COMMON FINAL
MATH - 115 (Winter - 2007)

COORDINATES - MATERIAL TO BE COVERED
DRAFT VERSION

COORDINATES : Common Final, Saturday April 28, 2007 at 9:00 a.m. in Pavilion, Odd Rows 1 - 23 (2.0 hours long).

Use of books, lecture notes, (mathematical) tables, laptops, calculators and formula sheets will not be allowed. (No aids allowed.)

MATERIAL TO BE COVERED : Material covered in LECTURES, (see the following sections of your 5th edition Calculus textbook), and material in the Homework-Assignments (if any) :

Ch. 5 : sections 5.1 to 5.5 inclusive,
Ch. 6 : sections 6.1 to 6.4 inclusive,
Ch. 7 : sections 7.1 to 7.7 inclusive,
Ch. 8 : sections 8.1 to 8.5 inclusive, 8.7, 8.8,
Ch. 9 : 9.1, 9.2,
Ch. 10 : 10.3, 10.4.

A NOTE ON FORMULAS :

For definitions and derivatives of inverse trigonometric and inverse hyperbolic functions, one can limit his or her study to the definitions and derivatives of :

inverse sine ($\sin^{-1}(x)$)
inverse cosine ($\cos^{-1}(x)$)
inverse tangent ($\tan^{-1}(x)$)
inverse hyperbolic sine ($\sinh^{-1}(x)$)
inverse hyperbolic cosine ($\cosh^{-1}(x)$)
inverse hyperbolic tangent ($\tanh^{-1}(x)$)

Also, a mention/reminder that one should know the (following) double angle formulas for the sine and cosine functions :

$$\sin(2x) = 2\sin(x)\cos(x), \quad \cos(2x) = \cos^2(x) - \sin^2(x)$$

Final Exam Math 115, Q1

INSTRUCTOR: Alexander Litvak

DATE: April 15, 1998

TIME: 9.00 - 11.00

INSTRUCTIONS:
There are 7 problems. **Explain** your answer. Show **all** your work for each question. Follow the regulations given in the examination booklet. Do not speak to or communicate with other students. Calculators, textbooks, memoranda are not allowed.

1. (15 pts.) Find the centroid of the region bounded by the curve:

$$y = x^{1/2} \cdot \sqrt{2-x} \qquad (0 \le x \le 2).$$

2. (10 pts.) Find the area of the surface obtained by rotating the curve $y = x^3/3$, $0 \le x \le 1$, about the x-axis.

3. (10 pts.) Find the length of the curve $y = \ln x$ between the points $(1,0)$ and $(e,1)$.

4. (20 pts.) Find the limits

$$\lim_{x\to 0} \frac{\tan x - \arctan(2x)}{x + \arcsin(2x)}; \qquad \lim_{x\to 1^+} \left(\frac{1}{\ln(1/x)} + \frac{1}{x-1} \right);$$

$$\lim_{x\to 0} (\sin x)^x, \qquad \lim_{x\to\infty} \left(\frac{e^{-x} - 1/x}{e^{-x} + 1} \right).$$

5. (10 pts.) Determine whether integral $\int_1^\infty \frac{|\sin(x)|}{x^2}\, dx$ is convergent or divergent. Give your reason.

6. (20 pts.) Evaluate the following integrals:

$$\int x^5 \sin(x^6) \cos(x^6) dx; \qquad \int \frac{x+1}{x(x^2+2x+2)} dx;$$

$$\int \frac{x dx}{\sqrt{1-6x-x^2}}; \qquad \int \sin(\ln(x))\, dx.$$

7. (15 pts.) Solve the following differential equation

$$\left(\sin y + (\sin y)^2 \right) \cdot y' = \cos y \cdot \ln x$$

Final Exam Math 115, S1

INSTRUCTOR: Alexander Litvak

DATE: April 26, 1999

TIME: 14.00 - 16.00

INSTRUCTIONS:
There are 6 problems. **Explain** your answer. Show **all** your work for each question. Follow the regulations given in the examination booklet. Do not speak to or communicate with other students. Calculators, textbooks, memoranda are not allowed.

1. (14 pts.) Solve the following differential equation

$$y \cdot e^x \cdot y' = \frac{1}{\ln y}.$$

2. (12 pts.) Find the area of the surface obtained by rotating the curve

$$y = 2x^3 , \qquad 0 \le x \le \sqrt[4]{2/9} ,$$

about the x-axis.

3. (10 pts.) Find the length of the curve

$$y = \frac{x^2}{4} - \frac{\ln x}{2} ,$$

between the points $(1,\ 1/4)$ and $(e,\ \frac{e^2}{4} - \frac{1}{2})$.

4. (24 pts.) Determine whether integrals

$$\int_{-\infty}^{0} xe^x \, dx; \qquad \int_0^{\pi/2} \tan x \, dx; \qquad \int_0^{\infty} \frac{|\sin x|}{x^{3/2}} \, dx$$

are convergent or divergent. Give your reason.

5. (24 pts.) Evaluate the following integrals:

$$\int \frac{1}{(x-1)(\sqrt[3]{x-1}+2)} \, dx ; \qquad \int \frac{dx}{7 + 5\sin x + 5\cos x} ; \qquad \int x^2 \cos x \, dx .$$

6. (16 pts.) Evaluate the following integrals:

$$\int \frac{5x^2 - 6x - 1}{(x+1)^2(x^2 - 4x + 5)} \, dx ; \qquad \int \frac{x+1}{\sqrt{-4x - x^2}} \, dx .$$

PREVIOUS FINAL EXAMINATION (MATH-115)

1. Find y' (or $\dfrac{dy}{dx}$) if : $y = (x)^{(\ln(x))}$
(you do not need to simplify your answer)

2. Find y' (or $\dfrac{dy}{dx}$) by implicit differentiation if :

$$x^2 - x\sinh(y) + (\sinh(y))^3 = 8$$

(y is assumed to be a differentiable function of x.)

3. Let g be the inverse function of f, with $f(2) = 10$, and $f'(2) = 20$, find $g'(10)$.

4. Find the area of the region bounded by the curves : $y = x^2 - 1$, and $y = 1 - x^2$.

5. Evaluate each of the following integrals :

(a) $\int_1^4 \ln(\sqrt{x})dx$

(b) $\int \dfrac{1}{x^4 - 1}\,dx$

6. Evaluate each of the following integrals :

(a) $\int \dfrac{1}{(x + x^{(3/4)})}\,dx$

(b) $\int \dfrac{1}{\sqrt{2x - x^2}}\,dx$

(c) $\int \sqrt{\dfrac{1+x}{1-x}}\,dx$

7. Evaluate each limit below, if the limit exists, or show that the limit does not exist.

(a) $\lim\limits_{x \to 0} \left(\dfrac{\sin(2x)}{\sinh(3x)}\right)$

(b) $\lim\limits_{x \to \infty} (x - \sqrt{x^2 - 1})$

(c) $\lim\limits_{x \to \infty} \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x$

8. Determine whether the integral is convergent or divergent for each of the following integrals. Evaluate the integral if it is convergent, or show that the integral is divergent.

(a) $\int_0^\infty \cos(x)dx$

(b) $\int_0^\infty xe^{(-x^2)}dx$

(c) $\int_0^2 \dfrac{1}{2x-3}dx$

9. Use the Comparison Theorem to determine whether the following integral is convergent or divergent :

$$\int_1^\infty \frac{1}{\sqrt{x^3+1}}\, dx$$

10. Use the Midpoint Rule with $n = 4$ subintervals to approximate the integral :

$$\int_{0.5}^{4.5} x^3\, dx$$

11. Find the length of the curve : $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$, for $1 \leq x \leq 2$.

12. The ellipse with equation :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants such that $a > b$, is rotated about the x-axis to form a surface called an ellipsoid. Find the surface area of this ellipsoid.

MATHEMATICS 115 (Q1)

PRACTICE PROBLEMS—FINAL EXAM (2 hours)

1. Answer the following questions.

   (a) Evaluate $\csc(\tan^{-1}\sqrt{3})$.

   (b) Find y' if $y = (\tanh x)^{\ln x}$.

   (c) Find y' if: $e^{x+y} = \sin(x - x^3)$.

   (d) Solve for x if: $\ln\left(\frac{x-2}{x-1}\right) = 1 + \ln\left(\frac{x-3}{x-1}\right)$.

   (e) Set up the integral (simplify it) for the surface area generated when the arc of the curve $y = e^{-x}$, $0 \le x \le 2$ is rotated about the x-axis.

2. Evaluate the following limits.

   (a) $\lim_{x\to\infty} x\big(\ln(x+3) - \ln(x-3)\big)$ (b) $\lim_{x\to 0^+} x^{\sin x}$ (c) $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

3. Evaluate the following integrals.

   (a) $\int_0^{\frac{\pi}{3}} \cos^4 x\, dx$ (b) $\int_1^4 \frac{\sqrt{x}}{\sqrt{x}+2}\, dx$

   (c) $\int \sqrt{e^{2x} - 9}\, dx$ (d) $\int \frac{x^2 - x + 1}{x^3 + 2x}\, dx$

4. Determine whether the following improper integrals converge or diverge.

   (a) $\int_1^{\infty} \frac{\ln x}{x^2}\, dx$

   (b) $\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx$

   (c) $\int_1^{\infty} \frac{x^2}{\sqrt{x^5 - \frac{1}{2}}}\, dx$ (Use the Comparison Theorem for improper integrals.)

5. Solve the following initial value problem. Solve for y explicitly.

$$\frac{dy}{dx} = -\frac{5x}{y\sqrt{x^2+1}}, \qquad y(0) = -1$$

6. (6 points) Find the arc length of the curve

$$y = \ln(1 - x^2), \qquad 0 \le x \le \frac{1}{3}.$$

MATHEMATICS 115 (Q1)

SOLUTIONS TO PRACTICE PROBLEMS (FINAL EXAM)

1. (a) Let $\theta = \tan^{-1}\sqrt{3}$. Then $\tan\theta = \sqrt{3}$. So $\theta = \frac{\pi}{3}$. Using the right triangle with $\sqrt{3}$ as the side opposite of the angle θ, we get $\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

(b) Taking natural logarithm of the function

$$\ln y = (\ln x)\ln(\tanh x) \implies \frac{1}{y}y' = \frac{1}{x}\ln(\tanh x) + \frac{\ln x}{\tanh x}(\operatorname{sech}^2 x)$$
$$y' = y\left(\frac{1}{x}\ln(\tanh x) + \frac{(\ln x)(\operatorname{sech}^2 x)}{\tanh x}\right).$$

(c) Differentiating implicitly,

$$e^{x+y}(1+y') = [\cos(x-x^3)](1-3x^2) \implies y' = \frac{(1-3x^2)\cos(x-x^3) - e^{x+y}}{e^{x+y}}.$$

(d)

$$\ln\left(\frac{x-2}{x-1}\cdot\frac{x-1}{x-3}\right) = 1 \implies \frac{x-2}{x-3} = e \implies x-2 = e(x-3)$$
$$\implies (1-e)x = 2-3e \implies x = \frac{2-3e}{1-e}.$$

(e) $y' = -e^{-x}$. So the surface area is

$$S = 2\pi\int_0^2 y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^2 e^{-x}\sqrt{1+e^{-2x}}\,dx = 2\pi\int_0^2 e^{-x}\sqrt{\frac{e^{2x}+1}{e^{2x}}}\,dx = 2\pi\int_0^2 \frac{\sqrt{e^{2x}+1}}{e^{2x}}\,dx.$$

2. (a) Has the form $\infty - \infty$. Rewrite as follows:

$$\lim_{x\to\infty} x(\ln(x+3)-\ln(x-3)) = \lim_{x\to\infty} x\ln\left(\frac{x+3}{x-3}\right) = \lim_{x\to\infty}\frac{\ln(x+3)-\ln(x-3)}{1/x} \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x+3}-\frac{1}{x-3}}{-1/x^2}$$
$$= \lim_{x\to\infty}\frac{-6}{x^2-9}\cdot(-x^2) = \lim_{x\to\infty}\frac{6}{1-\frac{9}{x^2}} = 6.$$

(b) Has the form 0^0. Let $y = x^{\sin x}$ so that $\ln y = (\sin x)\ln x$. Then

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}(\sin x)\ln x \qquad (0\cdot(-\infty))$$
$$= \lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} = -\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x} \qquad (0/0)$$
$$= -\lim_{x\to 0^+}\frac{2\sin x\cos x}{\cos x - x\sin x} = \frac{0}{1+0} = 0.$$
$$\therefore \quad \lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^0 = 1.$$

(c) Has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\left(\frac{x-1-\ln x}{(\ln x)(x-1)}\right) \overset{\text{l'H}}{=} \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$$
$$= \lim_{x\to 1}\frac{x-1}{x}\cdot\frac{x}{x-1+x\ln x} = \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} \overset{\text{l'H}}{=} \lim_{x\to 1}\frac{1}{1+\ln x+1} = \frac{1}{1+0+1} = \frac{1}{2}.$$

3. (a) Let I be the integral.

$$I = \int_0^{\frac{\pi}{3}} (\cos^2 x)^2\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} (1 + 2\cos 2x + \underbrace{\cos^2 2x}_{\frac{1}{2}(1+\cos 4x)})\,dx = \frac{1}{4}\int_0^{\frac{\pi}{3}}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right\vert_0^{\frac{\pi}{3}} = \frac{1}{4}\left(\frac{3}{2}\frac{\pi}{3} + \sin\frac{2\pi}{3} + \frac{1}{8}\sin\frac{4\pi}{3}\right)$$
$$= \frac{1}{4}\left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{16}\right) = \frac{1}{4}\left(\frac{\pi}{2} + \frac{7\sqrt{3}}{16}\right).$$

(b) Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$. Then

$$\int_1^4 \frac{\sqrt{x}}{\sqrt{x}+2}\,dx = \int_1^2 \frac{2u^2\,du}{u+2} = 2\int_1^2\left(u - 2 + \frac{4}{u+2}\right)du = 2\left[\frac{u^2}{2} - 2u + 4\ln(u+2)\right\vert_1^2$$
$$= 4 - 8 + 8\ln 4 - 1 + 4 - 8\ln 3 = 8\ln 4 - 8\ln 3 - 1 = 8\ln\frac{4}{3} - 1.$$

(c) Let $u = e^x$, $du = e^x\,dx$ so that $du/u = dx$. Then

$$\int \sqrt{e^{2x}-9}\,dx = \int \frac{\sqrt{u^2-9}}{u}\,du = \int \frac{\sqrt{9(\sec^2\theta - 1)}}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \quad (u = 3\sec\theta,\ du = \sec\theta\tan\theta\,d\theta)$$
$$= 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)d\theta = 3(\tan\theta - \theta) + C$$
$$= 3\left(\frac{\sqrt{u^2-9}}{3} - \sec^{-1}\frac{u}{3}\right) + C = \sqrt{e^{2x}-9} - 3\sec^{-1}\left(\frac{e^x}{3}\right) + C.$$

(d) Use partial fraction decompostion.

$$\frac{x^2-x+1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$
$$x^2 - x + 1 = A(x^2+2) + (Bx+C)x$$
$$\iff A = \frac{1}{2},\quad B = \frac{1}{2},\quad C = -1$$
$$\therefore\quad I = \int\left(\frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}x - 1}{x^2+2}\right)dx = \frac{1}{2}\int\left(\frac{1}{x} + \frac{x}{x^2+2} - \frac{2}{x^2+2}\right)dx$$
$$= \frac{1}{2}\left(\ln|x| + \frac{1}{4}\ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right) + C.$$

4. (a) We will evaluate the definite integral, using integration by parts,

$$\int_1^t x^{-2}\ln x\,dx = -\frac{\ln x}{x}\bigg|_1^t + \int_1^t x^{-2}\,dx \quad \left(u=\ln x,\ du=\tfrac{1}{x}\,dx,\ dv=x^{-2}dx,\ v=-x^{-1}\right)$$

$$= \left[-\frac{\ln x}{x}-\frac{1}{x}\right|_1^t = -\frac{\ln t}{t}-\frac{1}{t}+0+1;$$

$$\therefore \quad \int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\ln x\,dx = \lim_{t\to\infty}\left(-\frac{\ln t}{t}-\frac{1}{t}+0+1\right)$$

$$= 1 \qquad \left(\text{since } \lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{1}{t} = 0\right).$$

Converges.

(b)

$$\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{t\to 0^+}\int_t^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{t\to 0^+}\int_{\sin t}^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{u}} \quad (u=\sin x,\ du=\cos x\,dx)$$

$$= \lim_{t\to 0^+} 2u^{\frac{1}{2}}\bigg|_{\sin t}^{\frac{1}{\sqrt{2}}} = \lim_{t\to 0^+} 2\sqrt{\frac{1}{\sqrt{2}}} - 2\sqrt{\sin t} = 2\sqrt{\frac{1}{\sqrt{2}}} = \sqrt{2\sqrt{2}}.$$

Converges.

(c) For $x \geq 1$,

$$\sqrt{x^5-\frac{1}{2}} < \sqrt{x^5} \implies \frac{1}{\sqrt{x^5-\frac{1}{2}}} > \frac{1}{\sqrt{x^5}} \implies \frac{x^2}{\sqrt{x^5-\frac{1}{2}}} > \frac{x^2}{\sqrt{x^5}} = \frac{x^2}{x^{\frac{5}{2}}} = \frac{1}{x^{\frac{1}{2}}},$$

and $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}\,dx$ diverges since we showed that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for $p \leq 1$. So by the Comparison Thm. $\int_1^\infty \frac{x^2}{\sqrt{x^5-\frac{1}{2}}}\,dx$ diverges.

5. Separate the variables and integrate both sides:

$$\int y\,dy = -\int \frac{5x}{\sqrt{x^2+1}}\,dx \implies \frac{y^2}{2} = -\frac{5}{2}\int \frac{du}{\sqrt{u}} \qquad (u=x^2+1)$$

$$\implies \frac{y^2}{2} = -5u^{\frac{1}{2}}+C \implies y^2 = -10\sqrt{x^2+1}+C_1$$

$$y(0)=-1 \implies 1 = -10\sqrt{1}+C_1 \implies C_1 = 1+10 = 11$$

$$\therefore \quad y^2 = -10\sqrt{x^2+1}+11 \implies y = -\sqrt{11-10\sqrt{x^2+1}}$$

is the solution to the initial value problem.

6. $y' = (-2x)/(1-x^2)$. Therefore,

$$1 + (y')^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2} = \frac{x^4 + 2x^2 + 1}{(1-x^2)^2} = \frac{(x^2+1)^2}{(1-x^2)^2}.$$

So, the arc length L of the curve is

$$\begin{aligned} L &= \int_0^{\frac{1}{3}} \sqrt{1 + (y')^2}\, dx = \int_0^{\frac{1}{3}} \frac{1+x^2}{1-x^2}\, dx = \int_0^{\frac{1}{3}} \left(-1 + \frac{2}{1-x^2}\right) dx \\ &= \int_0^{\frac{1}{3}} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \Big[-x + \ln(1+x) - \ln(1-x) \Big|_0^{\frac{1}{3}} \\ &= -\frac{1}{3} + \ln\frac{4}{3} - \ln\frac{2}{3} = -\frac{1}{3} + \ln 2. \end{aligned}$$

Practise Exam, ¡ath 115, April 2007

1.(14 p) Evaluate the following limits

$$\lim_{x\to\infty} (1+\frac{1}{2x^2})^{x^2}, \qquad \lim_{x\to 0} \frac{(\sin x)^2}{\ln(1+x^2)}.$$

2.(40 p) Evaluate the integrals

$$\text{(a)} \quad \int \frac{\sec^4 x}{\sqrt{\tan x}}\, dx, \qquad \text{(b)} \quad \int \frac{(2x^2+x+4)}{x^3+4x}\, dx, \qquad \text{(c)} \quad \int \frac{\sqrt{x}}{x+2}\, dx,$$

$$\text{(d)} \quad \int x^2 \ln x\, dx, \qquad \text{(e)} \quad \int \frac{1}{x^2+2x+10}\, dx.$$

3.(14 p) Determine whether the improper integrals are convergent or divergent. Evaluate those which are convergent

$$\int_2^\infty x^3\, e^{-x^2}\, dx, \qquad \int_0^\pi \tan x\, dx.$$

4.(7 p) Find the solution of the initial problem

$$x\frac{dy}{dx} = \frac{1+x}{y}, \qquad x > 0, \qquad y(1) = 0.$$

5.(7 p) Solve the following equation for x:

$$\ln x + \ln(x-2) = \ln 3.$$

6.(9 p) Find the area of the surface obtained by revolving the curve about the x-axis

$$x = \frac{1}{2}y^2 + 1, \qquad 1 \le y \le 2.$$

6. (9 p) Find the volume of the solid generated by revolving the region bounded by the given curves about the line $y = 1$:

$$y = e^x, \qquad x = 0, \quad x = 3, \quad y = 1.$$

Practise Exam, ¡ath 115, April 2007; Solutions

There might be some typos and other small inaccuracies in the solutions

1.(14 p) Evaluate the following limits

$$\lim_{x\to\infty}(1+\frac{1}{2x^2})^{x^2}, \qquad \lim_{x\to 0}\frac{(\sin x)^2}{\ln(1+x^2)}.$$

Solutions (a) Set $y=(1+\frac{1}{2x^2})^{x^2}$. Then $\ln y = x^2\ln(1+\frac{1}{2x^2})$. We use l'Hospital's Rule

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{2x^2})}{x^{-2}} = \lim_{x\to\infty}\frac{\frac{-2x^{-3}/2}{1+\frac{1}{2x^2}}}{-2x^{-3}} = \lim_{x\to\infty}\frac{1}{2(1+\frac{1}{2x^2})} = \frac{1}{2}.$$

Thus $\lim\limits_{x\to\infty}\ln y = 1/2$, which implies $\lim\limits_{x\to\infty} y = e^{1/2}$.

(b) By l'Hospital's Rule,

$$\lim_{x\to 0}\frac{(\sin x)^2}{\ln(1+x^2)} = \lim_{x\to 0}\frac{2\sin x\cos x}{\frac{2x}{1+x^2}} = \lim_{x\to 0}\frac{2\sin x\cos x(1+x^2)}{2x} = \lim_{x\to 0}\cos x(1+x^2)\frac{\sin x}{x} = 1.$$

2.(40 p) Evaluate the integrals

$$\text{(a)}\quad \int\frac{\sec^4 x}{\sqrt{\tan x}}\,dx, \qquad \text{(b)}\quad \int\frac{2x^2+x+4}{x^3+4x}\,dx, \qquad \text{(c)}\quad \int\frac{\sqrt{x}}{x+2}\,dx,$$

$$\text{(d)}\quad \int x^2\ln x\,dx, \qquad \text{(e)}\quad \int\frac{1}{x^2+2x+10}\,dx.$$

Solutions (a) Substitute $u=\tan x$; so that $\sec^2 x\,dx = du$, $\sec^2 x = 1+u^2$. Then

$$\int\frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \int\frac{1+u^2}{\sqrt{u}}\,du = 2u^{1/2}+\frac{2}{5}u^{5/2}+C = 2(\tan x)^{1/2}+\frac{2}{5}(\tan x)^{5/2}+C.$$

(b) Since $x^3+4x = x(x^2+4)$ then

$$\frac{2x^2+x+4}{x^3+4x} = \frac{A}{x}+\frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}.$$

This gives the equations: $2=A+B$, $1=C$, $4=4A$, that have solutions $A=B=C=1$. Thus

$$\int\frac{2x^2+x+4}{x^3+4x}\,dx = \int\frac{1}{x}\,dx+\int\frac{x}{x^2+4}\,dx+\int\frac{1}{x^2+4}\,dx = \ln|x|+\frac{1}{2}\ln(x^2+4)+\frac{1}{2}\tan^{-1}(\frac{x}{2})+C.$$

(c) We substitute $u=\sqrt{x}$. Then $u^2=x$ so $2udu=dx$. Then

$$\int\frac{\sqrt{x}}{x+2}\,dx = \int\frac{\sqrt{2u^2}}{u^2+2}\,du = 2\int\left(1-\frac{2}{u^2+2}\right)\,du = 2u-2\sqrt{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C.$$

So the final answer is $2\sqrt{x}-2\sqrt{2}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{2}}\right)+C$.

(d) We integrate by parts, setting $u=\ln x$, $dv=x^2\,dx$. Then $du=(1/x)\,dx$, $v=(1/3)x^3$. We get

$$\int x^2\ln x\,dx=\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2\,dx=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C.$$

(e) Completing the square we get $x^2+2x+10=(x+1)^2+9=u^2+9$, where $u=x+1$.

$$\int\frac{1}{x^2+2x+10}\,dx=\int\frac{1}{u^2+9}\,du=\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)+C=\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right)+C.$$

3.(14 p) Determine whether the improper integrals are convergent or divergent. Evaluate those which are convergent

$$(\mathbf{a})\quad\int_2^{\infty}e^{-x/2}\,dx,\qquad\qquad(\mathbf{b})\quad\int_0^{\pi}\tan x\,dx.$$

Solutions (a) We have

$$\int_2^{\infty}e^{-x/2}\,dx=\lim_{t\to\infty}\int_2^t e^{-x/2}\,dx=\lim_{t\to\infty}\left(-2e^{-x/2}\Big|_2^t\right)=\lim_{t\to\infty}\left(-2e^{-t/2}+2e^{-1}\right)=2e^{-1},$$

and the integral is convergent.

(b) Since $\tan x$ is discontinuous at $\pi/2$ we split the integral $\int_0^{\pi}\tan x\,dx=\int_0^{\pi/2}\tan x\,dx+\int_{\pi/2}^{\pi}\tan x\,dx$. Note that $\int\tan x dx=-\ln|\cos x|+C$. Then

$$\int_0^{\pi/2}\tan x dx=\lim_{t\to\pi/2-}\int_0^t\tan x dx+=\lim_{t\to\pi/2-}\left(-\ln|\cos x|\Big|_0^t\right)=\lim_{t\to\pi/2-}\left(-\ln|\cos t|+\ln 1\right)=\infty.$$

Therefore $\int_0^{\pi/2}\tan x dx$ is divergent, and thus $\int_0^{\pi}\tan x dx$ is divergent too.

4.(7 p) Find the solution of the initial problem

$$x\frac{dy}{dx}=\frac{1+x}{y},\qquad\qquad x>0,\qquad y(1)=0.$$

Solutions We separate variables $ydy=\frac{1+x}{x}dx$. We evaluate $\int\frac{1+x}{x}dx=\int(\frac{1}{x}+1)dx$ $=\ln|x|+x+C$. Thus, after integrating both sides of the equation, $\frac{1}{2}y^2=\ln|x|+x+C$. Since $x>0$ we get $y=\sqrt{2\ln x+2x+C}$. In order to get $y(1)=0$ we put $C=-2$. Therefore, $y=\sqrt{2\ln x+2x-2}$.

5.(7 p) Solve the following equation for x:

$$\ln x+\ln(x-2)=\ln 3.$$

Solutions We get $\ln(x(x-2))=\ln 3$, which is equivalent to $x(x-2)=3$, equvalently, $x^2-2x-3=0$. Factoring we get $(x+1)(x-3)=0$, thus $x=-1$ or $x=3$. However the equation implicitely requires that $x>0$ and $x-2>0$, in order to have positive arguments under the logarithms. This means that from the two possibilities only one fits, so $x=3$.

6.(9 p) Find the area of the surface obtained by revolving the curve about the x-axis

$$x = \frac{1}{2}y^2 + 1, \qquad 1 \le y \le 2.$$

Solutions The surface area is $S = \int_1^2 2\pi y \, ds$. where $ds = \sqrt{1 + (\frac{dx}{dy})^2} \, dy = \sqrt{1 + y^2} \, dy$. Thus

$$S = \int_1^2 2\pi y \sqrt{1 + y^2} \, dy = \int_2^5 \pi \sqrt{u} \, du = \frac{2}{3}\pi u^{3/2} \Big|_2^5 = \frac{2}{3}\pi(5\sqrt{5} - 2\sqrt{2}).$$

Here we used the substitution $u = 1 + y^2$.

6. (9 p) Find the volume of the solid generated by revolving the region bounded by the given curves about the line $y = 1$:

$$y = e^x, \qquad x = 0, \quad x = 3, \quad y = 1.$$

Solutions We use the disc-washer emthod. The radius of a typical disc is $y - 1$, so the volume of such a disc is $\pi(y-1)^2 \, dx = \pi(e^x - 1)^2 \, dx$. We integrate it in x from 0 to 3, to get

$$V = \int_0^3 \pi(e^x - 1)^2 \, dx = \int_0^3 \pi(e^{2x} - 2e^x + 1) \, dx = \pi(\frac{1}{2}e^{2x} - 2e^x + x)\Big|_0^3 = \frac{\pi}{2}e^6 - 2\pi e^3 + \frac{11\pi}{2}.$$

1. $\lim_{\eta \to +\infty} \left(1+\frac{2}{\eta}\right)^{\eta} =$ 2a) $9^{\log_{0.25} 2} =$ 2b) $\frac{(\ln e)^3}{\left(\ln \frac{1}{e}\right)^2} =$

3. Find all solutions of $\log_2\left(\frac{x^2-2x+1}{x-2}\right) = 2 - \log_2\left(\frac{x^2-4x+4}{x-1}\right)$.

4.Find the volume of the solid obtained by rotating the circle $x^2-(y-1)^2=1$ about the y-axis. Use the method of slices or discs if you are in Y1. Use the method of cylindrical shells if you are in Z1.

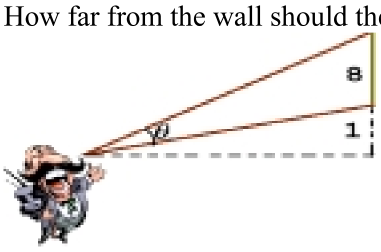
5. A worker on a construction site needs to raise 15kg of cement 3 meters above the ground by pulling on a cable weighing 3kg. How much work is required?

1.According to the Globe and Mail, March 1, 2002 in 1852 the salary of the highest paid university professor in Canada was about Ca\$500. Assuming that the salary of the highest paid university professor in 2002 is Ca\$ 180,000, determine the average annual percentage increase in the highest paid professor's salary? Assume that every year the salary has increased by the same percentage. You may need one of the following formulas: $360^{1/50}=1.12$, $360^{1/150}=1.04$, $360^{1/300}=1.02$, $180^{1/150}=1.035$, $180^{1/450}=.01$, $180^{1/150}=1.035$.

2. a) Use the definition of logarithm to evaluate $\log_{0.75} \frac{9}{16}$.

b) Use properties of exponential and logarithmic functions and the table provided to compute $11.9182 \times \sqrt[3]{17.4495}$.

3. A printing in an art gallery has height $8\ m$ and is hung so that its lower edge is $1\ m$ above the eye of the observer. How far from the wall should the observer stand to get the best view, i.e. to maximize the angle θ.



4. Use the properties of exponential and logarithmic functions and the method of differentials to compute $\log_{1.02} 1.03$.

5.a) Compute $\lim_{x\to 0^+} x^{\sin x}$. b) Find the derivative of $y = x^{\sin x}$.

1. The price of gasoline on January 1, 1985 was 30 cents per liter and on January 1, 2002 it was 60 cents per liter. Determine the average annual percentage increase in price of gasoline. You may need one of the following formulas: $1.04^{17}=2$, $\log_{30} 60=1.2$, $0.96^{17}=0.5$, $2^{1/17}=1.04$, $1.02^{17}=1.4$, $1.05^{20}=2.71$.

2. a) Use the definition of logarithm to solve $\log_{0.75} x = 3$.

b) Use properties of exponential and logarithmic functions and the table provided to compute $\frac{6428.7574^{0.5}}{19.19}$.

4. Use the properties of exponential and logarithmic functions and the method of differentials to compute $\frac{\log_{666} 1.06}{\log_{666} 1.04}$.

5.a) Compute $\lim_{x\to 0^+} (\sin x)^x$. b) Find the derivative of $y = (\sin x)^x$.

1. In May 27, 1990 The New York Times published an article "213 Years After Loan, Uncle Sam Is Dunned" by Lisa Belkin. According to the article in 1777 a wealthy Pennsylvanian merchant Jacon DeHaven lent \$450,000 to the Continental Congress to rescue troops at Valey Forgue. That loan was never repaid. Estimate how much will the descendants of Mr. DeHaven be owed 230 years later (that is in 2007) if we assume that the interest rate to be paid is 3% per annum compounded annually. By how much has the purchasing power of the original amount dropped, if we assume that the inflation rate has been 6% per year every year?

2. a) Use the definition of logarithms to evaluate $\log_{11} 121$, to solve $\log_{12} x = 144$.

b) Use properties of exponential and logarithmic functions and the table provided to compute $1138893.582 \times \sqrt[16]{4.594973} \times \frac{34.004}{129.12994}$.

4. Use the properties of exponential and logarithmic functions and the method of differentials to compute $\log_e 1.0666$.

5. a) Compute $\lim_{x\to 0^+}(x+\sin x)^{\sin x + x}$. b) Find the derivative of $y = (x + \sin x)^{\sin x + x}$.

2. The country of Qanada has $100 billion in paper currency in circulation, and each day $50 million comes into the country's banks. A KGB agent Kovalyov is sent to Qanada to destroy that country's economy. As part of his assignment he is to remove from circulation $50 million every day and to replace it with the same amount of fake money. Ten days after Kovalyov had started his money replacement scheme the Government of Qanada discovered the evil intentions and ordered the banks to sift through the currency each day and replace the fake money with real currency. What will happen? Hint: Solve the differential equation modeling the amount of fake money and take limit as x goes to infinity. What does this limit show? Who will prevail, the Government or Kovalyov?

3. Find the length of the curve $y = \frac{1}{3}\left(x^2+2\right)^{1.5}$, $0 \le x \le 1$.

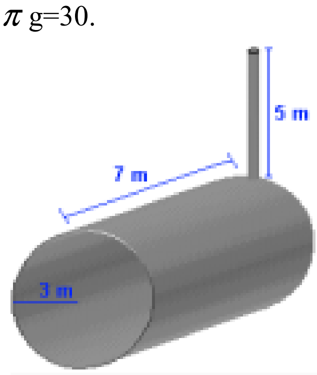
4. Use the substitution $z = \sin^2 x$ to evaluate $\int \frac{\sin x}{\cos^3 x}dx$.

5. Use the L'Hospital's rule to determine $\lim_{x\to 0}\frac{\int_0^x \cos t^2 dt}{\int_0^x (1+t)\ dt}$

6. Use the method of differentials to compute $\frac{\arcsin 0.1}{\cos 0.1}$. Hint: Apply the method of differentials to each term separately.

7. Use the method of simple fractions to evaluate $\int \frac{1+x^2}{1-x^2}dx$. Hint: First reduce the fraction to the appropriate form.

1. A tank shaped as a cylinder of radius 3 meters and length 7 meters is filled with water. A 5 meter long pipe is inserted into one end of the tank.How much energy is required to pump all that water out of the tank through the pipe ? Assume that π g=30.



1. A spherical tank of radius 10 meters is filled with water. Find the work done in pumping all of the water out through the top of the tank. Assume that π g=30 See picture above right.

2. The lake Inferior contains 1 million tons of water. The municipal government of the city of Badmenton had been dumping sewage into the lake for years until one day, when the amount of sewage reached 10,100 tons, the government decided to limit the amount of dumped sewage to 10 tons a day and build a water cleaning plant. The water cleaning plant is to process 10^5 tons of water a day and to remove all sewage from the processed water. How much sewage will be left in the lake 10 years later? Hint: for your computations you will need $1/e = e^{-1} = 0.369$.

3. Use the method of differentials to compute $e^{0.1} - \arctan 0.1$. Hint: Apply the method of differentials to each term separately.

4. Determine the function that satisfies the differential equation $f'(x) = 666 f(x)$ and the initial condition $f(0) = 0$.

5. Use the substitution $z = (\arcsin x)^2$ to evaluate $\int \frac{\arcsin x}{\sqrt{1-x^2}\ (1-\arcsin x)\ (1+\arcsin x)}dx$. Hint: you will need the formula: $(1-a)\ (1+a) = 1-a^2$.

6. Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$, $2 \le x \le 4$.

7. Use the method of integration by parts to evaluate $\int (xe^x + x^2 \ln x)\ dx$. Hint: You may need more than one step.

2. Research by a group of people (Meadows, Meadows, Randers and Behrens) indicates that the Earth has 3.2×10^9 acres of arable land available. The world population of 1950 required 10^9 acres to sustain it and the world population of 1980 needed 2×10^9 acres. If we assume that the population growth at a constant percentage rate, in what year will it reach the maximum sustainable size? Do you think the predictions of the research quoted are to come true? Hint: use that $\log_2 3.2 = 1.678$. Note this is not a mixing problem, it is a much simpler problem on exponents and logs.

3. Use the method of differentials to compute $\dfrac{\arctan 0.02}{\arcsin 0.04}$. Hint: Apply the method of differentials to each term separately.

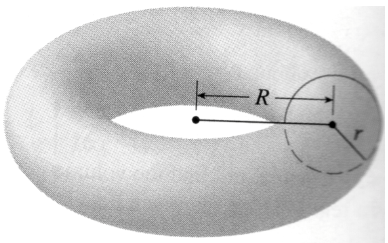
4. Determine the function that satisfies the differential equation $f'(x) = f(x) + 666$ and the initial condition $f(0) = 666$.

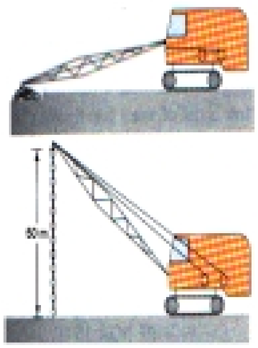
5. Use the substitution $z = \cos x$ to evaluate $\int \dfrac{\sin x}{\cos^{1001} x}\, dx$.

6. Find the length of the curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, $2 \le x \le 4$.

1. Determine domain of the function $y = |\ln x|$ if you are in Y1 or $y = 0.5|\ln x| -- 0.5 \ln|x|$ if you are in W1 and explain why the function is not one-to-one on the whole domain. What is the inverse of this function on the interval $0 < x < 1$.

2.Find the volume of the torus shown on the picture. Use the method of slices or discs if you are in W1. Use the method of cylindrical shells if you are in Y1. Assume that ***R=2r =2 cm***.





3. How much energy is required to raise a 50 meter long 400 kg chain from the ground to a vertical position?

4. Find solutions of $\log_2\left(\dfrac{x-7}{x-13}\right) = 1 - \log_4\left(\dfrac{x-13}{x-10}\right)^2$ and $\log_2\left(\dfrac{x-7}{x-666}\right) = 1 - \log_4\left(\dfrac{x-666}{x-10}\right)^2$.

1.Let $y = f(x)$ be the amount of US currency equivalent to x Canadian loonies. What does the inverse function show? What does $f^{-1}(1.00)$ show?

2. In May 27, 1990 The New York Times published an article "213 Years After Loan, Uncle Sam Is Dunned" by Lisa Belkin. According to the article in 1777 a wealthy Pennsylvanian merchant Jacon DeHaven lent $450,000 to the Continental Congress to rescue troops at Valey Forgue. That loan was never repaid. Estimate how much will the descendants of Mr. DeHaven be owed 230 years later (that is in 2007) if we assume that the interest rate is 6% per annum compounded daily and each year has 365 days? Will they be owed more or less than $$ 4.5×10^9? Hint: use that and that for small values of x, $(1+x)^{1/x} \approx e$.

3. Compute $\lim\limits_{x\to+\infty} \left(\dfrac{x}{x+1}\right)^x$

4. $\displaystyle\int_0^{\sin\left(\arccos\frac{3}{5}\right)} x\cos x\,dx =$

5. Compute derivative of the function $y = \sin(x^x) - (\sin^2 x + \cos^2 x)^x$. Hint: compute derivatives of x^x and first.

1. Let $f(t)$ = number of motor vehicles (in millions) in the world in the year t. Assuming that $f(t)$ is invertible, what does $f^{-1}(400)$ mean?

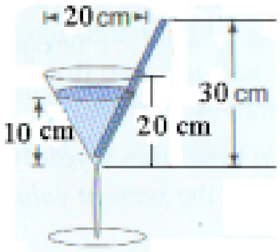
3. Compute $\lim\left(1 + \dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{3}{x^3}\right)^{-x}$ as x approaches plus infinity.

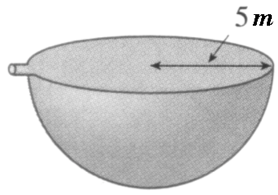
4. $\displaystyle\int_{\sin\arccos\frac{4}{5}}^{\cot\arccos\frac{3}{5}} x\sin x\,dx =$

5. Use logarithmic differentiation to compute derivative of the function $y = \dfrac{\tan(x^x)}{(\cot x)^x}$.

1. (a fable from Russian folklore, for kids under 10) Once upon a time there lived two brothers: Ivanuska the Silly and Einstein. One day Einstein said to his brother: "Ivanushka, how about this? I give you a penny today, two pennies tomorrow, 4 pennies the day after, etc., that is every day I give you twice as much money as the day before. In return every day you give me $30,000,000. We do it for 30 days. " Ivanushka agreed. Which brother will make money? Use *$\log_2 1024=10$.*

3. How much energy is required to sip the glass of coconut wine shown on the picture? Assume the density of the coconut wine is 1.0 gm/cm^3, $g \approx 19$ m/sec^2, $\pi \approx 3$, $15\sqrt{3} \approx 27$.

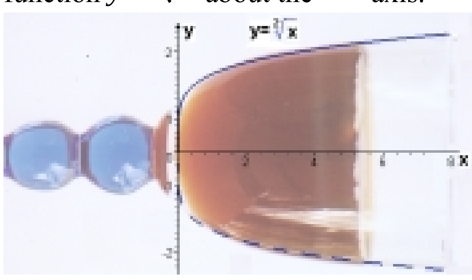
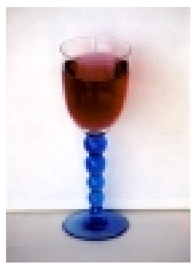
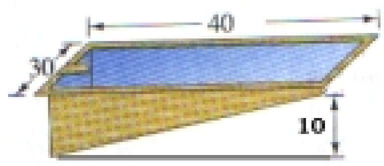




1. A tank, shaped as a hemisphere, is full of water. Find the work required to pump the water out of the outlet shown. Assume π g=30. see picture above right.

4. Using simple fractions evaluate $\int \frac{x+1}{x^2(x-1)}\,dx$. Hint: you will need three simple fractions.

5. Determine surface area of the whole wine glass shown on the picture? The glass is obtained by rotating the function $y=\sqrt[2]{x}$ about the $x-$axis.

3.The pool shown below is filled up with water. How much energy is required to pump all the water out of the pool? Assume that the density of water is 1.0 gm/ cm^3, g=10 m/sec^2. 7. Use integration by parts to compute $\int x^2 \sin x \, dx$.

8. Find all solutions of the differential equation $\frac{dy}{dx}e^{-y} = -\frac{dy}{dx}$.

1.In the Russian version of the game "Who wants to be a millionaire?" a contestant is paid $1,000 for a correct answer to the first question, $2,000 for a correct answer to the second question, $4,000 for a correct answer to the third question, etc. . In other words the amount of money paid to a contestant for a correct answer doubles each time. Using the formula $log_2 1024=10$, determine the minimum number of questions a contestant is required to answer to accumulate $1,000,000 or more. Exactly how much money will the contestant accumulate after answering that many questions?

2. Russian man Vaska drank a bottle of vodka containing 100 ounces of alcohol and went for a 5 hour walk. While walking he kept on continuously drinking beer and thus continuously consuming alcohol at a rate of 20 ounces of alcohol per hour. Assuming that his body eliminates alcohol at a rate of 20% per hour, determine how much alcohol will be left in his body after the walk? You may need some of these formulas: $\ln 0.8 \approx -0.2$, $1/\ln 0.8 \approx -5$ if you choose a certain way to solve this question, if you use another way you may not need them at all.

4. Using simple fractions evaluate $\int \frac{x^3+1}{x^2-1}\,dx$.

7. Use, if necessary, an appropriate substitution and integration by parts to evaluate $\int x^2 (e^x)^3\,dx$.

8. Find all solutions of the differential equation $\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} + 1 = 0$.

1. Nancy wants to compute the total number of her ancestors in the first 10 generations: the 1st generation consists of 2 parents, the second generation consists of 4 grandparents, etc. How many ancestors are in the 10 generations altogether?
2. A patient is being continuously given medicine at a rate of 10 mg per hour. Patient's liver removes the medicine from the body at a rate of 5% per hour. How much medicine will be left in the body after 20 hours of administering medicine? Use that for small values of x, . What differential equation describes the function that shows the amount of medicine in the body?
3.Solve: . $\ln\left(\frac{x-2}{x-1}\right) = 1 + \ln\left(\frac{x-3}{x-1}\right)$

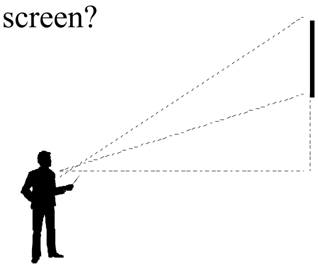
4.What is the equation of the tangent line to $\boldsymbol{y=x^x}$ at $\boldsymbol{x=1}$. Hint: Use logarithmic differentiation to compute derivative of the function $\boldsymbol{y=x^x}$ at $\boldsymbol{x=1}$ first.

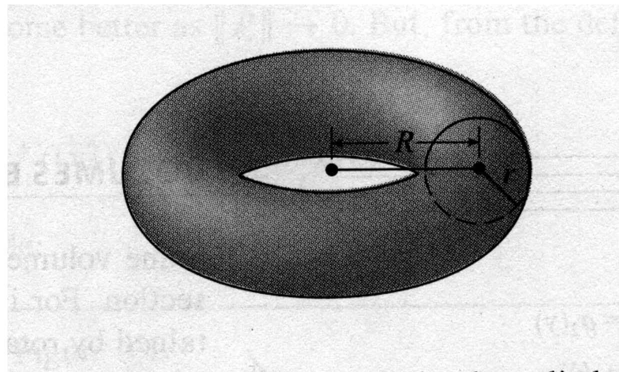
5.Evaluate: $\int \frac{\cos x}{1+\sin x}dx$

1. What is the definition of the number **e** ? Is the number **e** rational or irrational?

2. A cell of the E-coli bacterium divides into two cells every two minutes. If the initial population of the bacteria is 100 cells, when will the population reach 10,000 cells ?

3. A projection-TV screen of height 3 m is placed 3 m above the eye of a viewer. How far from the screen should the viewer stand to get the best view, that is to maximize the angle θ subtended at his eye by the screen?





1. Find the volume of the torus shown on the picture. Use method of discs if you are in Y1. Use method of cylindrical shells if you are in Z1. $R = 2r = 2$ ***cm***. See picture above.

3. Find the length of the curve $y = 0.5\,x^2 - 0.25 \ln x$, $2 < x < 4$.

4. Find the area of the surface obtained by rotating $x = \sqrt{2y - y^2}$, $0 < y < 1$ about the y-axis.

5. Which number is larger $\log_2 3$ or $\log_3 2$? Explain.

6.Compute $\lim\ (e^x - 1 - x)\, x^{-2}$

8.Two car dealers placed the following ads in a local newspaper:
Ad 1: We give you a 20% discount ***before*** we add the GST, this way you also save by paying the GST on a lower amount.
Ad 2: We give you a 21% discount ***after*** we add the GST, this way you also save by getting a discount on the GST.
Which ad is offering a better deal ? Explain your answer.

2. A tank contains1000 L of pure water. Brine that contains 0.05 kg/L of salt enters the tank at a rate of 5L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 5 L/min. How much salt is in the tank after 1 hour?

4. Evaluate $\int | \ln x |\, dx$

5. Find the arc-length of the curve $12\,x\,y = 4\,y^4 + 3$ between (7/12, 1) and (67/24, 2) . Hint: Use y as the independent variable.

6.Solve $\log_2 4^{x\ -2x+2} = \log_4 16$. Hint: Use properties of the exponential and logarithmic functions to simplify it first.

1. Nancy wants to compute the total number of her ancestors in the first 10 generations: the 1st generation consists of 2 parents, the second generation consists of 4 grandparents, etc. How many ancestors are in the 10 generations altogether?

I(6). Find y' if

(a) $y = x^x \sin^{-1}(x)$

(b) $y = \Big(\log_{10} x^2\Big)\Big(\tan^{-1}(x)\Big)^{x+1}$

II. (a)(4) Find the inverse function of $y = \dfrac{x-2}{x+2}$

(b)(4) Find the domain and range of the inverse function.

III(9). Evaluate

(a) $\displaystyle\lim_{x\to\infty} \frac{\sin x - x}{x^3}$

(b) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^{x+1/x}$

(c) $\displaystyle\lim_{x\to 1^+} \tan^{-1}\left(\frac{x+1}{x-1}\right)$

IV(9). Evaluate

(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{4}} \frac{1}{1+16x^2}dx$

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx$

(c) $\displaystyle\int_e^{2e} \frac{1}{x\Big(\ln(x)\Big)^3}dx$

V (8) A solid is formed by rotating the region bounded by $y = x$, $y = 4x - x^2$ about the line $x = 7$. Find its volume.

Midterm Math 115 (A1)

Date: Wednesday, November 20, 2000 50 minutes
Instructor: Y. Lin

Last Name: ____________________ First Name: ____________________ Initial: ______

Please show all your work!

MATH 115 (A1)
The Second Midterm Exam
Fall 2000

Student (Print) ______________________________
Last First Middle

Student (Sign) ______________________________

Student ID ______________________________

Instructor ______________________________

Section ______________________________

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2	
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TOTAL	

I (a)(5) $\int x^2 e^x dx$

(b)(5) $\int \cos^4(x) dx$

II. (a)(5) $\int \tan^5(x)\sec^7(x)dx$

(b)(5) $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

III Evaluate

(a)(5) $\int_0^1 x\ln(x)dx$

(b)(5) Use Midpoint rule and Simpson rule to find approximate values of $\int_0^2 (x^2 + x)dx$ with $n = 2$.

IV. (a) (5) $\int \frac{1}{(x-1)^2(x+4)}dx$

(b)(5) $\int \frac{1}{x^{1/2} + x^{1/3}} dx$

(5) 1. Find $f'(x)$ (or $\frac{df}{dx}$) if : $f(x) = x^2(\sin^{-1}(x))^5$.

(You do not need to simplify your answer)

$f'(x) = 2x(\sin^{-1}(x))^5 + x^2 \cdot 5(\sin^{-1}(x))^4 \cdot \frac{1}{\sqrt{1-x^2}}$

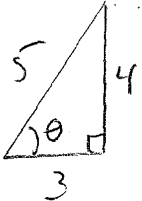
(5) 2. Find the equation of a line through the origin that is tangent to the graph of $y = \ln(x)$.

$\frac{dy}{dx} = \frac{1}{x}$; $y - y_0 = m(x - x_0) \Leftrightarrow (y - 0) = \frac{1}{x}(x - 0)$

$\Rightarrow y = 1 \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e$, thus at $(e, 1)$

TANGENT LINE EQ.: $y = \frac{x}{e}$ or $y - 1 = \frac{1}{e}(x - e)$.

(4) 3. Find the exact value (that is : a real number) of : $\csc(\cos^{-1}(\frac{3}{5}))$.



$\cos(\theta) = \frac{3}{5} \rightarrow \csc(\theta) = \frac{5}{4}$.

(4) 4. Find the error, if any, in the following calculation of a limit :

$$\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1} \frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1} \frac{6x - 2}{6x - 2} = 1.$$

Find the correct limit if the above value of the limit is incorrect.

(i) $\lim_{x\to 1} \frac{3x^2 - 2x + 1}{3x^2 - 2x} = \frac{2}{1}$, not of $\left(\frac{0}{0}\right)$ type

(ii) $\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x^3 - x^2} \overset{H}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 1} \left(\frac{3x^2 - 2x + 1}{3x^2 - 2x}\right) = \frac{2}{1} = 2$

(15) 5. Evaluate each of the following integrals :

(5) (a)

$$\int_0^1 \tan^{-1}(x)\,dx$$

$$\int_0^1 \underbrace{\tan^{-1}(x)}_{u}\,\underbrace{dx}_{dv} = \Big[\underbrace{\tan^{-1}(x)}_{u}\cdot\underbrace{x}_{v}\Big]_0^1 - \int_0^1 \underbrace{x}_{v}\cdot\underbrace{\frac{dx}{1+x^2}}_{du}$$

$$= \Big[x\tan^{-1}(x)\Big]_0^1 - \int_1^2 \frac{1}{2U}\,dU, \quad U = 1+x^2,\ dU = 2x\,dx, \quad x=0 \to u=1,\ x=1 \to u=2$$

$$= \Big[x\tan^{-1}(x)\Big]_0^1 - \frac{1}{2}\Big[\ln|U|\Big]_1^2$$

$$= \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln(2) = \frac{\pi}{4} - \ln(\sqrt{2}).$$

(5) (b)

$$\int \frac{1}{x^4\sqrt{x^2+4}}\,dx$$

Trig. Subst. : $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

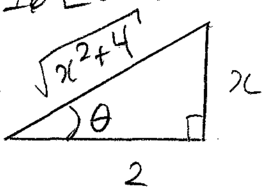
$$\int \frac{1}{x^4\sqrt{x^2+4}}\,dx = \frac{1}{16}\int \frac{\sec\theta}{\tan^4\theta}\,d\theta = \frac{1}{16}\int \frac{\cos^3\theta}{\sin^4\theta}\,d\theta$$

$$u = \frac{1}{16}\int \frac{(1-\sin^2\theta)}{(\sin^4\theta)}\cos\theta\, d\theta \text{ ; put } u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$$

$$u = \frac{1}{16}\int\left(\frac{1}{u^4} - \frac{1}{u^2}\right)du = \frac{1}{16}\left[-\frac{1}{3u^3} + \frac{1}{u}\right] + C$$

$$u = \frac{1}{16}\left[-\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta}\right] + C = \frac{1}{16}\left[-\frac{(x^2+4)^{3/2}}{3x^3} + \frac{\sqrt{x^2+4}}{x}\right] + C$$

Since:



5. Evaluate the following integral :

(5) (c)

$$\int \frac{(2x+4)}{(x^3-2x^2)}\,dx$$

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)}$$

$$\Rightarrow 2x+4 = Ax(x-2) + B(x-2) + Cx^2 \Rightarrow \begin{cases} A+C=0 \\ -2A+B=2 \\ -2B=4 \end{cases}$$

$$\therefore A=-2,\ B=-2,\ C=2$$

Thus:

$$\int \frac{2x+4}{(x^3-2x^2)}\,dx = \int -\frac{2}{x}\,dx + \int -\frac{2}{x^2}\,dx + \int \frac{2}{(x-2)}\,dx$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

(15) 6. Evaluate each of the following integrals :

(5) (a)

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$$

COMPLETION OF SQUARE: $3+2x-x^2 = 4-(x-1)^2$, put $\begin{cases} u = x-1 \\ du = dx \end{cases}$

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du \underset{\uparrow}{=} \int \frac{1}{2\sqrt{1-\sin^2\theta}}\,2\cos\theta\,d\theta = \int d\theta$$

$$\begin{pmatrix} u = 2\sin\theta \\ du = 2\cos\theta\,d\theta \end{pmatrix}$$

$$u = \theta + C = \sin^{-1}\left(\frac{u}{2}\right) + C = \sin^{-1}\left(\frac{x-1}{2}\right) + C$$

(5) (b)

$$\int \cos^4(x)\,dx = \int (\cos^2(x))^2\,dx$$

$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$$= \frac{1}{4}\int (1+\cos(2x))^2\,dx = \frac{1}{4}\int (1+2\cos(2x)+\cos^2(2x))\,dx$$

$$= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1}{2}+\frac{1}{2}\cos(4x)\right)dx, \text{ since } \cos^2(2x) = \frac{1}{2}(1+\cos(4x)).$$

$$= \frac{1}{4}\left[\frac{3x}{2}+\sin(2x)+\frac{1}{8}\sin(4x)\right]+C$$

(5) (c)

$$\int \cos(\sqrt{x})\, dx$$

$u = \sqrt{x} \Rightarrow 2u\,du = dx$

$\int \cos(\sqrt{x})\,dx = 2\int \underbrace{u}_{U}\underbrace{\cos(u)\,du}_{dV} = 2\left[\underbrace{u}_{U}\underbrace{\sin(u)}_{V} - \int \underbrace{\sin(u)}_{V}\underbrace{du}_{dU}\right]$

$= 2(u\sin(u) + \cos(u)) + C = 2(\sqrt{x}\sin(\sqrt{x}) + \cos(\sqrt{x})) + C$

(15) 7. Evaluate each limit below, if the limit exists, or show that the limit does not exist.

(5) (a) $\lim_{x\to\infty}\left(\frac{1}{x^5}\ln(x)\right) \overset{H}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{5x^4}\right) = \lim_{x\to\infty}\left(\frac{1}{5x^5}\right) = 0$

(5) (b) $\lim_{x\to 0^+} (\sin(x))^{(2/\ln(x))} = \lim_{x\to 0^+} e^{\ln\left(\sin(x)^{\frac{2}{\ln(x)}}\right)}$

$= \lim_{x\to 0^+} e^{\left(\frac{2}{\ln(x)} \ln(\sin(x))\right)}$, where:

$$\lim_{x\to 0^+} \left[\frac{2\ln(\sin(x))}{\ln(x)}\right] \overset{H}{=} \lim_{x\to 0^+} \frac{\left(2\cdot\frac{1}{\sin(x)}\cdot\cos(x)\right)}{\left(\frac{1}{x}\right)}$$

$$= 2\lim_{x\to 0^+} \left[\frac{\cos(x)}{\left(\frac{\sin(x)}{x}\right)}\right] = 2\cdot\frac{1}{1} = 2$$

thus: $\lim_{x\to 0^+} (\sin(x))^{(2/\ln(x))} = e^2$.

(5) (c) $\lim_{x\to\infty} (xe^{(1/x)} - x) = \lim_{x\to\infty} x(e^{1/x} - 1) = \lim_{x\to\infty} \frac{(e^{1/x} - 1)}{(1/x)}$

$\overset{H}{=} \lim_{x\to\infty} \left[\frac{e^{1/x} \cdot \left(-\frac{1}{x^2}\right) - 0}{\left(-\frac{1}{x^2}\right)}\right] = \lim_{x\to\infty} \left[\frac{e^{1/x}}{1}\right] = 1.$

(8) 8. Find the length of the curve: $y = \dfrac{x^4 + 48}{24x}$, on the interval $1 \leq x \leq 2$.

$$y = \frac{x^3}{24} + \frac{2}{x}, \quad 1 \leq x \leq 2$$

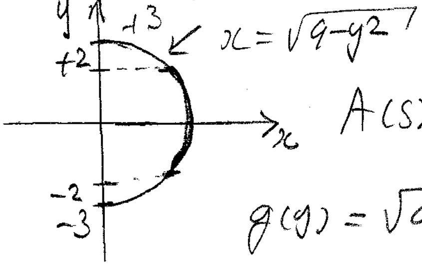
$$\Rightarrow y' = \frac{x^2}{8} - \frac{2}{x^2} \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2$$

$$1 + (y')^2 = 1 + \frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4} = \frac{x^4}{64} + \frac{1}{2} + \frac{4}{x^4} = \left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2$$

$$\Rightarrow L = \int_1^2 \sqrt{1 + (y')^2}\, dx = \int_1^2 \left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = \left[\frac{x^3}{24} - \frac{2}{x}\right]_1^2$$

$$L = \left(\frac{8}{24} - \frac{2}{2}\right) - \left(\frac{1}{24} - 2\right) = \frac{31}{24}.$$

(8) 9. Find the area of the surface obtained by revolving (rotating) the curve : $x = \sqrt{9 - y^2}$, about the y - axis from $y = -2$ to $y = 2$.



$$A(S) = \int_{-2}^{+2} 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy$$

$$g(y) = \sqrt{9-y^2} \Rightarrow g'(y) = \frac{-y}{\sqrt{9-y^2}}$$

$$\Rightarrow 1 + (g'(y))^2 = 1 + \frac{y^2}{9-y^2} = \frac{9}{9-y^2}$$

$$A(S) = \int_{-2}^{+2} 2\pi \sqrt{9-y^2} \cdot \frac{3}{\sqrt{9-y^2}}\, dy = 6\pi \int_{-2}^{+2} dy = 6\pi y \Big]_{-2}^{+2}$$

$$A(S) = 24\pi.$$

(10) 10. Determine whether the integral is convergent or divergent for each of the following integrals. Evaluate the integral if it is convergent, or show that the integral is divergent.

(5) (a)

$$\int_{-1}^{0} \frac{1}{(1+x)^{(1/3)}}\, dx$$

$$= \lim_{t \to -1^+} \int_t^0 \frac{1}{(1+x)^{1/3}}\, dx, \quad u = 1+x \Rightarrow du = dx$$

$$= \lim_{t \to 1^+} \left[\int_{t+1}^{1} \frac{1}{u^{1/3}}\, du \right] = \lim_{t \to -1^+} \left[\frac{3}{2} u^{2/3} \right]_{1+t}^{1}$$

$$= \lim_{t \to -1^+} \left[\frac{3}{2} - \frac{3}{2}(1+t)^{2/3} \right] = \frac{3}{2} \text{ (CONVERGENT)}$$

(5) (b)

$$\int_1^\infty \frac{x+1}{x^2+2x}\,dx$$

$$\int_1^\infty \left(\frac{x+1}{x^2+2x}\right) dx = \lim_{t\to\infty}\left[\int_1^t \left(\frac{x+1}{x^2+2x}\right) dx\right] \underset{\uparrow}{=} \lim_{t\to\infty}\left[\int_3^{t^2+2t} \frac{du}{2u}\right]$$

$$\left(\begin{array}{l} u = x^2+2x \\ du = 2(x+1)\,dx \end{array}\right)$$

$$u \quad = \lim_{t\to\infty}\left[\frac{1}{2}\ln|u|\right]_3^{t^2+2t} = \frac{1}{2}\lim_{t\to\infty}\left[\ln|t^2+2t| - \ln(3)\right] = +\infty$$

(DIVERGENT)

(6) 11. Determine a function $f(x)$ such that :

$$x = \sinh(\ln(f(x) + x)),$$

where $-\infty < x < \infty$. Show your work.

$$x = \frac{e^{\ln(f(x)+x)} - e^{-\ln(f(x)+x)}}{2}$$

$\Leftrightarrow$

$$2x = (f(x)+x) - \frac{1}{(f(x)+x)}$$

$\Leftrightarrow$

$$2x(f(x)+x) = (f(x)+x)^2 - 1$$

$\Leftrightarrow$

$$2xf(x) + 2x^2 = (f(x))^2 + 2xf(x) + x^2 - 1$$

$\Leftrightarrow \quad (f(x))^2 = x^2 + 1$

$\Rightarrow \quad f(x) = +\sqrt{x^2+1}$, since $(f(x)+x)$ is argument of ln.

(5) 12. A sphere with radius 1 m has a (uniform) temperature of 15^{0}C. It lies inside a concentric sphere with radius 2 m and (uniform) temperature 25^{0}C. Both spheres have common center. If the temperature $y(x)$ (measured in ^{0}C) at a distance x (measured in m) from the common center of the spheres satisfies the following differential equation :

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0,$$

for $x > 0$, solve (the differential equation and impose conditions) to find an expression for the temperature $y(x)$ between the spheres.

(1) $z = \frac{dy}{dx} \Rightarrow \frac{dz}{dx} + \frac{2}{x} z = 0 \Leftrightarrow \frac{dz}{z} = -2\frac{dx}{x}$, if $z \neq 0$

$\Rightarrow \ln|z| = -2\ln|x| + C \Rightarrow |z| = \left|\frac{1}{x^2}\right| e^C$

$\Leftrightarrow z = \pm \frac{e^C}{x^2}$, $x > 0$, but $z = \frac{A}{x^2}$, with A any real constant,

Since $z = 0$ is also solution of the d.e..

(2) $z = \frac{dy}{dx} = \frac{A}{x^2} \Leftrightarrow dy = \frac{A}{x^2}dx \Rightarrow \int dy = \int \frac{A}{x^2}dx$

$\Rightarrow y = -\frac{A}{x} + B$, where B is any real constant.

(3) CONDITIONS: $y(1) = 15 \Rightarrow y(1) = -A + B = 15$
$y(2) = 25 \Rightarrow y(2) = -\frac{A}{2} + B = 25$
} $A = 20$ and $B = 35$

$\therefore y(x) = -\frac{20}{x} + 35$, $1 \leq x \leq 2$.

MATH-115 : PREVIOUS FINAL SOLUTIONS

1. $y' = e^{(\ln(x))^2} \cdot 2\left(\frac{\ln(x)}{x}\right) = (x^{\ln(x)}) \cdot 2\frac{\ln(x)}{x}.$

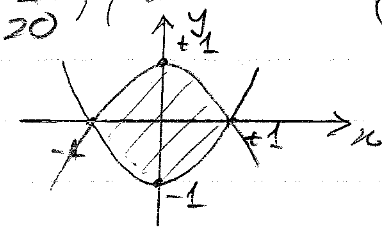
2. $\frac{d}{dx}\left(x^2 - x\sinh(y) + (\sinh(y))^3\right) = 0$

$\Leftrightarrow \frac{dy}{dx}\left(x\cosh(y) - 3\sinh^2(y)\cosh(y)\right) = 2x - \sinh(y)$

$\Rightarrow \frac{dy}{dx} = \frac{(2x - \sinh(y))}{(x\cosh(y) - 3\sinh^2(y)\cosh(y))}.$

3. $f(2) = 10,\ f'(2) = 20 \Rightarrow g'(10) = \frac{1}{f'(g(10))} = \frac{1}{f'(2)} = \frac{1}{20}$; $f(2) = 10 \Leftrightarrow 2 = g(10).$

4. Intersection : $1 - x^2 = x^2 - 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$



$A = \int_{-1}^{+1} [(1-x^2) - (x^2-1)]\,dx = 4\int_0^1 (1-x^2)\,dx$

$A = 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3}.$

5.(a) $\int_1^4 \ln(\sqrt{x})\,dx = \frac{1}{2}\int_1^4 \ln(x)\,dx$ $\left\{\begin{array}{l} u = \ln(x) \Rightarrow du = \frac{1}{x}dx \\ dv = dx \Leftarrow v = x \end{array}\right.$

$= \frac{1}{2}\left[x\ln(x)\right]_1^4 - \frac{1}{2}\int_1^4 dx = \frac{x\ln(x)}{2}\Big]_1^4 - \frac{x}{2}\Big]_1^4 = 4\ln(2) - \frac{3}{2}.$

(b) $\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x-1)(x+1)(x^2+1)}\,dx$; $\frac{1}{x^4-1} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2+1)}$

$\Rightarrow 1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$

$x = 0 : 1 = A - B - D \Rightarrow D = -1/2$

$\left.\begin{array}{l} x = +1 : 1 = 4A \\ x = -1 : 1 = -4B \end{array}\right\} \Rightarrow A = 1/4,\ B = -1/4$

$x = 2 : 1 = 15A + 5B + (2C+D)\cdot 3 \Rightarrow C = 0$

$\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\int\frac{dx}{(x-1)} - \frac{1}{4}\int\frac{dx}{(x+1)} - \frac{1}{2}\int\frac{dx}{(x^2+1)} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}(x) + C$

6. (a) $\int \frac{1}{(x+x^{3/4})}dx = \int \frac{4u^3}{(u^4+u^3)}du = 4\int \frac{du}{u+1} = 4\ln|u+1| + C = 4\ln|x^{1/4}+1| + C$

$(u = \sqrt[4]{x} \Rightarrow 4u^3 du = dx)$

(b) $\int \frac{1}{\sqrt{2x-x^2}}dx = \int \frac{1}{\sqrt{1-(x-1)^2}}dx = \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C = \sin^{-1}(x-1) + C$

$(u = x-1 \Rightarrow du = dx)$

(c) $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}(x) - \sqrt{1-x^2} + C$

7. (a) $\lim_{x\to 0}\left(\frac{\sin(2x)}{\sinh(3x)}\right) \overset{H}{=} \lim_{x\to 0}\left(\frac{\cos(2x)\cdot 2}{\cosh(3x)\cdot 3}\right) = \frac{2}{3}.$

(b) $\lim_{x\to\infty}(x-\sqrt{x^2-1})\frac{(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})} = \lim_{x\to\infty}\frac{(x^2-(x^2-1))}{(x+\sqrt{x^2-1})} = \lim_{x\to\infty}\frac{1}{(x+\sqrt{x^2-1})} = 0.$

(c) $\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}$; but $\lim_{x\to\infty} x\cdot\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)$

$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}{\left(\frac{1}{x}\right)} \overset{H}{=} \lim_{x\to\infty}\frac{\left(\frac{-3}{x^2}-\frac{10}{x^3}\right)\cdot\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^{-1}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{\left(3+\frac{10}{x}\right)}{\left(1+\frac{3}{x}+\frac{5}{x^2}\right)} = 3$

$\therefore \lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = e^3.$

8. (a) $\int_0^\infty \cos(x)dx = \lim_{x\to\infty}\int_0^t \cos(x)dx = \lim_{t\to\infty}(\sin(t)-0)$: does not exist (DIVERGENT)

(b) $\int_0^\infty x e^{-x^2}dx = \lim_{t\to\infty}\int_0^t x e^{-x^2}dx = \lim_{t\to\infty}\frac{1}{2}\int_0^{t^2} e^{-u}du =$

$(u = x^2 \Rightarrow du = 2x\,dx)$

$\lim_{t\to\infty}\frac{1}{2}\left[-e^{-u}\right]_0^{t^2} = \lim_{t\to\infty}\frac{1}{2}\left[1-e^{-t^2}\right] = \frac{1}{2}$ (CONVERGENT)

(c) $\int_0^2 \frac{1}{2x-3}dx = \int_0^{3/2}\frac{1}{2x-3}dx + \int_{3/2}^2 \frac{1}{2x-3}dx$, but $\int_0^{3/2}\frac{1}{(2x-3)}dx = \lim_{t\to 3/2^-}\int_0^t \frac{1}{2x-3}dx$

$= \lim_{t\to 3/2^-}\left[\frac{\ln|2x-3|}{2}\right]_0^t = \lim_{t\to 3/2^-}\frac{1}{2}\left[\ln|2t-3| - \ln(3)\right] = -\infty$ (DIVERGENT)

9. $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$, is CONVERGENT since $\frac{1}{\sqrt{x^3+1}} \le \frac{1}{x^{3/2}}$, on $[1,\infty)$.

and $\int_1^\infty x^{-3/2}\,dx = \lim_{t\to\infty}\left[-2x^{-1/2}\right]_1^t = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t}} + 2\right] = 2.$ CONVERGENT, BY COMPARISON THM.

10. MIDPOINT RULE: $n=4$; $\Delta x = (4.5-0.5)/4 = 1$

$$\int_{0.5}^{4.5} x^3\,dx \approx 1\cdot\left[(1)^3+(2)^3+(3)^3+(4)^3\right] = 100.$$

11. $L = \int_1^2 \sqrt{1+\left(x^3-\frac{1}{4x^3}\right)^2}\,dx = \int_1^2\left(x^3+\frac{1}{4x^3}\right)dx = \left[\frac{x^4}{4}-\frac{1}{8x^2}\right]_1^2 = \frac{123}{32}.$

12. $\frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2-x^2}} \Rightarrow 1+(y')^2 = \frac{a^4-a^2x^2+b^2x^2}{a^2(a^2-x^2)}$

$A = \int_{-a}^{+a} 2\pi\left(\frac{b}{a}\sqrt{a^2-x^2}\right)\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2(a^2-x^2)}}\,dx$, put: $\sqrt{a^2-b^2}\,x = a^2\sin\theta$ $(a>b)$

$A = \frac{4\pi b\,a^2}{\sqrt{a^2-b^2}}\int_0^{\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)}\cos^2\theta\,d\theta = \frac{2\pi b a^2}{\sqrt{a^2-b^2}}\left[\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)+b\frac{\sqrt{a^2-b^2}}{a^2}\right].$