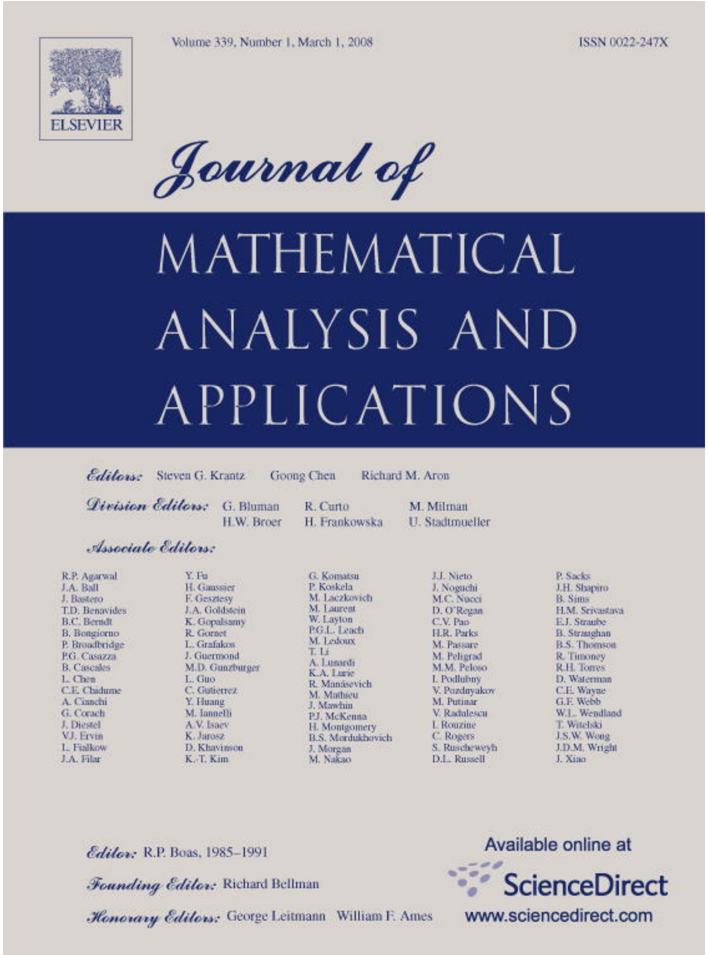
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# Remarks on the global regularity for the super-critical 2D dissipative quasi-geostrophic equation

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**Abstract**

In this article we apply the method used in the recent elegant proof by Kiselev, Nazarov and Volberg of the well-posedness of critically dissipative 2D quasi-geostrophic equation to the super-critical case. We prove that if the initial value satisfies $\|\nabla\theta_0\|_{L^\infty}^{1-2s}\|\theta_0\|_{L^\infty}^{2s} < c_s$ for some small number $c_s > 0$, where $s$ is the power of the fractional Laplacian, then no finite time singularity will occur for the super-critically dissipative 2D quasi-geostrophic equation.



*Keywords:* Quasi-geostrophic equation; Regularity conditions; Super-critically dissipative

## 1. Introduction

The study of global regularity or finite-time singularity of the two-dimensional dissipative/non-dissipative quasi-geostrophic equation (subsequently referred to as "2D QG equation" for convenience) has been an active research area in recent years. The 2D QG equation reads

$$\begin{aligned}
&\theta_t + \boldsymbol{u}\cdot\nabla\theta = -\kappa(-\Delta)^s\theta,\\
&\boldsymbol{u} = -\begin{pmatrix} -R_2 \\ R_1 \end{pmatrix}\theta,\\
&\theta|_{t=0} = \theta_0.
\end{aligned} \tag{1}$$

where $R_i = \partial_{x_i}(-\Delta)^{-1/2}$, $i = 1, 2$, are the Riesz transforms. When $\kappa = 0$ the system (1) becomes the 2D non-dissipative QG equation. When $\kappa > 0$, (1) is called "sub-critical" when $s > 1/2$, "critical" when $s = 1/2$ and "super-critical" when $s < 1/2$.

Ever since the pioneering works by Constantin, Majda and Tabak [7] and Constantin and Wu [9], which revealed close relations between dissipative/non-dissipative 2D QG equation and the 3D Navier–Stokes/Euler equations regarding global regularity or finite-time singularity, many results have been obtained by various researchers. See e.g. [2–6, 9,10,17–20,22–25,27–29] for the dissipative case, and [8,11–13] for the non-dissipative case. Among them, [9] settled

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the global regularity for the sub-critical case and [2,23] showed that smooth solutions for the critically dissipative QG equation will never blowup ([23] requires periodicity). On the other hand, whether solutions for the super-critically dissipative QG equation and the non-dissipative QG equation are globally regular is still unknown.

For the super-critical case, several small initial data results have been obtained. More specifically, global regularity has been shown when the initial data is small in spaces $B^{2-2s}_{2,1}$ [5], $H^r$ with $r > 2$ [10], or $B^r_{2,\infty}$ with $r > 2 - 2s$ [27], and when the product $\|\theta_0\|^{\beta}_{H^r}\|\theta_0\|^{1-\beta}_{L^p}$ with $r \geqslant 2 - 2\alpha$, and certain $p \in [1, \infty]$ and $\beta \in (0, 1]$ is small [22].

In this article, we derive a new global regularity result for smooth and periodic initial data which is small in certain sense using the method in [23] combined with a new representation formula for fractional Laplacians discovered by Caffarelli and Silvestre [1]. Our main theorem is the following.

**Theorem 1.** *For each* $s \in (0, 1/2)$*, there is a constant* $c_s > 0$ *such that the solution to the dissipative QG equation* (1) *with smooth periodic initial data remains smooth for all times when the initial data is small in the following sense*:

$$\|\nabla\theta_0\|^{1-2s}_{L^\infty}\|\theta_0\|^{2s}_{L^\infty} < c_s. \tag{2}$$

*Furthermore, when* (2) *is satisfied, we have the following uniform bound*

$$\|\nabla\theta\|_{L^\infty} < 2\|\nabla\theta_0\|_{L^\infty} \tag{3}$$

*for all* $t > 0$.

**Remark 1.** Our result is independent of previous small initial data results [5,10,22,27] in the sense that Theorem 1 can neither imply nor be implied by any of them. Furthermore the smallness condition (2) only involves the first derivative of $\theta_0$ while the smallness conditions in previous works all involve at least $2 - 2s$ derivatives. On the other hand, all previous results apply to the case $s = 0$ as well as non-periodic initial data too while our result does not.

**Remark 2.** It has been shown [26] that as long as

$$\int_0^T \|\nabla\theta\|_{L^\infty}(t)\,\mathrm{d}t < \infty,$$

the smooth solution $\theta$ can be extended beyond $T$. Therefore all we need to do is to show the uniform bound (3).

**Remark 3.** It can be shown that the conclusions of Theorem 1 remains true even if the initial data $\theta_0$ is not smooth and periodic. More specifically, we can show that Theorem 1 remains true for $\theta_0 \in \dot{H}^{2-2s}$. See Remark 5 in Section 8 for more explanation.

Our proof uses the same idea as [23]. More specifically, we show that for any smooth and periodic initial value $\theta_0$, there exists a modulus of continuity $\omega(\xi)$, such that $\omega(\xi)$ remains a modulus of continuity for $\theta(\boldsymbol{x}, t)$ for all $t > 0$. Once this is shown, the uniform bound of $\|\nabla\theta\|_{L^\infty}(t)$ can simply be taken as $\omega'(0)$.

Since our proof uses the same method as [23], and since the only difference between the critically and the super-critically dissipative QG equations is in the dissipation term, many arguments in [23] still work here. However we choose to repeat the main steps of these arguments for completeness and better readability of this paper.

## 2. Preliminaries

### 2.1. Modulus of continuity

**Definition 1** *(Modulus of continuity).* A modulus of continuity is a continuous, increasing and concave function $\omega : [0, +\infty) \mapsto [0, +\infty)$ with $\omega(0) = 0$. If for some function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$\big|f(\boldsymbol{x}) - f(\boldsymbol{y})\big| \leqslant \omega\big(|\boldsymbol{x} - \boldsymbol{y}|\big)$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we call $\omega$ a modulus of continuity for $f$.

**Remark 4.** There is another definition of modulus of continuity for a function $f$ in the context of classical Fourier analysis, referring to a specific function

$$\omega_M(\xi) = \sup_{|\boldsymbol{x}-\boldsymbol{y}|\leqslant\xi} \left| f(\boldsymbol{x}) - f(\boldsymbol{y}) \right|.$$

See e.g. [30]. $\omega_M(\xi)$ is increasing but not necessarily concave so may not satisfy the conditions in Definition 1. It turns out that when $f$ is periodic there is always a function $\omega$ satisfying

$$\frac{1}{2}\omega(\xi) \leqslant \omega_M(\xi) \leqslant \omega(\xi),$$

and furthermore the conditions in Definition 1 hold for $\omega$. See [16].

Note that when $\omega$ is a modulus of continuity of $\theta : \mathbb{R}^n \mapsto \mathbb{R}$, we always have

$$|\nabla\theta|(\boldsymbol{x}) \leqslant \omega'(0)$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. To see this, we take an arbitrary unit vector $\boldsymbol{e}$. By definition we have

$$\left|\theta(\boldsymbol{x} + h\boldsymbol{e}) - \theta(\boldsymbol{x})\right| \leqslant \omega(h)$$

for any $h > 0$. Recalling $\omega(0) = 0$, we have

$$|\boldsymbol{e} \cdot \nabla\theta|(\boldsymbol{x}) = \lim_{h\searrow 0} \left| \frac{\theta(\boldsymbol{x} + h\boldsymbol{e}) - \theta(\boldsymbol{x})}{h} \right| \leqslant \lim_{h\searrow 0} \frac{\omega(h) - \omega(0)}{h} = \omega'(0).$$

The conclusion follows from the arbitrariness of $\boldsymbol{e}$.

An important class of modulus of continuity is $\omega(\xi) = C\xi^\alpha$ for $\alpha \in (0, 1)$. It is easy to see that for a fixed $\alpha \in (0, 1)$, a function $f$ has a modulus of continuity $C\xi^\alpha$ for some $C > 0$ if and only if $f$ is $C^{0,\alpha}$ continuous. Therefore moduli of continuity can be seen as generalizations of Hölder continuity. In [23] it is shown that, similar to the Hölder semi-norms, moduli of continuity also enjoy nice properties under singular integral operators. In particular, we can obtain the following estimate for $\boldsymbol{u} = \binom{-R_2}{R_1}\theta$ in the 2D QG equation.

**Lemma 1** *(Estimate for Riesz transform [23]). If the function $\theta$ has modulus of continuity $\omega$, then $\boldsymbol{u} = \binom{-R_2}{R_1}\theta$ has modulus of continuity*

$$\Omega(\xi) = A\left[ \int_0^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta \right] \tag{4}$$

*with some universal constant $A > 0$.*

**Proof.** See Appendix of [23]. □

*2.2. Representation of the fractional Laplacian*

A key observation in [23] is the following representation formula

$$-(-\Delta)^{1/2}\theta = P_{2,h} * \theta,$$

where $P_{2,h}$ is the 2D Poisson kernel.

It turns out that the fractional Laplacian operators $-(-\Delta)^s$ for $s \neq 1/2$ also have similar representations, which have just been discovered by Caffarelli and Silvestre [1]. We summarize results from [1] that will be useful to our proof here.

Consider the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ for $s \in (0, 1)$. We define the following kernel

$$P_{n,h}(\boldsymbol{x}) = C_{n,s} \frac{h}{(|\boldsymbol{x}|^2 + 4s^2|h|^{1/s})^{\frac{n}{2}+s}} \tag{5}$$

where $C_{n,s}$ is a normalization constant making $\int P_{n,h}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. Then we have

$$\big[-(-\Delta)^s\theta\big](\boldsymbol{x}) = C\frac{\mathrm{d}}{\mathrm{d}h}[P_{n,h} * \theta](\boldsymbol{x})$$

where $C$ is a positive constant depending only on the dimension $n$ and the power $s$. The exact value of this constant $C$ is not important to our proof.

## 3. The breakthrough scenario

In [23], it is shown that if $\omega$ is a modulus of continuity for $\theta$ before some time $T$ but ceases to be so after $T$, then there exist two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ such that

$$\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T) = \omega\big(|\boldsymbol{x} - \boldsymbol{y}|\big)$$

when $\omega$ satisfies

(a) $\omega'(0)$ finite, and
(b) $\omega''(0+) = -\infty$, and
(c) $\omega(\xi) \to +\infty$ as $\xi \to \infty$.

We repeat the argument in [23] here for the completeness of this paper.

- Since $\omega$ is a modulus of continuity for $\theta$ for $t < T$, we have the uniform estimate

  $$\|\nabla\theta\|_{L^\infty}(t) \leqslant \omega'(0) \quad \text{for all } t < T.$$

  Thus $\theta$ remains smooth for a short time beyond $T$, and therefore $\omega$ remains a modulus of continuity for $\theta$ at $t = T$ due to the continuity of $|\theta(\boldsymbol{x}, t) - \theta(\boldsymbol{y}, t)|$ with respect to $t$.
- Now assume $|\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T)| < \omega(|\boldsymbol{x} - \boldsymbol{y}|)$ for any $\boldsymbol{x} \neq \boldsymbol{y}$. There are three cases. Let $\delta > 0$ be a very small number to be fixed.
  1. $|\boldsymbol{x} - \boldsymbol{y}| > \delta^{-1}$. Since $\omega(\xi)$ is unbounded, we can take $\delta$ so small that

     $$\omega\big(|\boldsymbol{x} - \boldsymbol{y}|\big) > \omega\big(\delta^{-1}\big) > 2\|\theta\|_{L^\infty}(T) + \varepsilon \geqslant \big|\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T)\big| + \varepsilon_0$$

     for some small $\varepsilon_0 > 0$. Thus there is $T_1 > T$ such that $|\theta(\boldsymbol{x}, t) - \theta(\boldsymbol{y}, t)| \leqslant \omega(|\boldsymbol{x} - \boldsymbol{y}|)$ for all $t \leqslant T_1$ for all $|\boldsymbol{x} - \boldsymbol{y}| > \delta^{-1}$.
  2. $|\boldsymbol{x} - \boldsymbol{y}| < \delta$. We first show that $\omega'(0) > \|\nabla\theta\|_{L^\infty}(T)$. Let $\boldsymbol{x}$ be an arbitrary point and $\boldsymbol{e}$ be an arbitrary direction, we have

     $$\big|\theta(\boldsymbol{x} + h\boldsymbol{e}, T) - \theta(\boldsymbol{x}, T)\big| < \omega(h) = \omega'(0)h + \frac{1}{2}\omega''(\xi)h^2.$$

     Note that the left-hand side is bounded from below by

     $$\big|\boldsymbol{e} \cdot \nabla\theta(\boldsymbol{x}, T)\big|h - \frac{1}{2}\big\|\nabla^2\theta\big\|_{L^\infty}(T)h^2.$$

     Since $\omega''(0+) = -\infty$, taking $h$ small enough gives $|\boldsymbol{e} \cdot \nabla\theta(\boldsymbol{x})| < \omega'(0)$. Now taking $\boldsymbol{e} = \frac{\nabla\theta(\boldsymbol{x},T)}{|\nabla\theta(\boldsymbol{x},T)|}$ we conclude $|\nabla\theta(\boldsymbol{x}, T)| < \omega'(0)$ for any $\boldsymbol{x}$. Since $\theta$ is periodic, we have $\omega'(0) > \|\nabla\theta\|_{L^\infty}(T)$.
     Thus there is $T_2 > T$ such that $\omega'(0) > \|\nabla\theta\|_{L^\infty}(t)$ for all $t \leqslant T_2$. Take $\delta$ so small that $\omega'(\delta) > \|\nabla\theta\|_{L^\infty}(t)$ for all $t \leqslant T_2$. This gives

     $$\omega\big(|\boldsymbol{x} - \boldsymbol{y}|\big) \geqslant \omega'(\delta)|\boldsymbol{x} - \boldsymbol{y}| > \|\nabla\theta\|_{L^\infty}(t)|\boldsymbol{x} - \boldsymbol{y}| \geqslant \big|\theta(\boldsymbol{x}, t) - \theta(\boldsymbol{y}, t)\big|$$

     for all $t \leqslant T_2$, where the first inequality is due to the concavity of $\omega$.
  3. $\delta \leqslant |\boldsymbol{x} - \boldsymbol{y}| \leqslant \delta^{-1}$. Since $|\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T)|$ is a periodic function in $\mathbb{R}^4$, there is $M > 0$ such that for any $\boldsymbol{x}, \boldsymbol{y}$, there are $|\boldsymbol{x}'|, |\boldsymbol{y}'| \leqslant M$, $|\boldsymbol{x}' - \boldsymbol{y}'| \geqslant \delta$ such that $|\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T)| = |\theta(\boldsymbol{x}', T) - \theta(\boldsymbol{y}', T)|$. Thus there is $\varepsilon_0 > 0$ such that

     $$\big|\theta(\boldsymbol{x}, T) - \theta(\boldsymbol{y}, T)\big| = \big|\theta(\boldsymbol{x}', T) - \theta(\boldsymbol{y}', T)\big| < \omega\big(|\boldsymbol{x}' - \boldsymbol{y}'|\big) - \varepsilon_0 \leqslant \omega\big(|\boldsymbol{x} - \boldsymbol{y}|\big) - \varepsilon_0$$

     due to the compactness of the region $|\boldsymbol{x}'|, |\boldsymbol{y}'| \leqslant M$, $|\boldsymbol{x}' - \boldsymbol{y}'| \geqslant \delta$ in $\mathbb{R}^2 \times \mathbb{R}^2$.

Therefore there is $T_3 > T$ such that $|\theta(\boldsymbol{x},t) - \theta(\boldsymbol{y},t)| < \omega(|\boldsymbol{x}-\boldsymbol{y}|)$ for all $t < T_3$.
In summary, when $|\theta(\boldsymbol{x},T) - \theta(\boldsymbol{y},T)| < \omega(|\boldsymbol{x}-\boldsymbol{y}|)$ for any $\boldsymbol{x} \neq \boldsymbol{y}$, $\omega$ will remain a modulus of continuity for $\theta$ for a short time beyond $T$.

- Therefore, if $\omega$ is a modulus of continuity for $t \leqslant T$ but ceases to be so for $t > T$, there must be two points $\boldsymbol{x}, \boldsymbol{y}$ such that

$$\left|\theta(\boldsymbol{x},T) - \theta(\boldsymbol{y},T)\right| = \omega\big(|\boldsymbol{x}-\boldsymbol{y}|\big).$$

By switching $\boldsymbol{x}$ and $\boldsymbol{y}$ if necessary, we reach

$$\theta(\boldsymbol{x},T) - \theta(\boldsymbol{y},T) = \omega\big(|\boldsymbol{x}-\boldsymbol{y}|\big). \tag{6}$$

We now set out to prove

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\theta(\boldsymbol{x},T) - \theta(\boldsymbol{y},T)\big] < 0$$

which implies

$$\theta(\boldsymbol{x},t) - \theta(\boldsymbol{y},t) > \omega\big(|\boldsymbol{x}-\boldsymbol{y}|\big)$$

for some $t < T$ but very close to $T$. This gives a contradiction.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\theta(\boldsymbol{x},T) - \theta(\boldsymbol{y},T)\big] = -\big[(\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{x},T) - (\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{y},T)\big] + \big[-(-\Delta)^s\theta\big](\boldsymbol{x},T) - \big[-(-\Delta)^s\theta\big](\boldsymbol{y},T)$$

all we need are good upper bounds of the convection term $-[(\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{y},T) - (\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{x},T)]$ and the dissipation term $[-(-\Delta)^s\theta](\boldsymbol{x},T) - [-(-\Delta)^s\theta](\boldsymbol{y},T)$. We perform such estimates in the following two sections.

In the following analysis we will suppress the time dependence since all estimates are independent of time.

## 4. Estimate of the convection term

We estimate the convection term in the same way as [23]. For completeness we repeat what they did here. Denote $\xi = |\boldsymbol{x}-\boldsymbol{y}|$.

We have

$$\theta\big(\boldsymbol{x} - h\boldsymbol{u}(\boldsymbol{x})\big) - \theta\big(\boldsymbol{y} - h\boldsymbol{u}(\boldsymbol{y})\big) \leqslant \omega\big(|\boldsymbol{x}-\boldsymbol{y}| + h\big|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{y})\big|\big) \leqslant \omega\big(\xi + h\Omega(\xi)\big).$$

Thus

$$\begin{aligned}
-\big[(\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{x}) - (\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{y})\big] &= \frac{\mathrm{d}}{\mathrm{d}h}\big[\theta\big(\boldsymbol{x} - h\boldsymbol{u}(\boldsymbol{x})\big) - \theta\big(\boldsymbol{y} - h\boldsymbol{u}(\boldsymbol{y})\big)\big]\big|_{h=0} \\
&= \lim_{h\searrow 0}\frac{1}{h}\big\{\big[\theta\big(\boldsymbol{x} - h\boldsymbol{u}(\boldsymbol{x})\big) - \theta\big(\boldsymbol{y} - h\boldsymbol{u}(\boldsymbol{y})\big)\big] - \big[\theta(\boldsymbol{x}) - \theta(\boldsymbol{y})\big]\big\} \\
&= \lim_{h\searrow 0}\frac{1}{h}\big\{\big[\theta\big(\boldsymbol{x} - h\boldsymbol{u}(\boldsymbol{x})\big) - \theta\big(\boldsymbol{y} - h\boldsymbol{u}(\boldsymbol{y})\big)\big] - \omega(\xi)\big\} \\
&\leqslant \lim_{h\searrow 0}\frac{1}{h}\big[\omega\big(\xi + h\Omega(\xi)\big) - \omega(\xi)\big] \\
&= \Omega(\xi)\omega'(\xi).
\end{aligned}$$

To summarize, we have the estimate

$$-\big[(\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{x}) - (\boldsymbol{u}\cdot\nabla\theta)(\boldsymbol{y})\big] \leqslant \Omega(\xi)\omega'(\xi) \tag{7}$$

for the convection term at the two particular points $\boldsymbol{x}, \boldsymbol{y}$ chosen in Section 3.

## 5. Estimate of the dissipation term

Now we estimate the dissipation term. Without loss of generality let $\boldsymbol{x} = (\frac{\xi}{2}, 0)$, $\boldsymbol{y} = (-\frac{\xi}{2}, 0)$ as in [23]

$$
\begin{aligned}
(P_{2,h} * \theta)(\boldsymbol{x}) - (P_{2,h} * \theta)(\boldsymbol{y}) &= \iint_{\mathbb{R}^2} \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \theta(\eta, \nu)\, \mathrm{d}\eta\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \theta(\eta, \nu)\, \mathrm{d}\eta \\
&\quad + \int_{\mathbb{R}} \mathrm{d}\nu \int_{-\infty}^0 \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \theta(\eta, \nu)\, \mathrm{d}\eta \\
&= \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \theta(\eta, \nu)\, \mathrm{d}\eta \\
&\quad + \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} + \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} + \eta, -\nu\right)\right] \theta(-\eta, \nu)\, \mathrm{d}\eta \\
&= \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \theta(\eta, \nu)\, \mathrm{d}\eta \\
&\quad - \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(-\frac{\xi}{2} + \eta, -\nu\right) - P_{2,h}\left(\frac{\xi}{2} + \eta, -\nu\right)\right] \theta(-\eta, \nu)\, \mathrm{d}\eta \\
&= \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \left[\theta(\eta, \nu) - \theta(-\eta, \nu)\right] \mathrm{d}\eta \\
&\leqslant \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \left[ P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) - P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)\right] \omega(2\eta)\, \mathrm{d}\eta \\
&= \int_0^\infty \left[ P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(-\frac{\xi}{2} - \eta\right)\right] \omega(2\eta)\, \mathrm{d}\eta
\end{aligned}
$$

where we have used the symmetry of the kernel $P_{n,h}$ and the fact that

$$
P_{2,h}\left(\frac{\xi}{2} - \eta, -\nu\right) \geqslant P_{2,h}\left(-\frac{\xi}{2} - \eta, -\nu\right)
$$

because $|\frac{\xi}{2} - \eta| \leqslant |-\frac{\xi}{2} - \eta|$ for $\xi, \eta \geqslant 0$. The last equality is because

$$
\int_{\mathbb{R}} P_{n,h}(x_1, \ldots, x_n)\, \mathrm{d}x_n = P_{n-1,h}(x_1, \ldots, x_{n-1})
$$

which can be checked directly.

Following the same argument as in [23] we have

$$
\left[-(-\Delta)^s \theta\right](\boldsymbol{x}) - \left[-(-\Delta)^s \theta\right](\boldsymbol{y}) = \frac{\mathrm{d}}{\mathrm{d}h}\left[(P_{2,h} * \theta)(\boldsymbol{x}) - (P_{2,h} * \theta)(\boldsymbol{y})\right]
$$

$$
\begin{aligned}
&= \lim_{h\searrow 0} \frac{\{[(P_{2,h} * \theta)(\boldsymbol{x}) - (P_{2,h} * \theta)(\boldsymbol{y})] - [\theta(\boldsymbol{x}) - \theta(\boldsymbol{y})]\}}{h} \\
&= \lim_{h\searrow 0} \frac{\{[(P_{2,h} * \theta)(\boldsymbol{x}) - (P_{2,h} * \theta)(\boldsymbol{y})] - \omega(\xi)\}}{h} \\
&= \lim_{h\searrow 0} \frac{1}{h}\left\{\int_0^\infty \left[P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(-\frac{\xi}{2} - \eta\right)\right]\omega(2\eta)\,\mathrm{d}\eta - \omega(\xi)\right\}.
\end{aligned}
$$

Now we simplify

$$
I = \int_0^\infty \left[P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(-\frac{\xi}{2} - \eta\right)\right]\omega(2\eta)\,\mathrm{d}\eta.
$$

We have

$$
\begin{aligned}
I &= \int_0^\infty \left[P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(\frac{\xi}{2} + \eta\right)\right]\omega(2\eta)\,\mathrm{d}\eta \\
&= \int_0^{\xi/2} P_{1,h}\left(\frac{\xi}{2} - \eta\right)\omega(2\eta)\,\mathrm{d}\eta + \int_{\xi/2}^\infty P_{1,h}\left(\frac{\xi}{2} - \eta\right)\omega(2\eta)\,\mathrm{d}\eta - \int_{\xi/2}^\infty P_{1,h}(\eta)\omega(2\eta - \xi)\,\mathrm{d}\eta \\
&= \int_0^{\xi/2} P_{1,h}(\eta)\omega(\xi - 2\eta)\,\mathrm{d}\eta + \int_0^\infty P_{1,h}(\eta)\omega(\xi + 2\eta)\,\mathrm{d}\eta - \int_{\xi/2}^\infty P_{1,h}(\eta)\omega(2\eta - \xi)\,\mathrm{d}\eta \\
&= \int_0^{\xi/2} P_{1,h}(\eta)\big[\omega(\xi - 2\eta) + \omega(\xi + 2\eta)\big]\,\mathrm{d}\eta + \int_{\xi/2}^\infty P_{1,h}(\eta)\big[\omega(2\eta + \xi) - \omega(2\eta - \xi)\big]\,\mathrm{d}\eta.
\end{aligned}
$$

On the other hand,

$$
\omega(\xi) = \int_0^\infty P_{1,h}(\eta)\big[2\omega(\xi)\big]\,\mathrm{d}\eta = \int_0^{\xi/2} P_{1,h}(\eta)\big[2\omega(\xi)\big]\,\mathrm{d}\eta + \int_{\xi/2}^\infty P_{1,h}(\eta)\big[2\omega(\xi)\big]\,\mathrm{d}\eta
$$

due to the fact that $\int_0^\infty P_{1,h}(\eta)\,\mathrm{d}\eta = \frac{1}{2}\int_{\mathbb{R}} P_{1,h}(\eta)\,\mathrm{d}\eta = \frac{1}{2}$.

Combining the above, and recalling the explicit formula (5) of $P_{n,h}$, we have

$$
\begin{aligned}
\big[-(-\Delta)^s\theta\big](\boldsymbol{x}) - \big[-(-\Delta)^s\theta\big](\boldsymbol{y}) &\leqslant \lim_{h\searrow 0} \frac{1}{h}\int_0^{\xi/2} P_{1,h}(\eta)\big[\omega(\xi - 2\eta) + \omega(\xi + 2\eta) - 2\omega(\xi)\big]\,\mathrm{d}\eta \\
&\quad + \lim_{h\searrow 0} \frac{1}{h}\int_{\xi/2}^\infty P_{1,h}(\eta)\big[\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)\big]\,\mathrm{d}\eta \\
&= C\int_0^{\xi/2} \frac{\omega(\xi - 2\eta) + \omega(\xi + 2\eta) - 2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta \\
&\quad + C\int_{\xi/2}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta
\end{aligned}
$$

for some positive constant $C$ depending on $s$ only.

Thus we obtain the following upper bound for the dissipation term:

$$C\kappa\left[\int_0^{\xi/2}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta+\int_{\xi/2}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta\right],\tag{8}$$

where $\kappa$ is the dissipation constant in (1). Note that since $\omega$ is taken to be strictly concave, both terms are negative.

## 6. Construction of the modulus of continuity

The task now is to choose a special $\omega(\xi)$ such that dissipation dominates, that is

$$A\left[\int_0^{\xi}\frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta+\xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta\right]\omega'(\xi)$$
$$+C\kappa\left[\int_0^{\xi/2}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta+\int_{\xi/2}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta\right]<0$$

for all $\xi\geqslant 0$.

We construct $\omega$ in the following way

$$\omega(\xi)=\xi-\xi^r\quad\text{when }0\leqslant\xi\leqslant\delta,\tag{9}$$

and

$$\omega'(\xi)=\frac{\gamma}{(\xi/\delta)^{\alpha}}\quad\text{when }\xi>\delta,\tag{10}$$

where $r\in(1,1+2s)$ and $\alpha\in(2s,1)$ are arbitrary constants. The other two constants $0<\delta\ll 1$ and $0<\gamma<1-r\delta^{r-1}$ are taken to be small enough.

We first check that $\omega$ satisfies the conditions in Definition 1. It is clear that $\omega$ is continuous and increasing. It is also clear that $\omega(0)=0$. Since $\omega'(\xi)$ is decreasing in $[0,\delta]$ and $(\delta,\infty)$ respectively, $\omega$ is concave as long as $\omega'(\delta-)\geqslant\omega'(\delta+)$. We compute

$$\omega'(\delta-)=1-r\delta^{r-1}$$

and

$$\omega'(\delta+)=\gamma.$$

Thus the concavity of $\omega$ is guaranteed when $\gamma<1-r\delta^{r-1}$.

We notice that $|\omega'(\xi)|\leqslant 1$ for all $\xi$, therefore $\omega(\xi)\leqslant\xi$ for all $\xi$. Also note that for the simplicity of formulas our $\gamma$ corresponds to the quantity $\gamma/\delta$ in [23]. We would also like to remark that our construction here does not extend to the critical case or the case $s=0$.

We discuss the cases $0\leqslant\xi\leqslant\delta$ and $\xi>\delta$ separately.

### *6.1. The case* $0\leqslant\xi\leqslant\delta$

- Convection term.

We have

$$\int_0^{\xi}\frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta=\xi-\frac{1}{r}\xi^r\leqslant\xi,$$

and

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta = \int_\xi^\delta \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta + \int_\delta^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta = \int_\xi^\delta \frac{\eta-\eta^r}{\eta^2}\,\mathrm{d}\eta - \int_\delta^\infty \omega(\eta)\,\mathrm{d}\left(\frac{1}{\eta}\right)$$
$$\leqslant \log\frac{\delta}{\xi} + \frac{\omega(\delta)}{\delta} + \int_\delta^\infty \frac{\omega'(\eta)}{\eta}\,\mathrm{d}\eta \leqslant \log\frac{\delta}{\xi} + 1 + \int_\delta^\infty \frac{\gamma\delta^\alpha}{\eta^{1+\alpha}}\,\mathrm{d}\eta$$
$$= \log\frac{\delta}{\xi} + 1 + \frac{\gamma}{\alpha} \leqslant \log\frac{\delta}{\xi} + 2$$

if we take $\gamma \leqslant \alpha \in (2s, 1)$.

Thus the convection term

$$A\left[\int_0^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta\right]\omega'(\xi)$$

can be estimated from above by

$$A\xi\left(3 + \log\frac{\delta}{\xi}\right) \tag{11}$$

since $0 < \omega'(\xi) \leqslant 1$.

- Dissipation term.

We have

$$\int_0^{\xi/2} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+2s}} \leqslant C\xi^{2-2s}\omega''(\xi) = -C\xi\xi^{r-(1+2s)}$$

for some constant $C > 0$ when $\delta$ is small enough, where the first inequality comes from Taylor expansion and the fact that $\omega''$ is increasing for the $\omega$ defined by (9)–(10).

Since the other term is always negative, the dissipation term

$$C\kappa\left[\int_0^{\xi/2} \frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta + \int_{\xi/2}^\infty \frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta\right]$$

can be estimated from above by

$$-\kappa\xi\xi^{r-(1+2s)}. \tag{12}$$

Now combining the above, we see that the sum of the convection and the dissipation terms for $\xi \leqslant \delta$ can be estimated from above by

$$\xi\left[A\left(3+\log\frac{\delta}{\xi}\right) - \kappa\xi^{r-(1+2s)}\right].$$

This is negative when $\delta$ and consequently $\xi$ is small enough since $r-(1+2s) < 0$.

### 6.2. The case $\xi > \delta$

- Convection term.

We have

$$\int_0^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta = \int_0^\delta \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta + \int_\delta^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta \leqslant \delta + \int_\delta^\xi \frac{\omega(\xi)}{\eta}\,\mathrm{d}\eta = \delta + \omega(\xi)\log\frac{\xi}{\delta} \leqslant \omega(\xi)\left(2+\log\frac{\xi}{\delta}\right)$$

where in the last step we have used $\omega(\xi) \geqslant \omega(\delta) = \delta - \delta^r > \delta/2$ when $\delta$ is small enough.

On the other hand,

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta = -\int_\xi^\infty \omega(\eta)\,\mathrm{d}\left(\frac{1}{\eta}\right) = \frac{\omega(\xi)}{\xi} + \int_\xi^\infty \frac{\omega'(\eta)}{\eta}\,\mathrm{d}\eta = \frac{\omega(\xi)}{\xi} + \gamma\delta^\alpha \int_\xi^\infty \frac{1}{\eta^{1+\alpha}}\,\mathrm{d}\eta = \frac{\omega(\xi)}{\xi} + \frac{\gamma}{\alpha}\delta^\alpha \xi^{-\alpha}$$
$$= \frac{1}{\xi}\left[\omega(\xi) + \frac{\gamma}{\alpha}\delta^\alpha \xi^{1-\alpha}\right].$$

Note that for $\xi > \delta$ we have

$$\omega(\xi) = \delta - \delta^r + \int_\delta^\xi \frac{\gamma\delta^\alpha}{\eta^\alpha}\,\mathrm{d}\eta = \delta - \delta^r + \frac{\gamma}{1-\alpha}\delta^\alpha\xi^{1-\alpha} - \frac{\gamma}{1-\alpha}\delta$$
$$\geqslant \left[\frac{1}{2} - \frac{\gamma}{1-\alpha}\right]\delta + \frac{\gamma}{1-\alpha}\delta^\alpha\xi^{1-\alpha} = \frac{\gamma}{1-\alpha}\delta^\alpha\xi^{1-\alpha}$$

if we take $\gamma$ to be less than $\frac{1-\alpha}{2} > 0$ so that $\frac{1}{2} - \frac{\gamma}{1-\alpha} > 0$.

Thus we have

$$\delta^\alpha\xi^{1-\alpha} \leqslant \frac{1-\alpha}{\gamma}\omega(\xi). \tag{13}$$

Therefore

$$\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta = \frac{1}{\xi}\left[\omega(\xi) + \frac{\gamma}{\alpha}\delta^\alpha\xi^{1-\alpha}\right] \leqslant \frac{1}{\alpha}\frac{\omega(\xi)}{\xi}.$$

Now we can estimate the convection term

$$A\left[\int_0^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta\right]\omega'(\xi)$$

as follows:

$$A\left[\int_0^\xi \frac{\omega(\eta)}{\eta}\,\mathrm{d}\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\,\mathrm{d}\eta\right]\omega'(\xi) \leqslant \omega(\xi)\left(C + \log\frac{\xi}{\delta}\right)\omega'(\xi) = \gamma\omega(\xi)\left(C + \log\frac{\xi}{\delta}\right)\left(\frac{\xi}{\delta}\right)^{-\alpha}. \tag{14}$$

- The dissipation term.

To estimate the dissipation term we notice that

$$\omega(2\eta+\xi) - \omega(2\eta-\xi) \leqslant \omega(2\xi) = \omega(\xi) + \int_\xi^{2\xi} \frac{\gamma\delta^\alpha}{\eta^\alpha}\,\mathrm{d}\eta = \omega(\xi) + \frac{\gamma(2^{1-\alpha}-1)}{1-\alpha}\delta^\alpha\xi^{1-\alpha}$$
$$\leqslant \omega(\xi) + \left(2^{1-\alpha}-1\right)\omega(\xi) \leqslant 2^{1-\alpha}\omega(\xi),$$

where we have used (13).

Notice that $2^{1-\alpha} < 2$ since $\alpha \in (2s, 1)$, there is $C > 0$ such that $\omega(2\xi) - 2\omega(\xi) \leqslant -C\omega(\xi)$ for all $\xi > \delta$.

Thus the dissipation term

$$C\kappa\left[\int_0^{\xi/2}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta+\int_{\xi/2}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta\right]$$

is bounded from above by

$$\begin{aligned}C\kappa\int_{\xi/2}^{\infty}\frac{\omega(2\eta+\xi)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta &\leqslant C\kappa\int_{\xi/2}^{\infty}\frac{\omega(2\xi)-2\omega(\xi)}{\eta^{1+2s}}\,\mathrm{d}\eta\\ &\leqslant -C\kappa\omega(\xi)\left(\int_{\xi/2}^{\infty}\frac{\mathrm{d}\eta}{\eta^{1+2s}}\right)\\ &= -C\kappa\xi^{-2s}\omega(\xi).\end{aligned}\tag{15}$$

for some positive constant $C$.

Combining the above, we see that the sum of the convection and the dissipation terms for $\xi>\delta$ is

$$\omega(\xi)\left[\gamma\left(C'+\log\frac{\xi}{\delta}\right)\left(\frac{\xi}{\delta}\right)^{-\alpha}-\kappa C\xi^{-2s}\right]=\omega(\xi)\left[\gamma\left(C'+\log\frac{\xi}{\delta}\right)\left(\frac{\xi}{\delta}\right)^{-\alpha}-\kappa C\delta^{2s}\left(\frac{\xi}{\delta}\right)^{-2s}\right]$$

which is negative if $\gamma\leqslant\kappa C\delta^{2s}$ and $\delta$ is small enough since $\alpha>2s$. Here we use $C'$ for the constant in the convection estimate. Note that since the constants are all independent of $\gamma$, we are free to make $\gamma$ small.

## 7. Global regularity

We have shown that the if $\theta_0$ has a modulus of continuity $\omega(\xi)$ as defined by (9)–(10), then $\omega(\xi)$ will remain to be a modulus of continuity for $\theta$ for all $t$. We now show that, due to the special scaling invariance $\theta(x,t)\mapsto\mu^{2s-1}\theta(\mu x,\mu^{2s}t)$ of the super-critically dissipative QG equation, there is $c_s>0$ such that whenever (2) is satisfied, i.e., $\|\nabla\theta_0\|_{L^\infty}^{1-2s}\|\theta_0\|_{L^\infty}^{2s}<c_s$, we can find $\mu$ such that a re-scaling of $\theta_0$ has modulus of continuity $\omega(\xi)$, thus proving Theorem 1.

### *7.1. Scaling invariance of super-critical dissipative QG equation*

We first recall that the super-critical dissipative QG equation has the following scaling invariance:

$$\theta(\boldsymbol{x},t)\mapsto\mu^{2s-1}\theta\big(\mu\boldsymbol{x},\mu^{2s}t\big).$$

More specifically, if $\theta_0(\boldsymbol{x})=\mu^{2s-1}\tilde\theta_0(\mu\boldsymbol{x},\mu^{2s}t)$, and $\tilde\theta$ solves

$$\tilde\theta_t+\tilde{\boldsymbol{u}}\cdot\nabla\tilde\theta=-(-)^s\tilde\theta$$

for some $s\in(0,\frac12)$ with initial data $\tilde\theta_0$, then

$$\theta(\boldsymbol{x},t)=\mu^{2s-1}\tilde\theta\big(\mu\boldsymbol{x},\mu^{2s}t\big)$$

solves the same equation with initial data $\theta_0$. Thus in particular, if $\tilde\theta$ is globally regular, so is $\theta$.

On the other hand, if $\tilde\omega(\xi)$ is a modulus of continuity for $\tilde\theta_0(\boldsymbol{x})$, then

$$\omega(\xi)\equiv\mu^{2s-1}\tilde\omega(\mu\xi)$$

is a modulus of continuity for $\theta_0(\boldsymbol{x})=\mu^{2s-1}\tilde\theta_0(\mu\boldsymbol{x},\mu^{2s}t)$.

Therefore, to establish global regularity for certain smooth and periodic initial data $\theta_0$, it suffices to find a scaling constant $\mu$ such that

$$\omega_\mu(\xi)=\mu^{2s-1}\omega(\mu\xi)$$

is a modulus of continuity for $\theta_0(\boldsymbol{x})$, where $\omega(\xi)$ is the particular modulus of continuity defined by (9)–(10).

*7.2. Rescaling $\omega(\xi)$*

Recall that $\omega(\xi)$ is defined by

$$\omega(\xi) = \xi - \xi^r \quad \text{when } 0 \leqslant \xi \leqslant \delta$$

and

$$\omega'(\xi) = \frac{\gamma \delta^\alpha}{\xi^\alpha} \quad \text{when } \xi > \delta.$$

We can easily compute

$$\omega_\mu\big(|\boldsymbol{x}-\boldsymbol{y}|\big) = \begin{cases} \mu^{2s}|\boldsymbol{x}-\boldsymbol{y}| - \mu^{2s-1+r}|\boldsymbol{x}-\boldsymbol{y}|^r & \text{when } \mu|\boldsymbol{x}-\boldsymbol{y}| \leqslant \delta, \\ \frac{\gamma\delta^\alpha}{1-\alpha}\mu^{2s-\alpha}|\boldsymbol{x}-\boldsymbol{y}|^{1-\alpha} + \mu^{2s-1}[\delta - \frac{\gamma}{1-\alpha}\delta - \delta^r] & \text{when } \mu|\boldsymbol{x}-\boldsymbol{y}| > \delta. \end{cases}$$

Now we take $\mu^{2s} = 2\|\nabla\theta_0\|_{L^\infty}$. Then for $\boldsymbol{x}, \boldsymbol{y}$ such that $|\boldsymbol{x}-\boldsymbol{y}| \leqslant \delta/\mu$, we have

$$\begin{aligned} \big|\theta_0(\boldsymbol{x}) - \theta_0(\boldsymbol{y})\big| &\leqslant \|\nabla\theta_0\|_{L^\infty}|\boldsymbol{x}-\boldsymbol{y}| = 2\|\nabla\theta_0\|_{L^\infty}|\boldsymbol{x}-\boldsymbol{y}|\frac{1}{2} = \mu^{2s}|\boldsymbol{x}-\boldsymbol{y}|\frac{1}{2} \\ &\leqslant \mu^{2s}|\boldsymbol{x}-\boldsymbol{y}|\big(1-\delta^{r-1}\big) \leqslant \mu^{2s}|\boldsymbol{x}-\boldsymbol{y}|\big(1-\mu^{r-1}|\boldsymbol{x}-\boldsymbol{y}|^{r-1}\big) = \omega_\mu\big(|\boldsymbol{x}-\boldsymbol{y}|\big) \end{aligned}$$

according to our choice of $\delta$.

On the other hand, for any $\boldsymbol{x}, \boldsymbol{y}$ with $|\boldsymbol{x}-\boldsymbol{y}| > \delta/\mu$, we have

$$\big|\theta_0(\boldsymbol{x}) - \theta_0(\boldsymbol{y})\big| \leqslant 2\|\theta_0\|_{L^\infty}.$$

So it is clear that $\omega_\mu$ is a modulus of continuity of $\theta_0$ as long as

$$2\|\theta_0\|_{L^\infty} \leqslant \omega_\mu\left(\frac{\delta}{\mu}\right) = \mu^{2s-1}\big(\delta - \delta^r\big) = 2^{\frac{2s-1}{2s}}\|\nabla\theta_0\|_{L^\infty}^{\frac{2s-1}{2s}}\big(\delta-\delta^r\big).$$

This can be simplified to

$$\|\nabla\theta_0\|_{L^\infty}^{1-2s}\|\theta_0\|_{L^\infty}^{2s} \leqslant \frac{1}{2}\big(\delta-\delta^r\big)^{2s}.$$

Now taking

$$c_s = \frac{1}{2}\big(\delta-\delta^r\big)^{2s}$$

we see that

$$\|\nabla\theta_0\|_{L^\infty}^{1-2s}\|\theta_0\|_{L^\infty}^{2s} \leqslant c_s$$

is sufficient for $\theta$ to stay smooth for all time.

## 8. Final remarks

Theorem 1 can be generalized to the case with non-periodic initial data. The smallness condition can also be improved slightly when the level sets are smooth around the "breakthrough" points.

**Remark 5.** In [14], the author proved that when the initial data $\theta_0 \in \dot{H}^{2-2s}$, the solution exists at least for finite time $[0, T)$, and furthermore the following decay estimates hold:

$$\sup_{0<t<T} t^{\beta/(2s)}\|\theta\|_{H^{2-2s+\beta}} < +\infty \quad \forall\beta \leqslant 0.$$

Using this and similar argument as in [15], one can prove that the breakthrough scenario as in Section 3 remains true for general non-periodic $\theta_0$. With the breakthrough scenario true, the conclusions of Theorem 1 can be established for general non-periodic initial data with the same argument as for the periodic case.

**Remark 6.** When the level sets around the two "breakthrough" points $\boldsymbol{x}$, $\boldsymbol{y}$, which satisfy (6), stay smooth, for example, when the unit tangent vector field around $\boldsymbol{x}$, $\boldsymbol{y}$ stays Lipschitz continuous, we can show that no finite singularity can occur when the Hölder semi-norm $|\theta_0|_{\dot{C}^{1-2s}}$ is small, thus weakening the smallness condition (2). Very recently Ju [21] proved global regularity for the 2D QG equation with critical dissipation under similar Lipschitz assumption on the unit tangent vector field in regions with large $|\nabla\theta|$. Although the result there holds for all initial data including non-periodic ones, the method there cannot be applied to the super-critical case.

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## References

[1] Luis Caffarelli, Luis Silvestre, An extension problem related to the fractional Laplacian, preprint, arXiv: math.AP/0608640 v1, August 2006.
[2] Luis Caffarelli, Alexis Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, preprint, arXiv: math.AP/0608447 v1, August 2006.
[3] Dongho Chae, The quasi-geostrophic equation in the Triebel–Lizorkin spaces, Nonlinearity 16 (2003) 479–495.
[4] Dongho Chae, On the regularity conditions for the dissipative quasi-geostrophic equations, SIAM J. Math. Anal. 37 (2006) 1649–1656.
[5] Dongho Chae, Jihoon Lee, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, Comm. Math. Phys. 233 (2003) 297–311.
[6] Peter Constantin, Diego Cordoba, Jiahong Wu, On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. 50 (2001) 97–107.
[7] Peter Constantin, Andrew J. Majda, Esteban Tabak, Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar, Nonlinearity 7 (1994) 1495–1533.
[8] Peter Constantin, Qing Nie, Norbert Schörghofer, Nonsingular surface quasi-geostrophic flow, Phys. Lett. A 241 (3) (1998) 168–172.
[9] Peter Constantin, Jiahong Wu, Behavior of solutions of 2d quasi-geostrophic equations, SIAM J. Math. Anal. 30 (1999) 937–948. E-HOST030
[10] Antonio Cordoba, Diego Cordoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004) 511–528.
[11] Diego Cordoba, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Ann. of Math. 148 (3) (1998) 1135–1152.
[12] Diego Cordoba, Charles Fefferman, Growth of solutions for QG and 2D Euler equations, J. Amer. Math. Soc. 15 (3) (2002) 665–670 (electronic).
[13] Jian Deng, Thomas Y. Hou, Ruo Li, Xinwei Yu, Level set dynamics and the non-blowup of the 2D quasi-geostrophic equation, Methods Appl. Anal. (2006), in press.
[14] Hongjie Dong, Higher regularity for the critical and super-critical dissipative quasi-geostrophic equations, preprint, arXiv: math/0701826, 2007.
[15] Hongjie Dong, Dapeng Du, Global well-posedness and a decay estimate for the critical quasi-geostrophic equation, preprint, arXiv: math/0701828, 2007.
[16] A.V. Efimov, Linear methods for the approximation of continuous periodic functions, Mat. Sb. 54 (1961) 51–90 (in Russian).
[17] Sadek Gala, Amina Lahmar-Benbernou, Dissipative quasi-geostrophic equations with initial data, preprint, arXiv: math.AP/0507492 v1, July 2005.
[18] Ning Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys. 251 (2) (2004) 365–376.
[19] Ning Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys. 255 (1) (2005) 161–181.
[20] Ning Ju, On the two-dimensional quasi-geostrophic equations, Indiana Univ. Math. J. 54 (3) (2005) 897–926.
[21] Ning Ju, Geometric constrains for global regularity of 2D quasi-geostrophic flows, J. Differential Equations 226 (2006) 54–79.
[22] Ning Ju, Global solutions to the two-dimensional quasi-geostrophic equation with critical or super-critical dissipation, Math. Ann. 334 (2006) 627–642.
[23] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical $2d$ dissipative quasi-geostrophic equation, Invent. Math. 167 (2007) 445–453.
[24] Hideyuki Miura, Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space, Comm. Math. Phys. 267 (2006) 141–157.
[25] Atanas Stefanov, Global well-posedness for the 2d quasi-geostrophic equation in a critical Besov space, preprint, arXiv: math.AP/0607320 v1, July 2006.
[26] Jiahong Wu, Dissipative quasi-geostrophic equations with $L^p$ data, Electron. J. Differential Equations 2001 (56) (2001) 1–13.
[27] Jiahong Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, SIAM J. Math. Anal. 36 (2005) 1014–1030.
[28] Jiahong Wu, Solutions of the 2D quasi-geostrophic equation in Hölder spaces, Nonlinear Anal. 62 (2005) 579–594.
[29] Jiahong Wu, The two-dimensional quasi-geostrophic equation with critical or supercritical dissipation, Nonlinearity 18 (2005) 139–154.
[30] Antoni Zygmund, Trigonometric Series, vol. 1, Cambridge Univ. Press, 1959.