

Then it follows that $g'(y) = -2y$ or $g(y) = -y^2$ The solution to the equation is given by

$$F(x, y) = x^2 + xy - y^2 = C$$

where C is a constant to be determined from the initial condition.

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$$\partial F/\partial x = ye^{xy} - 1/y$$

and after integration

$$F(x,y) = e^{xy} - x/y + g(y) \tag{1}$$

Differentiating with respect to y we get

$$\partial F/\partial y = xe^{xy} + x/y^2 + g'(y)$$

On the other hand we need that

$$\partial F/\partial y = xe^{xy} + x/y^2$$

and therefore $g'(y) = 0$. Thus we can choose $g(y) = 0$ because any constant can be absorbed in the right hand side constant of the solution which we easily obtain from (1) to be:

$$e^{xy} - \frac{x}{y} = const = C$$

Since we want $y(1) = 1$ it follows that

$$F(1,1) = e - 1 = C$$

and then the solution of the initial-value problem (IVP) is given by

$$e^{xy} - \frac{x}{y} = e - 1$$

Review problems:

18 The equation is of the type: $G(ax + by)$ with $a = 2$, $b = 1$. We substitute $v = 2x + y$ to obtain:

$$\frac{dv}{dx} = 2 + \frac{dy}{dx}$$

and therefore the equation becomes

$$\frac{dv}{dx} = (v-1)^2 + 2$$

which is a separable equation. Then

$$\frac{dv}{(v-1)^2+2} = dx$$

which after one integration yields

$$\frac{1}{\sqrt{2}}\arctan\frac{v-1}{\sqrt{2}} = x + c_1$$

or

$$v = \sqrt{2}\tan(\sqrt{2}x + c), \quad c = \sqrt{2}c_1$$

Substituting $v = 2x + y$ we get

$$y = 1 - 2x + \sqrt{2}\tan(\sqrt{2}x + c)$$

20 The equation is of a Bernoulli type with $n = -2$. Therefore we substitute $v = y^3$ which yields

$$\frac{dy}{d\theta} = 1/3y^{-2}\frac{dv}{d\theta}$$

The equation transforms into

$$1/3y^{-2}\frac{dv}{d\theta} + \frac{y}{\theta} = -4\theta y^{-2}$$

or (substituting $y^3 = v$)

$$\frac{dv}{d\theta} + \frac{3}{\theta}v = -12\theta$$

which is a linear equation. $\mu(x) = \theta^3$ and multiplying by it we get

$$\frac{d}{d\theta}[\theta^3 v] = -12\theta^4$$

which after one integration gives

$$v = -12/5\theta^2 + c/\theta^3$$

or

$$y = -12/5\theta^{2/3} + c_1/\theta$$

28 Some elementary algebra (as shown in p. 77 of the textbook) shows that the substitution $x = u - 2, y = v - 3$ reduces the equation to the homegeneous one:

$$\frac{dv}{du} = \frac{u - v}{u + v}$$

Therefore we substitute $v/u = w$ which yields

$$\frac{dv}{udu} - \frac{v}{u^2} = \frac{dw}{du}$$

Then

$$\frac{dw}{du}u = -w + \frac{1 - w}{1 + w}$$

After some elementary calculations we obtain

$$\frac{1 + w}{1 - 2w - w^2}dw = \frac{du}{u}$$

Integrating once both sides we get

$$\frac{1}{2}\int \frac{dp}{2 - p} = \ln |u| + c$$

where $p = (w + 1)^2$. After computing the integral we get

$$-1/2 \ln |(w + 1)^2 - 2| = \ln |u| + c$$

or after taking the exponent of both sides

$$(w + 1)^2 - 2 = Au^{-2}.$$

After substituting back w, u and v and some elementary algebra we obtain

$$(y + 3)^2 + 2(x + 2)(y + 3) - (x + 2)^2 = A$$

Math 201—Assignment 2 Solutions

Section 2.6:

16. The equation can be put in the form

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right).$$

This is a homogeneous equation and we need to substitute $u = y/x$. Then the equation in terms of u reads

$$x\frac{du}{dx} + u = u(\ln u + 1).$$

This is a separable equation and the separated form is

$$\frac{du}{u \ln u} = \frac{dx}{x}.$$

After one integration we obtain

$$\ln\ln|u| = \ln|x| + c_1$$

and taking the exponents of both sides

$$\ln|u| = cx.$$

This yields

$$y = xe^{cx}.$$

22. We first divide by y^3 and obtain

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Note that while dividing by y we presume that $y \neq 0$ but $y = 0$ is a solution to the initial equation which will not be a solution to the new equation. Therefore, we have to keep it in mind. We further substitute $y^{-2} = u$ and after some simple calculus ($du/dx = -2y^{-3}dy/dx$) obtain

$$\frac{du}{dx} + 2u = -2e^{2x}.$$

This is a linear equation and the integrating factor is $\mu(x) = e^{2x}$. Multiplying the equation with it we obtain

$$\frac{d}{dx}(e^{2x}u) = -2e^{4x}$$

and integrating once, and substituting back $u = y^{-2}$

$$y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}.$$

So the solutions are $y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}$ and $y = 0$.

Section 4.2:

6. The auxiliary equation is $r^2 - 5r + 6 = 0$. Use the quadratic formula to find roots at $r = 2$ and $r = 3$, so

$$y(x) = c_1e^{2x} + c_2e^{3x} \quad .$$

10. The auxiliary equation $4r^2 - 4r + 1$ has a repeated root at $r = 1/2$, so

$$y(x) = (c_1 + c_2x)\, e^{x/2} \quad .$$

18. The auxiliary equation has a repeated root at $r = 3$, so the general solution is

$$y(x) = (c_1 + c_2x)\, e^{3x} \quad .$$

The condition $y(0) = 2$ fixes $c_1 = 2$. Now

$$y'(x) = (3c_1 + c_2 + 3c_2x)\, e^{3x} \quad .$$

so

$$y'(0) = 3c_1 + c_2 = 6 + c_2 \quad .$$

The condition $y'(0) = 25/3$ fixes $c_2 = 7/3$ so the particular solution is

$$y(x) = \left(2 + \frac{7}{3}x\right) e^{3x} \quad .$$

Section 4.3:

6. The auxiliary equation $r^2 - 4r + 7 = 0$ has roots $r = 2 \pm i\sqrt{3}$, so

$$y(x) = e^{2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right) \quad .$$

10. The auxiliary equation $r^2 + 4r + 8 = 0$ has roots $r = -2 \pm 2i$, so

$$y(x) = e^{-2x}\left(c_1 \cos 2x + c_2 \sin 2x\right) \quad .$$

28. The auxiliary equations for $b = 5, 4, 2$ are

$$r^2 + 5r + 4 = 0, \quad r^2 + 4r + 4 = 0, \quad r^2 + 2r + 4 = 0$$

with roots

$r_1 = -1, r_2 = -4;\quad r_1 = r_2 = -2;\quad r_1 = -1 + \sqrt{(3)}i, r_2 = -1 - \sqrt{(3)}i;$

This implies the general solutions

$y(x) = c_1e^{-x} + c_2e^{-4x};\quad y(x) = c_1e^{-2x} + c_2xe^{-2x},\quad y(x) = e^{-x}(c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x))$

Imposing the initial conditions we obtain for the constants c_1, c_2 (in each of the cases considered above):

$$c_1 = 4/3, c_2 = -1/3; c_1 = 1, c_2 = 2; c_1 = 1, c_2 = 3^{-1/2}$$

Math 201 — Solutions to Assignment 3

1. (Sec. 4.4 (18)) *Find a particular solution to:* $y'' + 4y = 8\sin 2t$.

Solution: The auxillary equation to the associated homogeneous equation is $r^2 + 4 = 0$. So $r = \pm 2i$. Then

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

Then a particular solution has the form

$$y_p = At\cos 2t + Bt\sin 2t$$

and

$$\begin{aligned} y_p' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\ &= (A + 2Bt)\cos 2t + (B - 2At)\sin 2t \\ y_p'' &= 2B\cos 2t - 2(A + 2Bt)\sin 2t - 2A\sin 2t + 2(B - 2At)\cos 2t. \end{aligned}$$

Substituting into the differential equation, we obtain

$$\begin{aligned} &(4B - 4At)\cos 2t + (-4A - 4Bt)\sin 2t + 4At\cos 2t + 4Bt\sin 2t = 8\sin 2t \\ \Longrightarrow\ &4B\cos 2t - 4A\sin 2t = 8\sin 2t. \end{aligned}$$

Equating the coefficients of like terms:

$$B = 0 \quad \text{and} \quad -4A = 8 \implies A = -2.$$

Hence,

$$y_p(t) = -2t\cos 2t.$$

2. (Sec. 4.4 (22)) *Find a particular solution to* $x''(t) - 2x'(t) + x(t) = 24t^2e^t$.

Solution: The auxillary equation is $r^2 - 2r + 1 = 0 \implies (r-1)^2 = 0$. So $r = 1, 1$. Then $y_h = c_1e^t + c_2te^t$. So a particular solution has the form

$$x_p = t^2(At^2 + Bt + C)e^t = (At^4 + Bt^3 + Ct^2)e^t$$

and

$$x_p' = (4At^3 + 3Bt^2 + 2Ct)e^t + (At^4 + Bt^3 + Ct^2)e^t$$
$$x_p'' = (12At^2 + 6Bt + 2C)e^t + (4At^3 + 3Bt^2 + 2Ct)e^t$$
$$+ (4At^3 + 3Bt^2 + 2Ct)e^t + (At^4 + Bt^3 + Ct^2)e^t.$$

Then the d.e. becomes

$$\{At^4 + (8A + B)t^3 + (12A + 6B + C)t^2 + (6B + 4C)t + 2C\}e^t$$
$$-2(At^4 + (4A + B)t^3 + (3B + C)t^2 + 2Ct)e^t$$
$$+(At^4 + Bt^3 + Ct^2)e^t = 24t^2e^t.$$

This leads to

$$12At^2e^t + 6Bte^t + 2Ce^t = 24t^2e^t.$$

So

$$12A = 24 \implies A = 2; \quad 6B = 0 \implies B = 0; \quad 2C = 0 \implies C = 0.$$

Then

$$x_p(t) = 2t^4e^t.$$

3. (**Sec. 4.4 (32)**) *Determine the form of a particular solution for* $y'' - y' - 12y = 2t^6e^{-3t}$.

Solution: The auxillary equation is $r^2 - r - 12 = (r-4)(r+3) = 0$. So $r = 4, -3$. Then $y_h = c_1e^{4t} + c_2e^{-3t}$. Then a particular solution has the form

$$y_p = t(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G)e^{-3t}.$$

4. (**Sec. 4.5 (2(b))**) *Given that* $y_1(t) = \frac{1}{4}\sin 2t$ *is a solution to* $y'' + 2y' + 4y = \cos 2t$ *and that* $y_2(t) = \frac{t}{4} - \frac{1}{8}$ *is a solution to* $y'' + 2y' + 4y = t$, *use the Superposition principle to find solution to* $y'' + 2y' + 4y = 2t - 3\cos 2t$.

Solution: By the Superposition principle,

$$y(t) = -3y_1(t) + 2y_2(t)$$
$$= -3\left(\frac{1}{4}\sin 2t\right) + 2\left(\frac{t}{4} - \frac{1}{8}\right) = -\frac{3}{4}\sin 2t + \frac{t}{2} - \frac{1}{4}.$$

5. (Sec. 4.5 (18)) *Find a general solution to* $y'' - 2y' - 3y = 3t^2 - 5$.

Solution: The auxillary equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0$. So $r = 3, -1$. Then

$$y_h(t) = c_1 e^{3t} + c_2 e^{-t}.$$

For a particular solution, we have

$$y_p = At^2 + Bt + C \implies y_p' = 2At + B \implies y_p'' = 2A.$$

Substituting into the differential equation,

$$\begin{aligned}
&2A - 4At - 2B - 3At^2 - 3Bt - 3C = 3t^2 - 5\\
&-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) = 3t^2 - 5\\
\iff &-3A = 3, \quad -4A - 3B = 0, \quad 2A - 2B - 3C = -5\\
\implies &A = -1, \quad B = \frac{4}{3}, \quad C = \frac{1}{9}.
\end{aligned}$$

Therefore,

$$y_p(t) = -t^2 + \frac{4}{3}t + \frac{1}{9},$$

and the general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - t^2 + \frac{4}{3}t + \frac{1}{9}.$$

6. (Sec. 4.5 (26)) *Find the solution to the initial value problem:*

$$y'' + 9y = 27, \quad y(0) = 4, \quad y'(0) = 6.$$

Solution: The auxillary equation is $r^2 + 9 = 0$. So $r = \pm 3i$ and

$$y_h = c_1 \cos 3t + c_2 \sin 3t.$$

Now find a particular solution:

$$y_p = A \implies y_p' = 0 = y_p''.$$

Putting into the d.e., we obtain

$$0 + 9A = 27 \implies A = 3.$$

So the general solution is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + 3.$$

Then

$$y'(t) = -3c_1 \sin 3t + 3C_2 \cos 3t$$

$$y(0) = 4 \implies c_1 + 3 = 4 \implies c_1 = 1$$

$$y'(0) = 6 \implies 3c_2 = 6 \implies c_2 = 2$$

$$\therefore \quad y(t) = \cos 3t + 2 \sin 3t + 3.$$

7. (**Sec. 4.5 (28)**) *Same as #26 except:* $y'' + y' - 12y = e^t + e^{2t} - 1, \quad y(0) = 1,$ $y'(0) = 3.$

Solution: $r^2 + r - 12 = (r+4)(r-3) = 0 \implies r = -4, 3$. So $y_h = c_1 e^{-4t} + c_2 e^{3t}$. Now for y_p:

$$y_p = Ae^t + Be^{2t} + C$$
$$y_p' = Ae^t + 2Be^{2t}$$
$$y_p'' = Ae^t + 4Be^{2t}$$

Substituting into the d.e.,

$$Ae^t + 4Be^{2t} + Ae^t + 2Be^{2t} - 12Ae^t - 12Be^{2t} - 12C = e^t + e^{2t} - 1$$

$$\implies -10Ae^t - 6Be^{2t} - 12C = e^t + e^{2t} - 1$$

$$\implies -10A = 1 \quad \therefore \quad A = -\frac{1}{10}$$

$$-6B = 1 \quad \therefore \quad B = -\frac{1}{6}$$

$$-12C = -1 \quad \therefore \quad C = \frac{1}{12}$$

$$\therefore \quad y_p = -\frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}$$

and the general solution is

$$y = c_1 e^{-4t} + c_2 e^{3t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}.$$

Differentiating, we get

$$y' = -4c_1e^{-4t} + 3c_2e^{3t} - \frac{1}{10}e^t - \frac{1}{3}e^{2t}.$$

Using the initial values,

$$y(0) = c_1 + c_2 - \frac{1}{10} - \frac{1}{6} + \frac{1}{12} = 1 \implies c_1 + c_2 = \frac{71}{60}$$
$$y'(0) = -4c_1 + 3c_2 - \frac{1}{10} - \frac{1}{3} = 3 \implies -4c_1 + 3c_2 = \frac{103}{30}.$$

Solving the 2 equations simultaneously, we obtain

$$c_1 = \frac{1}{60} \quad \text{and} \quad c_2 = \frac{7}{6}.$$

Hence, the solution is

$$y(t) = \frac{1}{60}e^{-4t} + \frac{7}{6}e^{3t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}.$$

8. (Sec. 4.6 (12)) *Find a general solution to $y'' + y = \tan t + e^{2t} - 1$.*

Solution: The auxillary equation is $r^2 + 1 = 0$. Then $r = \pm i$. So $y_h = c_1 \cos t + c_2 \sin t$. Using variation of parameters, we set $y_p = v_1 \cos t + v_2 \sin t$ where v_1 and v_2 are functions to be determined using the equations

$$(1) \qquad (\cos t)v_1' + (\sin t)v_2' = 0$$
$$(2) \qquad -(\sin t)v_1' + (\cos t)v_2' = \tan t + e^{3t} - 1.$$

From eqn (1) $v_1' = -(\sin t)v_2'/\cos t$. Substituting this into (2) results in

$$\frac{\sin^2 t}{\cos t}v_2' + (\cos t)v_2' = \tan t + e^{3t} - 1$$
$$\left(\frac{\overbrace{\sin^2 t + \cos^2 t}^{1}}{\cos t}\right) v_2' = \tan t + e^{3t} - 1$$
$$\therefore \quad v_2' = \cos t(\tan t + e^{3t} - 1)$$
$$= \sin t + e^{3t}\cos t - \cos t$$
$$v_2 = \int (\sin t + e^{3t}\cos t - \cos t)\, dt.$$

We integrate the second expression using integration by parts twice:

$$\begin{aligned}\int e^{3t}\cos t\,dt &= e^{3t}\sin t - 3\int e^{3t}\sin t\,dt\\ &= e^{3t}\sin t - 3\Big(-e^{3t}\cos t + 3\int e^{3t}\cos t\,dt\Big)\\ &= e^{3t}(\sin t + 3\cos t) - 9\int e^{3t}\cos t\,dt\end{aligned}$$

$$\therefore\quad 10\int e^{3t}\cos t\,dt = e^{3t}(\sin t + 3\cos t)$$

$$\therefore\quad \int e^{3t}\cos t\,dt = \frac{1}{10}e^{3t}(\sin t + 3\cos t),$$

$$\therefore\quad v_2 = -\cos t - \sin t + \frac{1}{10}e^{3t}(\sin t + 3\cos t).$$

We now solve for v_1. We have

$$v_1' = -\frac{\sin t}{\cos t}(\sin t + e^{3t}\cos t - \cos t) = -\frac{\sin^2 t}{\cos t} - e^{3t}\sin t + \sin t.$$

Then

$$\begin{aligned}\int \frac{\sin^2 t}{\cos t}\,dt = \int \frac{1-\cos^2 t}{\cos t}\,dt &= \int(\sec t - \cos t)\,dt\\ &= \ln|\sec t + \tan t| - \sin t,\end{aligned}$$

$$\int e^{3t}\sin t\,dt = \frac{1}{10}e^{3t}(3\sin t - \cos t) \quad \text{by intn by parts}$$

$$\int \sin t\,dt = -\cos t.$$

Then putting them all together we get

$$v_1 = \sin t - \ln|\sec t + \tan t| - \frac{1}{10}e^{3t}(3\sin t - \cos t) - \cos t.$$

Since $y_p = (\cos t)v_1 + (\sin t)v_2$, (after cleaning up) we obtain

$$y_p = \frac{1}{10}e^{3t} - 1 - (\cos t)\ln|\sec t + \tan t|.$$

Then the general solution is

$$y(t) = c_1\cos t + c_2\sin t + \frac{1}{10}e^{3t} - 1 - (\cos t)\ln|\sec t + \tan t|.$$

9. (**Sec. 4.6 (18)**) *Find a general solution to* $y'' - 6y' + 9y = t^{-3}e^{3t}$.

Solution: The auxillary equation and the roots are: $r^2 - 6r + 9 = (r-3)^2 = 0, \quad r = 3, 3$. So $y_h = c_1e^{3t} + c_2te^{3t}$. y_p is of the form $y_p = v_1e^{3t} + v_2te^{3t}$. We solve

$$e^{3t}v_1' + te^{3t}v_2' = 0 \tag{1}$$

$$3e^{3t}v_1' + (e^{3t} + 3te^{3t})v_2' = t^{-3}e^{3t}. \tag{2}$$

Do the operation $(2) - 3 \times (1)$ to get

$$e^{3t}v_2' = t^{-3}e^{3t} \implies v_2' = t^{-3} \implies v_2 = -\frac{1}{2t^2}.$$

From eqn (1),

$$e^{3t}v_1' + te^{3t}t^{-3} = 0 \implies v_1' = -t^{-2} \implies v_1 = \frac{1}{t}.$$

Therefore,

$$y_p = \frac{1}{t}e^{3t} - \frac{1}{2t^2}te^{3t} = \frac{1}{t}e^{3t} - \frac{1}{2t}e^{3t} = \frac{1}{2t}e^{3t},$$

and the general solution is

$$y = c_1e^{3t} + c_2te^{3t} + \frac{1}{2t}e^{3t}.$$

Math 201 — Solutions to Assignment 4

1. (Sec. 4.6 (10)) *Find a general solution to:* $y'' + 4y' + 4y = e^{-2t}\ln t$.

Solution: The auxillary equation is $r^2 + 4r + 4 = (r+2)^2 = 0$. So $r = -2, -2$. Then

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t} \qquad \text{and} \qquad y_p = v_1 e^{-2t} + v_2 t e^{-2t} \quad \text{(variation of parameters)}.$$

We solve the equations

$$e^{-2t} v_1' + (te^{-2t}) v_2' = 0 \tag{1}$$

$$-2e^{-2t} v_1' + (e^{-2t} - 2te^{-2t}) v_2' = e^{-2t}\ln t. \tag{2}$$

From eqn (1), $v_1' = -tv_2'$. From eqn (2),

$$\begin{aligned} 2tv_2' + (1-2t)v_2' = \ln t &\implies v_2' = \ln t \\ \implies v_2 = \int \ln t\, dt &= t\ln t - t \end{aligned}$$

(using integration by parts with $u = \ln t,\ du = \frac{1}{t}dt,\ dv = dt,\ v = t$). Using (1) again, we obtain

$$\begin{aligned} v_1' &= -t\ln t \\ v_1 &= -\int t\ln t\, dt = -\left(\frac{t^2}{2}\ln t - \frac{1}{2}\int t\, dt\right) \\ &= -\frac{t^2}{2}\ln t + \frac{1}{4}t^2 \quad (u = \ln t,\ dv = t\,dt,\ u = (1/t)\,dt,\ v = t^2/2), \\ \therefore \quad y_p &= \left(-\frac{t^2}{2}\ln t + \frac{1}{4}t^2\right)e^{-2t} + (t\ln t - t)te^{-2t} \\ &= e^{-2t}t^2\left(\frac{1}{2}\ln t - \frac{3}{4}\right), \end{aligned}$$

and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + t^2 e^{-2t}\left(\frac{1}{2}\ln t - \frac{3}{4}\right).$$

2. **(Sec. 4.6 (14))** *Find a general solution to $y''(\theta) + y(\theta) = \sec^3\theta$.*

Solution: The auxillary equation is $r^2 + 1 = 0 \implies r = \pm i$. So $y_h = c_1\cos\theta + c_2\sin\theta$. So a particular solution has the form $y_p = v_1\cos\theta + v_2\sin\theta$. We have

$$(\cos\theta)v_1' + (\sin\theta)v_2' = 0 \tag{1}$$

$$-(\sin\theta)v_1' + (\cos\theta)v_2' = \sec^3\theta. \tag{2}$$

Then eqn (1) gives

$$v_1' = -\frac{\sin\theta}{\cos\theta}\,v_2',$$

and using eqn (2)

$$\begin{aligned}\frac{\sin^2\theta}{\cos\theta}v_2' + (\cos\theta)v_2' &= \sec^3\theta\\ \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}\,v_2' &= \sec^3\theta\\ \implies v_2' = \sec^2\theta \implies v_2 &= \tan\theta.\end{aligned}$$

This leads to

$$\begin{aligned}v_1' &= -\frac{\sin\theta}{\cos\theta}\cdot\sec^2\theta = -\frac{\sin\theta}{\cos^3\theta},\\ v_1 &= -\int\frac{\sin\theta}{\cos^3\theta}\,d\theta = \int\frac{du}{u^3} \quad (u = \cos\theta)\\ &= \frac{u^{-2}}{-2} = -\frac{1}{2}\sec^2\theta,\\ \therefore \quad y_p &= -\frac{1}{2}\sec^2\theta\cos\theta + \sin\theta\tan\theta = -\frac{1}{2}\sec\theta + \sin\theta\tan\theta,\end{aligned}$$

and the general solution is

$$y = c_1\cos\theta + c_2\sin\theta - \frac{1}{2}\sec\theta + \sin\theta\tan\theta.$$

3. **(Sec. 4.8 (2))** *A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. Neglecting damping, find the equation of motion of the mass along with the amplitude, period and frequency. How long after release does the mass pass through the equilibrium position?*

Solution: The differential equation is $2y'' + 50y = 0$ with $y(0) = -\frac{1}{4}$, $y'(0) = -1$. The auxillary equation is $r^2 + 25 = 0$. So $r = \pm 5i$. Then

$$y(t) = c_1 \cos 5t + c_2 \sin 5t \implies y'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t.$$

Using the initial conditions we obtain

$$c_1 = -\frac{1}{4}, \quad c_2 = -\frac{1}{5} \implies y = -\frac{1}{4}\cos 5t - \frac{1}{5}\sin 5t.$$

We can also put it in an alternate form using $A = \sqrt{c_1^2 + c_2^2} = \sqrt{41}/20$ and $\tan\varphi = c_1/c_2 = 5/4$ where φ is in quadrant III, i.e., $\varphi = -\pi + \tan^{-1}(5/4)$. Hence,

$$y(t) = \frac{\sqrt{41}}{20}\sin(5t + \varphi).$$

Then the amplitude is $\sqrt{41}/20$, period is $2\pi/5$ and frequency $5/2\pi$.

To find the time the mass passes through the equilibrium posiiton, set $y = 0$, i.e.,

$$\begin{aligned}
\sin(5t + \varphi) &= 0 \\
5t - \pi + \tan^{-1}\left(\frac{5}{4}\right) &= n\pi, \quad (n \text{ is an integer}) \\
t &= \frac{1}{5}(\pi - \tan^{-1}(5/4) + n\pi) = \frac{1}{5}((n+1)\pi - \tan^{-1}(5/4)) \\
\therefore \quad t &= \frac{1}{5}(\pi - \tan^{-1}(5/4)) \approx 0.449 \quad (\text{with } n = 0).
\end{aligned}$$

4. **(Sec. 4.8 (10))** *A 1/4-kg mass is attached to a spring with stiffness 8 N/m. The damping constant for the system is 1/4 N-sec/m. If the mass is moved 1 m to the left of equilibrium and released, what is the maximum displacement to the right that it will attain?*

Solution: The equation is $\frac{1}{4}y'' + \frac{1}{4}y' + 8y = 0$, i.e., $y'' + y' + 32y = 0$ with $y(0) = -1$ and $y'(0) = 0$ The auxillary equation is $r^2 + r + 32 = 0$ where

$$r = \frac{-1 \pm \sqrt{1 - 128}}{2} = \frac{-1 \pm i\sqrt{127}}{2}.$$

Then

$$y(t) = e^{-\frac{1}{2}t}\left(c_1 \cos\frac{\sqrt{127}}{2}t + c_2 \sin\frac{\sqrt{127}}{2}t\right),$$

and

$$y'(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 \cos\frac{\sqrt{127}}{2}t + c_2 \sin\frac{\sqrt{127}}{2}t\right)$$
$$+ e^{-\frac{1}{2}t}\left(-\frac{\sqrt{127}}{2}c_1 \sin\frac{\sqrt{127}}{2}t + \frac{\sqrt{127}}{2}c_2 \cos\frac{\sqrt{127}}{2}t\right).$$

Using $y(0) = -1$ and $y'(0) = 0$ we obtain

$$c_1 = -1 \quad \text{and} \quad c_2 = -\frac{1}{\sqrt{127}}.$$

Hence the solution is

$$y = e^{-\frac{1}{2}t}\left(-\cos\frac{\sqrt{127}}{2}t - \frac{1}{\sqrt{127}}\sin\frac{\sqrt{127}}{2}t\right).$$

We now find y' to determine the critical points.

$$\begin{aligned}
y' &= -\frac{1}{2}e^{-\frac{1}{2}t}\left(-\cos\frac{\sqrt{127}}{2}t - \frac{1}{\sqrt{127}}\sin\frac{\sqrt{127}}{2}t\right) \\
&\qquad + e^{-\frac{1}{2}t}\left(\frac{\sqrt{127}}{2}\sin\frac{\sqrt{127}}{2}t - \frac{1}{2}\cos\frac{\sqrt{127}}{2}t\right) \\
&= -\frac{1}{2}e^{-\frac{1}{2}t}\left(-\frac{1}{\sqrt{127}}\sin\frac{\sqrt{127}}{2}t - \sqrt{127}\sin\frac{\sqrt{127}}{2}t\right) \\
&= \frac{\sqrt{127}}{2}e^{-\frac{1}{2}t}\sin\frac{\sqrt{127}}{2}t = 0 \\
\iff \frac{\sqrt{127}}{2}t = n\pi &\iff t = \frac{2\pi}{\sqrt{127}} \quad (n = 1).
\end{aligned}$$

(We can check that y is a maximum at $t = 2\pi/\sqrt{127}$ using the first derivative test.) So the maximum displacement is

$$y\left(\frac{2\pi}{\sqrt{127}}\right) = e^{-\frac{\pi}{\sqrt{127}}} \approx 0.755.$$

5. **(Sec. 4.9 (4))** *Determine the equation of motion for an undamped system at resonance governed by*

$$y'' + y = 5\cos t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: The auxillary equation is $r^2 + 1 = 0$, so $r = \pm i$ and $y_h = c_1 \cos t + c_2 \sin t$. A particular solution is of the form:

$$y_p = At\cos t + Bt\sin t$$

and

$$
\begin{aligned}
y_p' &= A\cos t - At\sin t + B\sin t + Bt\cos t = (A+Bt)\cos t + (B-At)\sin t \\
y_p'' &= B\cos t - (A+Bt)\sin t - A\sin t + (B-At)\cos t \\
&= (2B - At)\cos t - (2A+Bt)\sin t \\
\Longrightarrow &\ (2B-At)\cos t + (-2A-Bt)\sin t + At\cos t + Bt\sin t = 5\cos t \\
\Longrightarrow &\ 2B\cos t - 2A\sin t = 5\cos t.
\end{aligned}
$$

Therefore,

$$2B = 5 \Longrightarrow B = \frac{5}{2} \qquad \text{and} \qquad -2A = 0 \Longrightarrow A = 0.$$

Then $y_p = \frac{5}{2}t\sin t$ and hence

$$y(t) = c_1\cos t + c_2\sin t + \frac{5}{2}t\sin t$$

so that

$$
\begin{aligned}
&y'(t) = -c_1\sin t + c_2\cos t + \frac{5}{2}\sin t + \frac{5}{2}t\cos t \\
&y(0) = 0 \Longrightarrow c_1 = 0 \quad \text{and} \quad y'(0) = 1 \Longrightarrow c_2 = 1 \\
&\therefore \quad y(t) = \sin t + \frac{5}{2}t\sin t = \left(1 + \frac{5}{2}t\right)\sin t.
\end{aligned}
$$

6. (Sec. 7.2 (12)) *Use Definition 1 to determine the Laplace transform of*

$$f(t) = \begin{cases} e^{2t}, & 0 < t < 3 \\ 1, & t > 3. \end{cases}$$

Solution:

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)\,dt = \int_0^3 e^{-st}e^{2t}\,dt + \int_3^\infty e^{-st}\,dt \\
&= \int_0^3 e^{-(s-2)t}\,dt + \lim_{b\to\infty}\int_3^b e^{-st}\,dt = -\frac{1}{s-2}e^{-(s-2)t}\Big|_0^3 + \lim_{b\to\infty}\left(-\frac{1}{s}e^{-st}\Big|_3^b\right. \\
&= -\frac{1}{s-2}e^{-(s-2)3} + \frac{1}{s-2} + \lim_{b\to\infty}\Big(\underbrace{-\frac{1}{s}e^{-sb}}_{\to 0\ (s>0)} + \frac{1}{s}e^{-3s}\Big) \\
&= \frac{1}{s-2}\left(1 - e^{-3(s-2)}\right) + \frac{e^{-3s}}{s}, \qquad s > 2.
\end{aligned}
$$

7. (**Sec. 7.2 (16)**) *Use the Laplace transform table and the linearity of the Laplace transform to determine* $\mathcal{L}\{f(t)\}$.

Solution:

$$\begin{aligned}\mathcal{L}\{t^2 - 3t - 2e^{-t}\sin 3t\} &= \mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} - 2\mathcal{L}\{e^{-t}\sin 3t\} \\ &= \frac{2!}{s^3} - 3\left(\frac{1}{s^2}\right) - 2\left(\frac{3}{(s+1)^2+9}\right) \\ &= \frac{2}{s^3} - \frac{3}{s^2} - \frac{6}{(s+1)^2+9}, \quad s > 0.\end{aligned}$$

8. (**Sec. 7.2 (20)**) *Same as #16*

Solution:

$$\begin{aligned}\mathcal{L}\{e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}\} &= \mathcal{L}\{e^{-2t}\cos\sqrt{3}t\} - \mathcal{L}\{t^2e^{-2t}\} \\ &= \frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}, \quad s > -2.\end{aligned}$$

Solutions to assignment 5, Math 201, winter of 2005.

§7.5 #10 $y''-4y=4t-8e^{-2t}$, $y(0)=0$, $y'(0)=5$.

Apply the Laplace transform to the equation

$\mathcal{L}\{y''\}-4\mathcal{L}\{y\}=4\mathcal{L}\{t\}-8\mathcal{L}\{e^{-2t}\}$

Use $Y(s)=\mathcal{L}\{y\}$, from table 7.2, page 364, $\mathcal{L}\{y''\}=s^2Y-5$ Also from table 7.1, page 358, $\mathcal{L}\{t\}=\frac{1}{s^2}$, $\mathcal{L}\{e^{-2t}\}=\frac{1}{s+2}$.

Thus $(s^2-4)Y=5+\frac{4}{s^2}-\frac{8}{s+2}$ and

$$Y=\frac{5}{s^2-4}+\frac{4}{s^2(s^2-4)}-\frac{8}{(s+2)(s^2-4)}=$$

$$\frac{2}{(s+2)^2}-\frac{1}{s+2}+\frac{1}{s-2}-\frac{1}{s^2}$$, where

we used that $\frac{5}{s^2-4}=\frac{5}{4(s-2)}-\frac{5}{4(s+2)}$,

$\frac{4}{s^2(s^2-4)}=-\frac{1}{s^2}+\frac{1}{4(s-2)}-\frac{1}{4(s+2)}$ and

$$-\frac{8}{(s+2)(s^2-4)}=-\frac{1}{2(s-2)}+\frac{1}{2(s+2)}+\frac{2}{(s+2)^2}$$

$$y=\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2}-\frac{1}{s+2}+\frac{1}{s-2}-\frac{1}{s^2}\right\}$$

$$=e^{2t}+(2t-1)e^{-2t}-t$$

§7.2 #10. $f = \begin{cases} 1-t, & 0<t<1 \\ 0, & t>0. \end{cases}$

$$\mathcal{L}\{f\} = \int_0^t f(t)e^{-st}dt = \int_0^1 (1-t)e^{-st}dt =$$

$$e^{-st}\left[-\frac{1}{s} + \frac{t}{s} + \frac{1}{s^2}\right]\Bigg|_0^1 = \frac{e^{-s}}{s^2} + \frac{1}{s} - \frac{1}{s^2}$$

§7.3 #10 According to table 7.1 page 358

$\mathcal{L}\{e^{2t}\cos 5t\} = \frac{s-2}{(s-2)^2+25}$. Using theorem 6, page 363, $\mathcal{L}\{te^{2t}\cos 5t\} = -\frac{d}{ds}\left[\frac{s-2}{(s-2)^2+25}\right]$

$$= \frac{(s-2)^2-25}{[(s-2)^2+25]^2}$$

§7.3 #20. $\mathcal{L}\{\sin 2t \cos 5t\} = \frac{1}{2}\mathcal{L}\{\sin 7t - \sin 3t\}$

$$= \frac{1}{2}\,\frac{7}{s^2+49} - \frac{1}{2}\,\frac{3}{s^2+9}$$

Using theorem 6, page 363 $\mathcal{L}\{t\sin 2t\cos 5t\}$

$$= -\frac{d}{ds}\left\{\frac{1}{2}\,\frac{7}{s^2+49} - \frac{1}{2}\,\frac{3}{s^2+9}\right\} =$$

$$= \frac{7s}{(s^2+49)^2} - \frac{3s}{(s^2+9)^2}$$

§7.3 # 20, **CORRECTED.**

$$\mathcal{L}\{\sin 2t \sin 5t\} = \frac{1}{2}\mathcal{L}\{\cos 3t - \cos 7t\}$$

$$= \frac{1}{2}\frac{s}{s^2+9} - \frac{1}{2}\frac{s}{s^2+49}$$

Using theorem 6, page 363

$$\mathcal{L}\{t \sin 2t \cos 5t\} =$$

$$= -\frac{d}{ds}\left\{\frac{1}{2}\frac{s}{s^2+9} - \frac{1}{2}\frac{s}{s^2+49}\right\}$$

$$= \frac{1}{2}\frac{49-s^2}{(s^2+49)^2} - \frac{1}{2}\frac{9-s^2}{(s^2+9)^2}$$

§7.4 #10. $\mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{2\left[\left(s+\frac{1}{4}\right)^2+\frac{47}{16}\right]}\right\}$

$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}\right\} - \frac{5}{8}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{47}{16}}\right\}$

Using table 7.1, page 358 this equals to

$\frac{1}{2}e^{-\frac{1}{4}t}\cos\frac{\sqrt{47}}{4}t - \frac{5}{2\sqrt{47}}e^{-\frac{1}{4}t}\sin\frac{\sqrt{47}}{4}t$.

§7.4 #20 $\frac{s}{(s-1)(s^2-1)} = \frac{s}{(s-1)^2(s+1)} =$

$\frac{1}{2(s-1)^2} + \frac{1}{4(s-1)} - \frac{1}{4(s+1)}$

§7.4 #30 $F(s) = \frac{2s+5}{(s-1)(s^2+2s+1)} = \frac{2s+5}{(s-1)(s+1)^2}$

$= -\frac{3}{2(s+1)^2} + \frac{7}{4(s-1)} - \frac{7}{4(s+1)}$

$\mathcal{L}^{-1}\{F\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$-\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -\frac{3}{2}te^{-t} + \frac{7}{4}e^{t} - \frac{7}{4}e^{-t}$

§7.5 #20 $y'' + 3y = t^3$, $y(0) = y'(0) = 0$.

Apply the Laplace transform to the equation

$\mathcal{L}\{y''\} + 3\mathcal{L}\{y\} = \mathcal{L}\{t^3\}$ Use $Y(s) = \mathcal{L}\{y\}$,

$\mathcal{L}\{y''\} = s^2 Y$, $\mathcal{L}\{t^3\} = \frac{6}{s^4}$. Then

$$(s^2+3)Y = \frac{6}{s^4}$$

$$Y = \frac{6}{s^4(s^2+3)} = \frac{2}{3(s^2+3)} + \frac{2}{s^4} - \frac{2}{3s^2}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{3(s^2+3)} + \frac{2}{s^4} - \frac{2}{3s^2}\right\}$$

$$= -\frac{2}{3}t + \frac{2}{3\sqrt{3}}\sin\sqrt{3}t + \frac{t^3}{3}.$$

Solutions to assignmen 6.

§7.6 #6. $g(t) = u(t-2)(t+1)$

$$\mathcal{L}\{g\} = \int_2^{+\infty} (t+1) e^{-st} dt = \int_0^{+\infty} (\tau+3) e^{-s(\tau+2)} d\tau$$

Where $\tau = t-2$ or $t = \tau + 2$. Furthermore

$$\mathcal{L}\{g\} = e^{-2s} \int_0^{+\infty} \tau e^{-s\tau} d\tau + 3\, e^{-2s} \int_0^{+\infty} e^{-s\tau} d\tau$$

From table 7.1, page 358,

$$\int_0^{+\infty} \tau e^{-s\tau} d\tau = \frac{1}{s^2}, \quad \int_0^{+\infty} e^{-s\tau} d\tau = \frac{1}{s}.$$ Thus

$$\mathcal{L}\{g\} = \frac{e^{-2s}}{s^2} + \frac{3e^{-2s}}{s}$$

§7.6 #14 By formula (5) of theorem 8, page 387, $\frac{e^{-3s}}{s^2+9}$ is the Laplace transform of $u(t-3) f(t-3)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin t$. Thus $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+9}\right\} =$

$$= \frac{1}{3}\sin(t-3).$$

§7.6 #26 $\mathcal{L}\{f\} = \frac{1}{1-e^{-sa}} \int_0^a \frac{t}{a} e^{-st}\, dt$,

where formula (12) of theorem 9, page 391 was used. The eq-n of $f(t)$ on $[0, a]$ is $f(t) = \frac{1}{a} t$. Computing the integral

$$\mathcal{L}\{f\} = \frac{1}{a(1-e^{-sa})} \int_0^a t e^{-st}\, dt$$

Integrating by parts

$$\int_0^a t e^{-st}\, dt = -\left(\frac{t}{s} + \frac{1}{s^2}\right) e^{-st} \Big|_0^a$$

$$= \frac{1}{s^2} - \left(\frac{a}{s} + \frac{1}{s^2}\right) e^{-sa}$$

Thus

$$\mathcal{L}\{f\} = \frac{1}{a(1-e^{-sa})} \cdot \left[\frac{1}{s^2} - \left(\frac{a}{s} + \frac{1}{s^2}\right) e^{-sa}\right]$$

$$= \frac{1}{a(e^{sa}-1)} \left[\frac{1}{s^2} e^{sa} - \left(\frac{a}{s} + \frac{1}{s^2}\right)\right]$$

$$= \frac{e^{sa} - as - 1}{as^2(e^{sa}-1)}$$

§7.7. #4 $y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 1$

Apply Laplace transform to the eq-n:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{g\}$$

$$Y = \mathcal{L}\{y\}, \quad \mathcal{L}\{y''\} = s^2 Y - 1, \quad G = \mathcal{L}\{g\}$$

$$s^2 Y - 1 + Y = G \Rightarrow Y(1 + s^2) = G + 1$$

$$Y = \frac{1}{s^2+1} + \frac{1}{1+s^2} G$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}\left\{\frac{1}{1+s^2} \cdot G\right\}$$

$$= \sin t + \int_0^t \sin(t-v) g(v)\, dv$$

§7.7. #20 $y' + \int_0^t (t-v) y(v)\, dv = t, \quad y(0) = 0$

Apply Laplace transform: $\mathcal{L}\{y'\} + \mathcal{L}\{t * y\} = \mathcal{L}\{t\}$

$$\mathcal{L}\{y\} = Y, \quad \mathcal{L}\{y'\} = sY, \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$sY + \frac{1}{s^2} \cdot Y = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^3+1}$$

$$Y = \frac{1}{3(s+1)} + \frac{1}{2\left[(s-\frac{1}{2})^2 + \frac{3}{4}\right]} - \frac{s - \frac{1}{2}}{3\left[(s-\frac{1}{2})^2 + \frac{3}{4}\right]}$$. Using

$\mathcal{L}^{-1}\left\{\frac{1}{3(s+1)}\right\} = \frac{1}{3} e^{-t}$ and table 7.1, page 358

$$y = \frac{1}{3} e^{-t} + \frac{1}{\sqrt{3}} e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t - \frac{1}{3} e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t$$

§7.6#32 $y''+y = 3\sin 2t - 3(\sin 2t)u(t-2)$,

$y(0)=1,\ y'(0)=-2$.

Apply Laplace transform to the eq-n

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\} = 3\mathcal{L}\{\sin 2t\} - 3\mathcal{L}\{(\sin 2t)u(t-2)\}$$

$Y=\mathcal{L}\{y\}$, $\mathcal{L}\{y''\} = s^2 - s + 2$

$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, $\mathcal{L}\{(\sin 2t)u(t-2\pi)\} = \frac{2e^{-2\pi s}}{s^2+4}$

$$s^2Y - s + 2 + Y = \frac{6}{s^2+4} - \frac{6}{s^2+4}e^{-2\pi s}$$

$$Y = -\frac{2-s}{s^2+1} + \frac{6}{(s^2+4)(s^2+1)} - \frac{6}{(s^2+4)(s^2+1)}e^{-2\pi s}$$

$$= \frac{s}{s^2+1} - \frac{2}{s^2+4} - \left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right]e^{-2\pi s}$$

According to table 7.1, page 358

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1} - \frac{2}{s^2+4}\right\} = \cos t - \sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+1} - \frac{2}{s^2+4}\right\} = 2\sin t - \sin 2t$$

By theorem 8, page 387

$$\mathcal{L}^{-1}\left\{\left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right]e^{-2\pi s}\right\} = (2\sin t - \sin 2t)u(t-2\pi)$$

Thus

$$y = \cos t - \sin 2t - (2\sin t - \sin 2t)u(t-2\pi)$$