Remarks on the global regularity for the super-critical 2D dissipative quasi-geostrophic equation

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Abstract

In this article we apply the method used in the recent elegant proof by Kiselev, Nazarov and Volberg of the well-posedness of critically dissipative 2D quasi-geostrophic equation to the super-critical case. We prove that if the initial value satisfies 
\[ \| \nabla \theta_0 \|_{L^\infty}^2 \| \theta_0 \|_{L^\infty} < c_s \]
for some small number \( c_s > 0 \), where \( s \) is the power of the fractional Laplacian, then no finite time singularity will occur for the super-critically dissipative 2D quasi-geostrophic equation.

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1. Introduction

The study of global regularity or finite-time singularity of the two-dimensional dissipative/non-dissipative quasi-geostrophic equation (subsequently referred to as “2D QG equation” for convenience) has been an active research area in recent years. The 2D QG equation reads

\[ \begin{align*}
\theta_t + u \cdot \nabla \theta &= -\kappa (-\Delta)^s \theta, \\
u &= -\begin{pmatrix} -R_2 \\ R_1 \end{pmatrix} \theta, \\
\theta|_{t=0} &= \theta_0,
\end{align*} \]

where \( R_i = \partial_{x_i} (-\Delta)^{-1/2}, \ i = 1, 2, \) are the Riesz transforms. When \( \kappa = 0 \) the system (1) becomes the 2D non-dissipative QG equation. When \( \kappa > 0 \), (1) is called “sub-critical” when \( s > 1/2 \), “critical” when \( s = 1/2 \) and “super-critical” when \( s < 1/2 \).

Ever since the pioneering works by Constantin, Majda and Tabak [7] and Constantin and Wu [9], which revealed close relations between dissipative/non-dissipative 2D QG equation and the 3D Navier–Stokes/Euler equations regarding global regularity or finite-time singularity, many results have been obtained by various researchers. See e.g. [2–6, 9,10,17–20,22–25,27–29] for the dissipative case, and [8,11–13] for the non-dissipative case. Among them, [9] settled
the global regularity for the sub-critical case and [2,23] showed that smooth solutions for the critically dissipative QG equation will never blowup ([23] requires periodicity). On the other hand, whether solutions for the super-critically dissipative QG equation and the non-dissipative QG equation are globally regular is still unknown.

For the super-critical case, several small initial data results have been obtained. More specifically, global regularity has been shown when the initial data is small in spaces $B_{2,1}^{2-2\alpha}$ [5], $H^r$ with $r > 2$ [10], or $B_{2,\infty}^r$ with $r > 2 - 2\alpha$ [27], and when the product $\|\theta_0\|_{H^r}^p \|\theta_0\|_{L^p}^{1-p}$ with $r \geq 2 - 2\alpha$, and certain $p \in [1, \infty]$ and $\beta \in (0, 1]$ is small [22].

In this article, we derive a new global regularity result for smooth and periodic initial data which is small in certain sense using the method in [23] combined with a new representation formula for fractional Laplacians discovered by Caffarelli and Silvestre [1]. Our main theorem is the following.

**Theorem 1.** For each $s \in (0, 1/2)$, there is a constant $c_s > 0$ such that the solution to the dissipative QG equation (1) with smooth periodic initial data remains smooth for all times when the initial data is small in the following sense:

$$\|\nabla \theta_0\|_{L^\infty}^{1-2s} \|\theta_0\|_{L^\infty}^{2s} < c_s.$$  

Furthermore, when (2) is satisfied, we have the following uniform bound

$$\|\nabla \theta\|_{L^\infty} < 2\|\nabla \theta_0\|_{L^\infty}$$  

for all $t > 0$.

**Remark 1.** Our result is independent of previous small initial data results [5,10,22,27] in the sense that Theorem 1 can neither imply nor be implied by any of them. Furthermore the smallness condition (2) only involves the first derivative of $\theta_0$ while the smallness conditions in previous works all involve at least 2 $- 2\alpha$ derivatives. On the other hand, all previous results apply to the case $s = 0$ as well as non-periodic initial data too while our result does not.

**Remark 2.** It has been shown [26] that as long as

$$\int_0^T \|\nabla \theta\|_{L^\infty}(t) \, dt < \infty,$$

the smooth solution $\theta$ can be extended beyond $T$. Therefore all we need to do is to show the uniform bound (3).

**Remark 3.** It can be shown that the conclusions of Theorem 1 remains true even if the initial data $\theta_0$ is not smooth and periodic. More specifically, we can show that Theorem 1 remains true for $\theta_0 \in \dot{H}^{2-2\alpha}$. See Remark 5 in Section 8 for more explanation.

Our proof uses the same idea as [23]. More specifically, we show that for any smooth and periodic initial value $\theta_0$, there exists a modulus of continuity $\omega(\xi)$, such that $\omega(\xi)$ remains a modulus of continuity for $\theta(x, t)$ for all $t > 0$. Once this is shown, the uniform bound of $\|\nabla \theta\|_{L^\infty}(t)$ can simply be taken as $\omega'(0)$.

Since our proof uses the same method as [23], and since the only difference between the critically and the super-critically dissipative QG equations is in the dissipation term, many arguments in [23] still work here. However we choose to repeat the main steps of these arguments for completeness and better readability of this paper.

2. Preliminaries

2.1. Modulus of continuity

**Definition 1 (Modulus of continuity).** A modulus of continuity is a continuous, increasing and concave function $\omega : [0, +\infty) \mapsto [0, +\infty)$ with $\omega(0) = 0$. If for some function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$

$$|f(x) - f(y)| \leq \omega(|x - y|)$$

holds for all $x, y \in \mathbb{R}^n$, we call $\omega$ a modulus of continuity for $f$.  


Remark 4. There is another definition of modulus of continuity for a function \( f \) in the context of classical Fourier analysis, referring to a specific function

\[
\omega_M(\xi) = \sup_{|x-y| \leq \xi} |f(x) - f(y)|.
\]

See e.g. [30]. \( \omega_M(\xi) \) is increasing but not necessarily concave so may not satisfy the conditions in Definition 1. It turns out that when \( f \) is periodic there is always a function \( \omega \) satisfying

\[
\frac{1}{2} \omega(\xi) \leq \omega_M(\xi) \leq \omega(\xi),
\]

and furthermore the conditions in Definition 1 hold for \( \omega \). See [16].

Note that when \( \omega \) is a modulus of continuity of \( \theta : \mathbb{R}^n \mapsto \mathbb{R} \), we always have

\[
|\nabla \theta(x)| \leq \omega'(0)
\]

for all \( x \in \mathbb{R}^n \). To see this, we take an arbitrary unit vector \( e \). By definition we have

\[
|\theta(x + he) - \theta(x)| \leq \omega(h)
\]

for any \( h > 0 \). Recalling \( \omega(0) = 0 \), we have

\[
|e \cdot \nabla \theta(x) = \lim_{h \searrow 0} \frac{\theta(x + he) - \theta(x)}{h} | \leq \lim_{h \searrow 0} \frac{\omega(h) - \omega(0)}{h} = \omega'(0).
\]

The conclusion follows from the arbitrariness of \( e \).

An important class of modulus of continuity is \( \omega(\xi) = C\xi^\alpha \) for \( \alpha \in (0, 1) \). It is easy to see that for a fixed \( \alpha \in (0, 1) \), a function \( f \) has a modulus of continuity \( C\xi^\alpha \) for some \( C > 0 \) if and only if \( f \) is \( C^{0,\alpha} \) continuous. Therefore moduli of continuity can be seen as generalizations of Hölder continuity. In [23] it is shown that, similar to the Hölder semi-norms, moduli of continuity also enjoy nice properties under singular integral operators. In particular, we can obtain the following estimate for \( u = (−\Delta_\Omega^R)\theta \) in the 2D QG equation.

Lemma 1 (Estimate for Riesz transform [23]). If the function \( \theta \) has modulus of continuity \( \omega \), then \( u = (−\Delta_\Omega^R)\theta \) has modulus of continuity

\[
\Omega(\xi) = A \left[ \int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta \right]
\]

with some universal constant \( A > 0 \).

Proof. See Appendix of [23]. □

2.2. Representation of the fractional Laplacian

A key observation in [23] is the following representation formula

\[-(−\Delta)^{1/2} \theta = P_{2,h} \ast \theta,\]

where \( P_{2,h} \) is the 2D Poisson kernel.

It turns out that the fractional Laplacian operators \( -(−\Delta)^s \) for \( s \neq 1/2 \) also have similar representations, which have just been discovered by Caffarelli and Silvestre [1]. We summarize results from [1] that will be useful to our proof here.

Consider the fractional Laplacian \( (−\Delta)^s \) in \( \mathbb{R}^n \) for \( s \in (0, 1) \). We define the following kernel

\[
P_{n,h}(x) = C_{n,s} \frac{h}{(||x|^2 + 4s^2|h|^{1/s})^{n/2+s}}
\]

where \( C_{n,s} \) is a constant depending on \( n \) and \( s \).
where $C_{n,s}$ is a normalization constant making $\int P_{n,h}(x) \, dx = 1$. Then we have

$$\left[-(-\Delta)^s \theta\right](x) = C \frac{d}{dh} [P_{n,h} \ast \theta](x)$$

where $C$ is a positive constant depending only on the dimension $n$ and the power $s$. The exact value of this constant $C$ is not important to our proof.

3. The breakthrough scenario

In [23], it is shown that if $\omega$ is a modulus of continuity for $\theta$ before some time $T$ but ceases to be so after $T$, then there exist two points $x, y \in \mathbb{R}^2$ such that

$$\theta(x, T) - \theta(y, T) = \omega(|x - y|)$$

when $\omega$ satisfies

(a) $\omega'(0)$ finite, and
(b) $\omega''(0+)$ $= -\infty$, and
(c) $\omega(\xi) \to +\infty$ as $\xi \to \infty$.

We repeat the argument in [23] here for the completeness of this paper.

- Since $\omega$ is a modulus of continuity for $\theta$ for $t < T$, we have the uniform estimate

$$\|\nabla \theta\|_{L^\infty}(t) \leq \omega'(0) \quad \text{for all } t < T.$$  

Thus $\theta$ remains smooth for a short time beyond $T$, and therefore $\omega$ remains a modulus of continuity for $\theta$ at $t = T$ due to the continuity of $|\theta(x, t) - \theta(y, t)|$ with respect to $t$.

- Now assume $|\theta(x, T) - \theta(y, T)| = \omega(|x - y|)$ for any $x \neq y$. There are three cases. Let $\delta > 0$ be a very small number to be fixed.

1. $|x - y| > \delta^{-1}$. Since $\omega(\xi)$ is unbounded, we can take $\delta$ so small that

$$\omega(|x - y|) > \omega(\delta^{-1}) > 2\|\nabla \theta\|_{L^\infty}(T) + \epsilon \geq |\theta(x, T) - \theta(y, T)| + \epsilon_0$$

for some small $\epsilon_0 > 0$. Thus there is $T_1 > T$ such that $|\theta(x, t) - \theta(y, t)| \leq \omega(|x - y|)$ for all $t \leq T_1$ for all $|x - y| > \delta^{-1}$.

2. $|x - y| < \delta$. We first show that $\omega'(0) > \|\nabla \theta\|_{L^\infty}(T)$. Let $x$ be an arbitrary point and $e$ be an arbitrary direction, we have

$$|\theta(x + he, T) - \theta(x, T)| < \omega(h) = \omega'(0)h + \frac{1}{2} \omega''(\delta) h^2.$$  

Note that the left-hand side is bounded from below by

$$|e \cdot \nabla \theta(x, T)|h - \frac{1}{2}\|\nabla^2 \theta\|_{L^\infty}(T) h^2.$$  

Since $\omega''(0+) = -\infty$, taking $h$ small enough gives $|e \cdot \nabla \theta(x)| < \omega'(0)$. Now taking $\epsilon = \frac{\nabla \theta(x, T)}{|\nabla \theta(x, T)|}$ we conclude $|\nabla \theta(x, T)| < \omega'(0)$ for any $x$. Since $\theta$ is periodic, we have $\omega'(0) > \|\nabla \theta\|_{L^\infty}(T)$.

Thus there is $T_2 > T$ such that $\omega'(0) > \|\nabla \theta\|_{L^\infty}(t)$ for all $t \leq T_2$. Take $\delta$ so small that $\omega'(\delta) > \|\nabla \theta\|_{L^\infty}(t)$ for all $t \leq T_2$. This gives

$$\omega(|x - y|) \geq \omega'(\delta)|x - y| > \|\nabla \theta\|_{L^\infty}(t)|x - y| \geq |\theta(x, t) - \theta(y, t)|$$

for all $t \leq T_2$, where the first inequality is due to the concavity of $\omega$.

3. $\delta \leq |x - y| \leq \delta^{-1}$. Since $|\theta(x, T) - \theta(y, T)|$ is a periodic function in $\mathbb{R}^4$, there is $M > 0$ such that for any $x, y$, there are $|x'|, |y'| \leq M, |x' - y'| \geq \delta$ such that $|\theta(x, T) - \theta(y, T)| = |\theta(x', T) - \theta(y', T)|$. Thus there is $\epsilon_0 > 0$ such that

$$|\theta(x, T) - \theta(y, T)| = |\theta(x', T) - \theta(y', T)| < \omega(|x' - y'|) - \epsilon_0 \leq \omega(|x - y|) - \epsilon_0$$

due to the compactness of the region $|x'|, |y'| \leq M, |x' - y'| \geq \delta$ in $\mathbb{R}^2 \times \mathbb{R}^2$.  

Therefore there is $T_3 > T$ such that $|\theta(x, t) - \theta(y, t)| < \omega(|x - y|)$ for all $t < T_3$.

In summary, when $|\theta(x, T) - \theta(y, T)| < \omega(|x - y|)$ for any $x \neq y$, $\omega$ will remain a modulus of continuity for $\theta$ for a short time beyond $T$.

Therefore, if $\omega$ is a modulus of continuity for $t \leq T$ but ceases to be so for $t > T$, there must be two points $x$, $y$ such that

$$|\theta(x, T) - \theta(y, T)| = \omega(|x - y|).$$

By switching $x$ and $y$ if necessary, we reach

$$\theta(x, T) - \theta(y, T) = \omega(|x - y|).$$  \hspace{1cm} (6)

We now set out to prove

$$\frac{d}{dt} [\theta(x, T) - \theta(y, T)] < 0$$

which implies

$$\theta(x, t) - \theta(y, t) > \omega(|x - y|)$$

for some $t < T$ but very close to $T$. This gives a contradiction.

Since

$$\frac{d}{dt} [\theta(x, T) - \theta(y, T)] = -[(u \cdot \nabla \theta)(x, T) - (u \cdot \nabla \theta)(y, T)] + [(-\Delta)^{1/2}(x, T) - (-\Delta)^{1/2}(y, T)]$$

all we need are good upper bounds of the convection term $-[(u \cdot \nabla \theta)(y, T) - (u \cdot \nabla \theta)(x, T)]$ and the dissipation term $-[(\Delta)^{1/2}(x, T) - (\Delta)^{1/2}(y, T)]$. We perform such estimates in the following two sections.

In the following analysis we will suppress the time dependence since all estimates are independent of time.

4. **Estimate of the convection term**

We estimate the convection term in the same way as [23]. For completeness we repeat what they did here. Denote $\xi = |x - y|$.

We have

$$\theta(x - hu(x)) - \theta(y - hu(y)) \leq \omega(|x - y| + h|u(x) - u(y)|) \leq \omega(\xi + h\Omega(\xi)).$$

Thus

$$-[(u \cdot \nabla \theta)(x) - (u \cdot \nabla \theta)(y)] = \frac{d}{dh} \left[ \theta(x - hu(x)) - \theta(y - hu(y)) \right]_{h=0}
\leq \lim_{h \searrow 0} \frac{1}{h} \left[ \theta(x - hu(x)) - \theta(y - hu(y)) \right] - \left[ \theta(x) - \theta(y) \right]
\leq \lim_{h \searrow 0} \frac{1}{h} \left[ \omega(\xi + h\Omega(\xi)) - \omega(\xi) \right]
\leq \Omega(\xi) \omega'(\xi).$$

To summarize, we have the estimate

$$-[(u \cdot \nabla \theta)(x) - (u \cdot \nabla \theta)(y)] \leq \Omega(\xi) \omega'(\xi) \hspace{1cm} (7)$$

for the convection term at the two particular points $x$, $y$ chosen in Section 3.
5. Estimate of the dissipation term

Now we estimate the dissipation term. Without loss of generality let \( x = \left( \frac{\xi}{2}, 0 \right) \), \( y = \left( -\frac{\xi}{2}, 0 \right) \) as in [23]

\[
(P_{2,h} \ast \theta)(x) - (P_{2,h} \ast \theta)(y) = \int \int_{\mathbb{R}^2} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \theta(\eta, v) \, d\eta \, dv
\]

\[
= \int dv \int_{\mathbb{R}}^{\infty} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \theta(\eta, v) \, d\eta
\]

\[
+ \int dv \int_{-\infty}^{0} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \theta(\eta, v) \, d\eta
\]

\[
= \int dv \int_{\mathbb{R}}^{\infty} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \theta(\eta, v) \, d\eta
\]

\[
- \int dv \int_{\mathbb{R}}^{\infty} \left[ P_{2,h}\left( -\frac{\xi}{2} + \eta, -v \right) - P_{2,h}\left( \frac{\xi}{2} + \eta, -v \right) \right] \theta(-\eta, v) \, d\eta
\]

\[
= \int dv \int_{\mathbb{R}}^{\infty} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \theta(\eta, v) \, d\eta
\]

\[
\leq \int dv \int_{\mathbb{R}}^{\infty} \left[ P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) - P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right) \right] \omega(2\eta) \, d\eta
\]

\[
= \int_{0}^{\infty} \left[ P_{1,h}\left( \frac{\xi}{2} - \eta \right) - P_{1,h}\left( -\frac{\xi}{2} - \eta \right) \right] \omega(2\eta) \, d\eta
\]

where we have used the symmetry of the kernel \( P_{n,h} \) and the fact that

\[
P_{2,h}\left( \frac{\xi}{2} - \eta, -v \right) \geq P_{2,h}\left( -\frac{\xi}{2} - \eta, -v \right)
\]

because \(|\frac{\xi}{2} - \eta| \leq |\frac{-\xi}{2} - \eta|\) for \( \xi, \eta \geq 0 \). The last equality is because

\[
\int_{\mathbb{R}} P_{n,h}(x_1, \ldots, x_n) \, dx_n = P_{n-1,h}(x_1, \ldots, x_{n-1})
\]

which can be checked directly.

Following the same argument as in [23] we have

\[
[-(-\Delta)^{\nu} \theta](x) - [\frac{d}{dh} (P_{2,h} \ast \theta)(x) - (P_{2,h} \ast \theta)(y)]
\]
We have 

\[ \frac{1}{h} \lim_{h \to 0} \left\{ ((P_{2,h} \ast \theta)(x) - (P_{2,h} \ast \theta)(y)) - \theta(x) - \theta(y) \right\} \]

for some positive constant \( C \) depending on \( s \) only.

Now we simplify

\[ I = \int_0^\infty \left[ P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(-\frac{\xi}{2} - \eta\right) \right] \omega(2\eta) \, d\eta. \]

We have

\[
\begin{align*}
I &= \int_0^{\xi/2} \left[ P_{1,h}\left(\frac{\xi}{2} - \eta\right) - P_{1,h}\left(-\frac{\xi}{2} + \eta\right) \right] \omega(2\eta) \, d\eta \\
&\quad + \int_{\xi/2}^{\infty} \left[ P_{1,h}(\eta) \omega(2\eta) \, d\eta - \int_{\xi/2}^{\infty} P_{1,h}(\eta) \omega(2\eta - \xi) \, d\eta \right] \\
&= \int_0^{\xi/2} P_{1,h}(\eta) \omega(\xi - 2\eta) \, d\eta + \int_{\xi/2}^{\infty} P_{1,h}(\eta) \omega(\xi + 2\eta) \, d\eta - \int_{\xi/2}^{\infty} P_{1,h}(\eta) \omega(2\eta - \xi) \, d\eta \\
&= \int_0^{\xi/2} P_{1,h}(\eta) \left[ \omega(\xi - 2\eta) + \omega(\xi + 2\eta) \right] \, d\eta + \int_{\xi/2}^{\infty} P_{1,h}(\eta) \left[ \omega(2\eta + \xi) - \omega(2\eta - \xi) \right] \, d\eta.
\end{align*}
\]

On the other hand,

\[ \omega(\xi) = \int_0^{\infty} P_{1,h}(\eta) \left[ 2\omega(\xi) \right] \, d\eta = \int_0^{\xi/2} P_{1,h}(\eta) \left[ 2\omega(\xi) \right] \, d\eta + \int_{\xi/2}^{\infty} P_{1,h}(\eta) \left[ 2\omega(\xi) \right] \, d\eta \]

due to the fact that \( \int_0^{\infty} P_{1,h}(\eta) \, d\eta = \frac{1}{2} \int_{\mathbb{R}} P_{1,h}(\eta) \, d\eta = \frac{1}{2} \).

Combining the above, and recalling the explicit formula (5) of \( P_{n,h} \), we have

\[
\begin{align*}
\left[-(-\Delta)^s \theta\right](x) - \left[-(-\Delta)^s \theta\right](y) \leq \lim_{h \to 0} \frac{1}{h} \int_0^{\xi/2} P_{1,h}(\eta) \left[ \omega(\xi - 2\eta) + \omega(\xi + 2\eta) - 2\omega(\xi) \right] \, d\eta \\
&\quad + \lim_{h \to 0} \frac{1}{h} \int_{\xi/2}^{\infty} P_{1,h}(\eta) \left[ \omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi) \right] \, d\eta \\
&= C \int_0^{\xi/2} \frac{\omega(\xi - 2\eta) + \omega(\xi + 2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \\
&\quad + C \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta
\end{align*}
\]

for some positive constant \( C \) depending on \( s \) only.
Thus we obtain the following upper bound for the dissipation term:

\[
C\kappa \left[ \int_0^{\xi/2} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta + \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(\xi - \eta)}{\eta^{1+2s}} \, d\eta \right], \tag{8}
\]

where \( \kappa \) is the dissipation constant in (1). Note that since \( \omega \) is taken to be strictly concave, both terms are negative.

6. Construction of the modulus of continuity

The task now is to choose a special \( \omega(\xi) \) such that dissipation dominates, that is

\[
A \left[ \int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta \right] \omega'(\xi) + C\kappa \left[ \int_0^{\xi/2} \frac{\omega(\xi+2\eta) + \omega(\xi-2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta + \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(\xi - \eta)}{\eta^{1+2s}} \, d\eta \right] < 0
\]

for all \( \xi \geq 0 \).

We construct \( \omega \) in the following way

\[
\omega(\xi) = \xi - \xi^r \quad \text{when} \quad 0 \leq \xi \leq \delta, \quad \tag{9}
\]

and

\[
\omega'(\xi) = \frac{\gamma}{(\xi/\delta)^\alpha} \quad \text{when} \quad \xi > \delta, \quad \tag{10}
\]

where \( r \in (1, 1+2s) \) and \( \alpha \in (2s, 1) \) are arbitrary constants. The other two constants \( 0 < \delta \ll 1 \) and \( 0 < \gamma < 1 - r\delta^{-1} \) are taken to be small enough.

We first check that \( \omega \) satisfies the conditions in Definition 1. It is clear that \( \omega \) is continuous and increasing. It is also clear that \( \omega(0) = 0 \). Since \( \omega'(\xi) \) is decreasing in \([0, \delta]\) and \((\delta, \infty)\) respectively, \( \omega \) is concave as long as \( \omega'(\delta-) \geq \omega'(\delta+) \). We compute

\[
\omega'(\delta-) = 1 - r\delta^{-1}
\]

and

\[
\omega'(\delta+) = \gamma.
\]

Thus the concavity of \( \omega \) is guaranteed when \( \gamma < 1 - r\delta^{-1} \).

We notice that \( |\omega'(\xi)| \leq 1 \) for all \( \xi \), therefore \( \omega(\xi) \leq \xi \) for all \( \xi \). Also note that for the simplicity of formulas our \( \gamma \) corresponds to the quantity \( \gamma/\delta \) in [23]. We would also like to remark that our construction here does not extend to the critical case or the case \( s = 0 \).

We discuss the cases \( 0 \leq \xi \leq \delta \) and \( \xi > \delta \) separately.

6.1. The case \( 0 \leq \xi \leq \delta \)

- Convection term.

We have

\[
\int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta = \xi - \frac{1}{r} \xi^r \leq \xi,
\]

and
\[ \int_\xi^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta = \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta + \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta - \int_\delta^\infty \omega(\eta) \, d\left(\frac{1}{\eta}\right) \]

\[ \leq \log \frac{\delta}{\xi} + \frac{\omega(\delta)}{\delta} + \int_\delta^\infty \frac{\omega'(\eta)}{\eta} \, d\eta \leq \log \frac{\delta}{\xi} + 1 + \int_\delta^\infty \frac{\gamma \delta^u}{\eta^{1+u}} \, d\eta \]

\[ = \log \frac{\delta}{\xi} + 1 + \frac{\gamma}{\alpha} \leq \log \frac{\delta}{\xi} + 2 \]

if we take \( \gamma \leq \alpha \in (2s, 1) \).

Thus the convection term
\[ A\left( A\left( 3 + \log \frac{\delta}{\xi}\right) \right) \]

can be estimated from above by
\[ A\xi \left( 3 + \log \frac{\delta}{\xi}\right) \quad (11) \]

since \( 0 < \omega'(\xi) \leq 1 \).

- Dissipation term.

We have
\[ \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \leq C\xi^{2-2s} \omega''(\xi) = -C\xi^{r-(1+2s)} \]

for some constant \( C > 0 \) when \( \delta \) is small enough, where the first inequality comes from Taylor expansion and the fact that \( \omega'' \) is increasing for the \( \omega \) defined by (9)–(10).

Since the other term is always negative, the dissipation term
\[ C\kappa \left[ \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta + \int_{\xi/2}^\infty \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \right] \]

can be estimated from above by
\[ -\kappa \xi^{r-(1+2s)}. \quad (12) \]

Now combining the above, we see that the sum of the convection and the dissipation terms for \( \xi \leq \delta \) can be estimated from above by
\[ \xi \left[ A\left( 3 + \log \frac{\delta}{\xi}\right) - \kappa \xi^{r-(1+2s)} \right]. \]

This is negative when \( \delta \) and consequently \( \xi \) is small enough since \( r - (1 + 2s) < 0 \).

6.2. The case \( \xi > \delta \)

- Convection term.

We have
\[
\int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta = \int_0^\delta \frac{\omega(\eta)}{\eta} \, d\eta + \int_\delta^\xi \frac{\omega(\eta)}{\eta} \, d\eta \leq \delta + \frac{\xi}{\delta} \omega(\xi) \log \frac{\xi}{\delta} \leq \omega(\xi) \left(2 + \left(2 + \log \frac{\xi}{\delta}\right)\right)
\]

where in the last step we have used \(\omega(\xi) \geq \omega(\delta) = \delta - \delta' > \delta/2\) when \(\delta\) is small enough.

On the other hand,
\[
\int_\delta^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta = -\int_\xi^\infty \frac{\omega(\eta)}{\eta} \, d \left(\frac{1}{\eta}\right) = \frac{\omega(\xi)}{\xi} + \gamma \delta^\alpha \int_\delta^\infty \frac{1}{\eta^{1+\alpha}} \, d\eta = \frac{\omega(\xi)}{\xi} + \frac{\gamma}{\alpha} \delta^\alpha \xi^{1-\alpha}
\]

Note that for \(\xi > \delta\) we have
\[
\omega(\xi) = \delta - \delta' + \int_\delta^\xi \frac{\gamma \delta^\alpha}{\eta^\alpha} \, d\eta = \delta - \delta' + \frac{\gamma}{1-\alpha} \delta^\alpha \xi^{1-\alpha} - \frac{\gamma}{1-\alpha} \delta
\]

if we take \(\gamma\) to be less than \(\frac{1-\alpha}{2} > 0\) so that \(\frac{1}{\xi} - \frac{\gamma}{1-\alpha} > 0\).

Thus we have
\[
\delta^\alpha \xi^{1-\alpha} \leq \frac{1-\alpha}{\gamma} \omega(\xi).
\]

Therefore
\[
\int_\xi^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta = \frac{1}{\xi} \left[\omega(\xi) + \frac{\gamma}{\alpha} \delta^\alpha \xi^{1-\alpha}\right] \leq \frac{1}{\alpha} \omega(\xi).
\]

Now we can estimate the convection term
\[
A \left[\int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta + \frac{\xi}{\delta} \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta\right] \omega'(\xi)
\]

as follows:
\[
A \left[\int_0^\xi \frac{\omega(\eta)}{\eta} \, d\eta + \frac{\xi}{\delta} \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} \, d\eta\right] \omega'(\xi) \leq \omega(\xi) \left(C + \log \frac{\xi}{\delta}\right) \omega'(\xi) = \gamma \omega(\xi) \left(C + \log \frac{\xi}{\delta}\right) \left(\frac{\xi}{\delta}\right)^{-\alpha}.
\]

- The dissipation term.

To estimate the dissipation term we notice that
\[
\omega(2\eta + \xi) - \omega(2\eta - \xi) \leq \omega(2\xi) = \omega(\xi) + \int_\xi^{2\xi} \frac{\gamma \delta^\alpha}{\eta^\alpha} \, d\eta = \omega(\xi) + \frac{\gamma (2^{1-\alpha} - 1) \delta^\alpha \xi^{1-\alpha}}{1-\alpha}
\]

\[
\leq \omega(\xi) + (2^{1-\alpha} - 1) \omega(\xi) \leq 2^{1-\alpha} \omega(\xi),
\]

where we have used (13).

Notice that \(2^{1-\alpha} < 2\) since \(\alpha \in (2s, 1)\), there is \(C > 0\) such that \(\omega(2\xi) - 2\omega(\xi) \leq -C \omega(\xi)\) for all \(\xi > \delta\).
Thus the dissipation term
\[
C\kappa \left[ \int_0^{\xi/2} \frac{\omega(\xi + 2\eta) + \omega(\xi - 2\eta) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta + \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \right]
\]
is bounded from above by
\[
C\kappa \int_{\xi/2}^{\infty} \frac{\omega(2\eta + \xi) - \omega(2\eta - \xi) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \leq C\kappa \int_{\xi/2}^{\infty} \frac{\omega(2\xi) - 2\omega(\xi)}{\eta^{1+2s}} \, d\eta \\
\leq -C\kappa \alpha \left( \int_{\xi/2}^{\infty} \frac{d\eta}{\eta^{1+2s}} \right) \\
= -C\kappa \xi^{-2s} \omega(\xi).
\]
for some positive constant \( C \).

Combining the above, we see that the sum of the convection and the dissipation terms for \( \xi > \delta \)
\[
\omega(\xi) \left[ \gamma \left( C' + \log \left( \frac{\xi}{\delta} \right) \left( \frac{\xi}{\delta} \right)^{-\alpha} - \kappa C \xi^{-2s} \right) \right] = \omega(\xi) \left[ \gamma \left( C' + \log \left( \frac{\xi}{\delta} \right) \left( \frac{\xi}{\delta} \right)^{-\alpha} - \kappa C \xi^{-2s} \right) \right]
\]
is negative if \( \gamma \leq \kappa C \delta^{2s} \) and \( \delta \) is small enough since \( \alpha > 2s \). Here we use \( C' \) for the constant in the convection estimate. Note that since the constants are all independent of \( \gamma \), we are free to make \( \gamma \) small.

7. Global regularity

We have shown that the if \( \theta_0 \) has a modulus of continuity \( \omega(\xi) \) as defined by (9)–(10), then \( \omega(\xi) \) will remain to be a modulus of continuity for \( \theta \) for all \( t \). We now show that, due to the special scaling invariance \( \theta(x, t) \mapsto \mu^{2s-1} \theta(\mu x, \mu^{2s} t) \) of the super-critically dissipative QG equation, there is \( c_s > 0 \) such that whenever (2) is satisfied, i.e., \( \|\nabla \theta_0\|_{L^\infty} \|\theta_0\|_{L^\infty} < c_s \), we can find \( \mu \) such that a re-scaling of \( \theta_0 \) has modulus of continuity \( \omega(\xi) \), thus proving Theorem 1.

7.1. Scaling invariance of super-critical dissipative QG equation

We first recall that the super-critical dissipative QG equation has the following scaling invariance:
\[
\theta(x, t) \mapsto \mu^{2s-1} \theta(\mu x, \mu^{2s} t).
\]

More specifically, if \( \theta_0(x) = \mu^{2s-1} \tilde{\theta}_0(\mu x, \mu^{2s} t) \), and \( \tilde{\theta} \) solves
\[
\tilde{\theta}_t + \tilde{u} \cdot \nabla \tilde{\theta} = -(-\xi)^s \tilde{\theta}
\]
for some \( s \in (0, \frac{1}{2}) \) with initial data \( \tilde{\theta}_0 \), then
\[
\theta(x, t) = \mu^{2s-1} \tilde{\theta}(\mu x, \mu^{2s} t)
\]
solves the same equation with initial data \( \theta_0 \). Thus in particular, if \( \tilde{\theta} \) is globally regular, so is \( \theta \).

On the other hand, if \( \omega(\xi) \) is a modulus of continuity for \( \tilde{\theta}_0(x) \), then
\[
\omega(\xi) = \mu^{2s-1} \tilde{\omega}(\mu \xi)
\]
is a modulus of continuity for \( \tilde{\theta}_0(x) = \mu^{2s-1} \tilde{\theta}_0(\mu x, \mu^{2s} t) \).

Therefore, to establish global regularity for certain smooth and periodic initial data \( \theta_0 \), it suffices to find a scaling constant \( \mu \) such that
\[
\omega_\mu(\xi) = \mu^{2s-1} \omega(\mu \xi)
\]
is a modulus of continuity for \( \theta_0(x) \), where \( \omega(\xi) \) is the particular modulus of continuity defined by (9)–(10).
7.2. Rescaling $\omega(\xi)$

Recall that $\omega(\xi)$ is defined by

$$\omega(\xi) = \xi - \xi^r$$

when $0 \leq \xi \leq \delta$

and

$$\omega'(\xi) = \frac{\gamma \delta^\alpha}{\xi^\alpha}$$

when $\xi > \delta$.

We can easily compute

$$\omega_\mu(|x - y|) = \begin{cases} 
\mu^{2s} |x - y| - \mu^{2s-1+r} |x - y|^r & \text{when } \mu |x - y| \leq \delta, \\
\gamma \delta^\alpha \mu^{2s-\alpha} |x - y|^{1-\alpha} + \mu^{2s-1} [\delta - \frac{\gamma}{1-\alpha} \delta - \delta^r] & \text{when } \mu |x - y| > \delta.
\end{cases}$$

Now we take $\mu^{2s} = 2 \| \nabla \theta_0 \|_{L^\infty}$. Then for $x, y$ such that $|x - y| \leq \delta/\mu$, we have

$$|\theta_0(x) - \theta_0(y)| \leq \| \nabla \theta_0 \|_{L^\infty} |x - y| = 2 \| \nabla \theta_0 \|_{L^\infty} |x - y| \frac{1}{2} = \mu^{2s} |x - y| \frac{1}{2}$$

$$\leq \mu^{2s} |x - y|(1 - \delta^r - 1) \leq \mu^{2s} |x - y|(1 - \mu^{r-1} |x - y|^r - 1) = \omega_\mu(|x - y|)$$

according to our choice of $\delta$.

On the other hand, for any $x, y$ with $|x - y| > \delta/\mu$, we have

$$|\theta_0(x) - \theta_0(y)| \leq 2 \| \theta_0 \|_{L^\infty}.$$ 

So it is clear that $\omega_\mu$ is a modulus of continuity of $\theta_0$ as long as

$$2 \| \theta_0 \|_{L^\infty} \leq \omega_\mu \left( \frac{\delta}{\mu} \right) = \mu^{2s-1}(\delta - \delta^r) = 2 \| \nabla \theta_0 \|_{L^\infty}^\frac{2s-1}{2s} (\delta - \delta^r).$$

This can be simplified to

$$\| \nabla \theta_0 \|_{L^\infty}^{1-2s} \| \theta_0 \|_{L^\infty}^{2s} \leq \frac{1}{2} (\delta - \delta^r)^{2s}.$$ 

Now taking

$$c_s = \frac{1}{2} (\delta - \delta^r)^{2s}$$

we see that

$$\| \nabla \theta_0 \|_{L^\infty}^{1-2s} \| \theta_0 \|_{L^\infty}^{2s} \leq c_s$$

is sufficient for $\theta$ to stay smooth for all time.

8. Final remarks

Theorem 1 can be generalized to the case with non-periodic initial data. The smallness condition can also be improved slightly when the level sets are smooth around the “breakthrough” points.

**Remark 5.** In [14], the author proved that when the initial data $\theta_0 \in H^{2-2s}$, the solution exists at least for finite time $[0, T)$, and furthermore the following decay estimates hold:

$$\sup_{0 < t < T} t^{\beta/(2s)} \| \theta \|_{H^{2-2s+\beta}} < +\infty \quad \forall \beta \leq 0.$$ 

Using this and similar argument as in [15], one can prove that the breakthrough scenario as in Section 3 remains true for general non-periodic $\theta_0$. With the breakthrough scenario true, the conclusions of Theorem 1 can be established for general non-periodic initial data with the same argument as for the periodic case.
Remark 6. When the level sets around the two “breakthrough” points $x, y$, which satisfy (6), stay smooth, for example, when the unit tangent vector field around $x, y$ stays Lipschitz continuous, we can show that no finite singularity can occur when the Hölder semi-norm $|\theta_0|_{C^{1,\alpha}}$ is small, thus weakening the smallness condition (2). Very recently Ju [21] proved global regularity for the 2D QG equation with critical dissipation under similar Lipschitz assumption on the unit tangent vector field in regions with large $|\nabla \theta|$. Although the result there holds for all initial data including non-periodic ones, the method there cannot be applied to the super-critical case.

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