Math 115 HW# 10

9.1

8. $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4} \;\Rightarrow\; \dfrac{dy}{dx} = x - \dfrac{1}{4x} \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = x^2 + \dfrac{1}{2} + \dfrac{1}{16x^2}$. So

$$L = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}$$

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14. $y = \ln x \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{x} \;\Rightarrow\; \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{1+x^2}}{x}$. So $L = \displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\,dx$. Now let $v = \sqrt{1+x^2}$, so $v^2 = 1 + x^2$ and $v\,dv = x\,dx$. Thus

$$L = \int_{\sqrt{2}}^2 \frac{v}{v^2-1}\,v\,dv = \int_{\sqrt{2}}^2 \left(1 + \frac{1/2}{v-1} - \frac{1/2}{v+1}\right) dv = \left[v + \tfrac{1}{2}\ln|v-1| - \tfrac{1}{2}\ln|v+1|\right]_{\sqrt{2}}^2$$

$$= \left[v - \frac{1}{2}\ln\left|\frac{v+1}{v-1}\right|\right]_{\sqrt{2}}^2 = 2 - \sqrt{2} - \tfrac{1}{2}\ln 3 + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2 - \sqrt{2} + \ln\left(\sqrt{2}+1\right) - \tfrac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

9.2

12. $2y = 3x^{2/3}$, $y = \frac{3}{2}x^{2/3} \;\Rightarrow\; dy/dx = x^{-1/3} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + x^{-2/3}$. So

$$S = 2\pi \int_1^8 \tfrac{3}{2}x^{2/3}\sqrt{1 + x^{-2/3}}\,dx = 3\pi \int_1^2 u^2\sqrt{1 + 1/u^2}\,3u^2\,du \quad (u = x^{1/3},\, x = u^3,\, dx = 3u^2\,du)$$

$$= 9\pi \int_1^2 u^3\sqrt{u^2+1}\,du = \tfrac{9\pi}{2}\int_1^2 u^2\sqrt{u^2+1}\,2u\,du = \tfrac{9\pi}{2}\int_2^5 (y-1)\sqrt{y}\,dy \quad (y = u^2+1,\, dy = 2u\,du)$$

$$= \tfrac{9\pi}{2}\int_2^5 \left(y^{3/2} - y^{1/2}\right) dy = \tfrac{9\pi}{2}\left[\tfrac{2}{5}y^{5/2} - \tfrac{2}{3}y^{3/2}\right]_2^5 = 9\pi\left[\left(\tfrac{1}{5}\cdot 5^{5/2} - \tfrac{1}{3}\cdot 5^{3/2}\right) - \left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{1}{3}\cdot 2^{3/2}\right)\right]$$

$$= 9\pi\left[5\sqrt{5} - \tfrac{5\sqrt{5}}{3} - \tfrac{4\sqrt{2}}{5} + \tfrac{2\sqrt{2}}{3}\right] = \tfrac{3\pi}{5}\left(50\sqrt{5} - 2\sqrt{2}\right)$$

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18. $x = \dfrac{1}{2\sqrt{2}}\left(y^2 - \ln y\right) \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{1}{2\sqrt{2}}\left(2y - \dfrac{1}{y}\right) \;\Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{8}\left(2y - \frac{1}{y}\right)^2 = 1 + \frac{1}{8}\left(4y^2 - 4 + \frac{1}{y^2}\right) = \frac{1}{8}\left(4y^2 + 4 + \frac{1}{y^2}\right) = \left[\frac{1}{2\sqrt{2}}\left(2y + \frac{1}{y}\right)\right]^2$$

So

$$S = 2\pi \int_1^2 \frac{1}{2\sqrt{2}}\left(y^2 - \ln y\right)\frac{1}{2\sqrt{2}}\left(2y + \frac{1}{y}\right) dy = \frac{\pi}{4}\int_1^2 \left(2y^3 + y - 2y\ln y - \frac{\ln y}{y}\right) dy$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{2}y^4 + \tfrac{1}{2}y^2 - y^2\ln y + \tfrac{1}{2}y^2 - \tfrac{1}{2}(\ln y)^2\right]_1^2 = \tfrac{\pi}{8}\left[y^4 + 2y^2 - 2y^2\ln y - (\ln y)^2\right]_1^2$$

$$= \tfrac{\pi}{8}\left[16 + 8 - 8\ln 2 - (\ln 2)^2 - 1 - 2\right] = \tfrac{\pi}{8}\left[21 - 8\ln 2 - (\ln 2)^2\right]$$

10.3

8. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^t e^z \;\Rightarrow\; \int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C$

$\Rightarrow\; \dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = -\ln\left(e^t - C\right)$

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12. $x + 2y\sqrt{x^2+1}\dfrac{dy}{dx} = 0 \Rightarrow x\,dx + 2y\sqrt{x^2+1}\,dy = 0,\ y(0) = 1.\ \int 2y\,dy = -\int \dfrac{x\,dx}{\sqrt{x^2+1}} \Rightarrow$
$y^2 = -\sqrt{x^2+1} + C.\ y(0) = 1 \Rightarrow 1 = -1 + C \Rightarrow C = 2$, so $y^2 = 2 - \sqrt{x^2+1}$.

32. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10} 0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10 - x} = 0.005\,dt \Rightarrow \dfrac{-dx}{10 - x} = -0.005\,dt \Rightarrow \ln(10 - x) = -0.005t + c \Rightarrow 10 - x = Ce^{-0.005t}$, where $C = e^c \Rightarrow x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10\left(1 - e^{-0.005t}\right)$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.
$9 = 10\left(1 - e^{-0.005t}\right) \Rightarrow 0.9 = 1 - e^{-0.005t} \Rightarrow e^{-0.005t} = 0.1 \Rightarrow -0.005t = -\ln 10 \Rightarrow$
$t = 200\ln 10 \approx 460.517$ days ≈ 1.26 years.

10.4

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 400,\ y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get $e^{6k}/e^{2k} = 25{,}600/400 \Rightarrow e^{4k} = 64 \Rightarrow 4k = \ln 64 = 6\ln 2 \Rightarrow k = \frac{3}{2}\ln 2 = \frac{1}{2}\ln 8$. Thus, $y(0) = 400/e^{2k} = 400/e^{\ln 8} = \frac{400}{8} = 50$.

(b) $y(t) = y(0)e^{kt} = 50e^{(\ln 8)t/2}$ or $y = 50 \cdot 8^{t/2}$

(c) $y(t) = 50e^{(3\ln 2)t/2} = 100 \Leftrightarrow e^{(3\ln 2)t/2} = 2 \Leftrightarrow (3\ln 2)t/2 = \ln 2 \Leftrightarrow t = 2/3$ h $= 40$ min

(d) $50e^{(\ln 8)t/2} = 100{,}000 \Leftrightarrow e^{(\ln 8)t/2} = 2000 \Leftrightarrow (\ln 8)t/2 = \ln 2000 \Leftrightarrow t = (2\ln 2000)/\ln 8 \approx 7.3$ h.

16. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 500$, $r = 0.14$, and $t = 2$, we have:

(i) Annually: $n = 1$; $A = 500(1.14)^2 = \$649.80$

(ii) Quarterly: $n = 4$; $A = 500\left(1 + \frac{0.14}{4}\right)^8 = \$658.40$

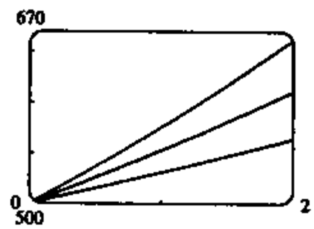
(iii) Monthly: $n = 12$; $A = 500\left(1 + \frac{0.14}{12}\right)^{24} = \$660.49$

(iv) Daily: $n = 365$; $A = 500\left(1 + \frac{0.14}{365}\right)^{2\cdot 365} = \$661.53$

(v) Hourly: $n = 365 \cdot 24$; $A = 500\left(1 + \frac{0.14}{365\cdot 24}\right)^{2\cdot 365\cdot 24} = \$661.56$

(vi) Continuously: $A = 500e^{(0.14)2} = \$661.56$

(b)



$A_{0.14}(2) = \$661.56$,
$A_{0.10}(2) = \$610.70$, and
$A_{0.06}(2) = \$563.75$.

18. $A_0e^{0.06t} = 2A_0 \Leftrightarrow e^{0.06t} = 2 \Leftrightarrow 0.06t = \ln 2 \Leftrightarrow t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.