LOCALIZED NON-BLOWUP CONDITIONS FOR THE 3D INCOMPRESSIBLE EULER EQUATIONS

JIAN DENG, THOMAS Y. HOU, XINWEI YU

ABSTRACT. Whether the 3D incompressible Euler equations can develop a finite time singularity from smooth initial data is an outstanding open problem. Here we review some existing computational and theoretical work on the possible finite blow-up of the 3D Euler equations. Further, we show that there is a sharp relationship between the geometric properties of the vortex filament and the maximum vortex stretching. By exploring this geometric property of the vorticity field, we have obtained global existence of the 3D incompressible Euler equations under some mild localized regularity assumption on vortex filaments. Our assumption on the local geometric regularity of vortex filaments seems consistent with numerical computations.

1. INTRODUCTION

The well-posedness of the 3D Euler equation in the whole space $\mathbb{R}^3$

$$\begin{aligned} u_t + (u \cdot \nabla) u &= -\nabla p \\ \nabla \cdot u &= 0 \\ u \mid_{t=0} &= u_0 \end{aligned} \tag{1.1}$$

is one of the most outstanding open problems in applied mathematics. In particular, the answer to the following "Euler singularity problem" is still missing:

Euler singularity problem: Given a smooth enough initial value u_0 with finite energy, will there be a finite time T^* such that the solution u ceases to satisfy the Euler equation (1.1) in the classical sense at time T^*?

Besides being an mathematically intriguing open problem, what adds much to the importance of the above Euler singularity problem is its possible relation to the onset of turbulence. Due to its mathematical and physical importance, many interesting results have been obtained by various mathematicians for this problem. For example Beale-Kato-Majda [BKM84], Ebin-Fischer-Marsden [EFM70], Caflisch [Caf93], Constantin-Fefferman-Majda [CFM96], Tadmor [Tad01], and Babin-Mahalov-Nicolaenko [BMN01]. In particular, the so-called BKM criterion, proved in Beale-Kato-Majda [BKM84], states that the smooth solution $u(x,t)$ for the 3D Euler equation blows up at some finite time T^* if and only if $\int_0^{T^*} \|\omega(\cdot,t)\|_\infty \, dt = \infty$, where $\omega = \nabla \times u$ is the vorticity. In recent years, improvements to the BKM criterion have been obtained. For example Konzono-Taniuchi [KT00] and Chae [Cha02].

Date: 9th June 2005.

In light of the BKM criterion, it is important to study the evolution of the vorticity magnitude $|\omega|$. In 1994, Constantin derived its evolution equation ([Con94]):

$$D_t |\omega| \equiv |\omega|_t + u \cdot \nabla |\omega| = \alpha (x, t) |\omega| \tag{1.2}$$

where the stretching factor $\alpha (x,t) \equiv \xi \cdot (\nabla u) \cdot \xi$, with $\xi \equiv \frac{\omega}{|\omega|}$ being the unit vorticity vector.

By the Biot-Savart law, we have $\nabla u = \nabla (-\triangle)^{-1} \nabla \times \omega$. Thus we can write $\nabla u = R(\omega)$, with R being a Riesz operator of degree zero. As a result, we know that $\alpha (x, t)$ is formally of the same order as $|\omega|$. This formal argument implies that the vorticity evolution equation may be subject to quadratic nonlinear growth, which yields finite time singularities. On the other hand, it was observed by Constantin in [Con94] that depletion of nonlinearity may occur when the vorticity directions align with one another. Based on this observation, Constantin-Fefferman-Majda [CFM96] proved that, for an $O(1)$ region that is carried by the flow, there will be no blow-up as long as the following three conditions are satisfied:

1. The maximum velocity $\|u\|_\infty$ is uniformly bounded in time.
2. There is enough alignment of the vorticity directions such that $\int_0^{T^*} \|\nabla \xi\|_\infty^2 \, dt < \infty$, where $\xi \equiv \omega / |\omega|$.
3. The maximum vorticity in a larger neighborhood of the $O(1)$ region is always controlled by the maximum vorticity in a smaller neighborhood, namely
$$\sup_{B_{3r}(X(W_0,t))} |\omega| \leq m \sup_{B_r(X(W_0,t))} |\omega|$$
for some constants r and m.

It turned out that the above theorem has little overlap with the observations from recent numerical computations (Kerr [Ker93, Ker95, Ker96, Ker97, Ker98], Pelz [Pel01], Grauer-Marliani-Germanschewski [GMG98]). In these computations, a growth rate of $(T^* - t)^{-1}$ is observed for the maximum vorticity. These computations also reveal that large vorticity resides in small regions which always shrink to one point. Since alignment of vorticity vectors is observed only inside a region that shrinks rapidly to a single point, new theorems with localized non-blowup conditions are needed. In Deng-Hou-Yu [DHY05, DHY04a] we propose one way to obtain such localized non-blowup theorems.

The key to our approach is the following understanding. When putting the Euler equation in the Lagrangian form, the growth of $|\omega|$ is directly related to the stretching of small vortex line segments transported by the flow. Note that these vortex line segments are allowed to shrink to one point. Careful study of the interaction of the stretching of these vortex line segments and local properties of the velocity and vorticity vector fields reveals subtle cancellations that have not been observed before. It is these cancellations that would hinder the formation of finite time singularities.

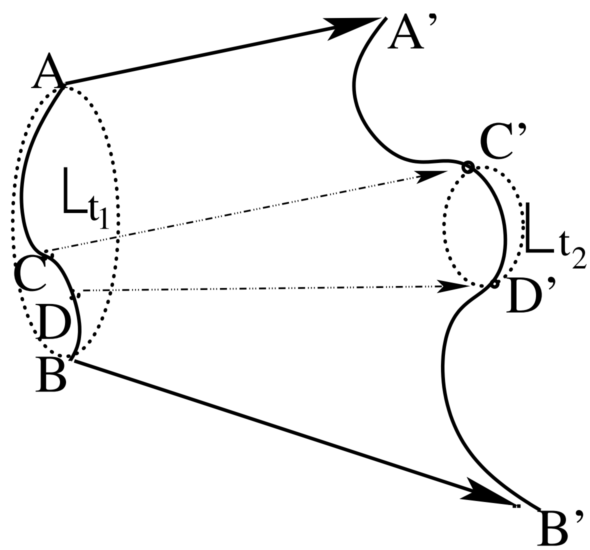
More specifically, we assume that at each time t there exists some vortex line segment L_t on which the maximum vorticity is comparable to the global maximum vorticity. A vortex line is defined as a 3D curve whose tangential vector is parallel to the vorticity direction. We denote by $L(t)$ the arc length of this vortex line segment. We prove non-blowup assuming the time integrability of the maximum normal and tangential velocity components, and the boundedness of both $\int_{L_t} \nabla \cdot \xi \, ds$ and $L(t) \max_{L_t} |\kappa|$ where κ denotes the curvature. Here the tangential and normal velocity components refer to the velocity components that are tangential or normal to the vorticity direction respectively. It is worth mentioning that the length of the local vortex line segment, $L(t)$, is allowed to shrink to zero. As we will see later, these localized non-blowup conditions are essentially consistent with the numerical observations.

2. Localized Non-blowup Conditions for the 3D Euler Equations

In this section we present our results on the non-blowup of the 3D Euler equations. We use $\Omega(t)$ to denote the global maximum vorticity, and consider, at time t, a single vortex line segment L_t along which the maximum vorticity is comparable to $\Omega(t)$. Denote by $L(t)$ the arc length of L_t, and ξ, n the tangential and normal unit vectors of L_t respectively. Note that by the definition of vortex lines, $\xi = \omega / |\omega|$. We further define $U_\xi(t) \equiv \max_{x,y \in L_t} |(u \cdot \xi)(x,t) - (u \cdot \xi)(y,t)|$, and $U_n(t) \equiv \max_{L_t} |u \cdot n|$.

We should point out that in general L_t is just a subset of the flow map image of $L_{t'}$ for $t' < t$. This can be illustrated by the following plot.

Figure 2.1. Illustration of L_t



As we can see from the above figure, we consider a family of shrinking regions in which maximum vorticity resides and certain alignment occurs. We illustrate the regions at times t_1 and t_2 by the dotted ellipsis. Then we consider L_t to be the vortex line segment that passes the maximum vorticity. Let $X(\alpha, t_0, t)$ be the Lagrangian flow map defined as follows:

$$\frac{\partial X}{\partial t} = u(X,t), \quad X(\alpha, t_0, t)|_{t=t_0} = \alpha.$$

Instead of taking L_{t_2} to be the flow image of L_{t_1} (that would be $A'B' = X(L_{t_1}, t_1, t_2)$), we take it to be the small section of L_{t_1}'s flow image that is contained in the shrinking region (that is we take $L_{t_2} = C'D'$ instead of $A'B'$).

With the above settings, we present our main theorem in Deng-Hou-Yu [DHY05].

Theorem 2.1. *Assume that there is a family of vortex line segments L_t and $T_0 \in [0, T^*)$, such that $X(L_{t_1}, t_1, t_2) \supseteq L_{t_2}$, for all $T_0 \le t_1 < t_2 < T^*$. Also assume that $\Omega(t)$ is monotonically increasing and $\Omega_L(t) \equiv \max_{L_t} |\omega| \ge c_0 \Omega(t)$ for some $c_0 > 0$ when $t \in [T_0, T^*)$. Furthermore, we assume that there are positive constants C_U, C_0, c_L such that*

1. $\int_{L_t} |\nabla \cdot \xi| \; ds, \; L(t) \max_{L_t} |\kappa| \le C_0$.
2. $U_\xi(t) + C_0 U_n(t) \le C_U (T^* - t)^{-A}$ *for some* $A > 0$,
3. $L(t) \ge c_L (T^* - t)^B$ *for some* $B > 0$.

Then, as long as $A + B < 1$, there will be no blowup in the 3D incompressible Euler flow up to time T^*.

Remark 2.2. In most numerical observations, $\Omega(t) \sim (T^* - t)^{-1}$, which bounds the maximum velocity by $(T^* - t)^{-3/5}$ according to Lemma 4 in Deng-Hou-Yu [DHY05]. Therefore in those cases, A would be no more than $3/5$. On the other hand, intuitively $L(t)$ should shrink at the same rate as the size of the region containing maximum vorticity.

Theorem 2.1 allows the length of the vortex line segment $L(t)$ to shrink and the maximum velocity to blow-up. However, in some numerical computations, the scaling $B = 1 - A = 1/2$ is observed, which is a critical case for Theorem 2.1 (e.g., Kerr [Ker93, Ker95, Ker96, Ker97, Ker98]). In Deng-Hou-Yu [DHY04a], we improved Theorem 2.1 to cover this critical case. We proved that no blow-up can occur when $A = B = 1/2$ as long as the scaling constants, C_0, C_U and c_L satisfy an algebraic inequality. More specifically, we have the following theorem.

Theorem 2.3. *Under the same assumptions as in Theorem 2.1, there will be no blow-up in the 3D incompressible Euler flow up to time T^* in the case $B = 1 - A = 1/2$, as long as*

$$C_U < f(C_0, c_0, c_L).$$

Here the function f is defined as

$$f(C_0, c_0, c_L) \equiv 2e^{-3C_0} c_L c_0^2 \cdot y_1 \left(2c_0^{1/2} e^{-C_0/2} / 3^{3/2} \right),$$

where $y_1(m)$ denotes the smallest positive y such that

$$\frac{y}{(y+1)^{3/2}} = m.$$

In the following we will discuss how we can apply Theorem 2.3 to Kerr's numerical computations. In a series of papers ([Ker93, Ker95, Ker96, Ker97, Ker98]), Kerr reported several observations for finite singularity formation of the Euler flow which can be summarized as follows.

1. Large vorticity ($\ge 0.6 \|\omega\|_\infty$) concentrates in small "active regions" in which vortex lines are "relatively straight".
2. "Active regions" look like two vortex sheets, each scaling like $(T^* - t)^{1/2} \times (T^* - t)^{1/2} \times (T^* - t)$, meeting at an angle. In particular, the scaling in the vorticity direction is $(T^* - t)^{1/2}$.
3. The maximum velocity $\|u\|_\infty$ also blows up, at the rate $(T^* - t)^{-1/2}$.

By (1), $\nabla\xi$, when rescaled to an $O(1)$ plot, should be bounded by some small constant C_0. Therefore we have $\nabla\cdot\xi, \kappa \le C_0/L(t)$ in the original frame. This implies $\int_{L_t} |\nabla\cdot\xi|\ ds \le C_0$ and $L(t)\max_{L_t}\kappa \le C_0$. (2) implies that the length of the vortex line segment that we can take should scale like $(T^*-t)^{1/2}$, which means $L(t) \ge c_L (T^*-t)^{1/2}$ for some constant c_L. Finally, since U_ξ and U_n are bounded by $\|u\|_\infty$ up to a constant factor, we immediately have $U_\xi + C_0 U_n \le C_U (T^*-t)^{-1/2}$ for some constant C_U. Thus we see that our major assumptions are essentially consistent with Kerr's numerical computations. If we consider the case L_t passing through the maximum vorticity, and take $C_0 = 0.1$ as a reasonable interpretation of vortex lines being "relative straight" inside the active regions, simple calculations based on our algebraic constraint on the scaling constants implies no blow-up up to time T^* if

$$\frac{C_U}{c_L} \le 0.4341.$$

In [DHY05], we argue that U_ξ is smaller than the maximum velocity field due to the local cancellation in the velocity kernel. Thus it is reasonable to expect that C_U is small. Currently there are no numerical measurements of C_U and c_L available. Whether the above condition is satisfied is still unknown. In a subsequent work, we plan to perform careful numerical studies to obtain accurate measurements for these scaling constants.

3. Proof of Main Theorems

In this section, we present the sketch of the proof of Theorem 2.1. The proof of Theorem 2.3 argues in a similar way, but is much more technical.

The key idea of our analysis is the representation of vorticity growth by the stretching of vortex line segments. A first step is the following lemma:

Lemma 3.1. *Let s and β be the arc length parameters of a vortex line at time t and t_0. Then for any point $\alpha\in\mathbb{R}^3$ at time $t_0 < t$ such that $\omega(\alpha,t_0)\ne 0$, we have*

$$\frac{\partial s}{\partial\beta}\left(X(\alpha,t_0,t),t\right) = \frac{|\omega(X(\alpha,t_0,t),t)|}{|\omega(\alpha,t_0)|}. \tag{3.1}$$

Proof. The key to this proof is the following well-known representation of the vorticity (Chorin-Marsden [CM93]):

$$\omega(X(\alpha,t_0,t),t) = \nabla_\alpha X(\alpha,t_0,t)\cdot\omega(\alpha,t_0).$$

Defining the short hand $\omega(t) = \omega(X(\alpha,t_0,t),t)$, we have

$$\begin{aligned}
|\omega(t)| &= \xi(t)\cdot\omega(t) = \xi(t)\cdot\nabla_\alpha X\cdot\omega(0)\\
&= \xi(t)\cdot\nabla_\alpha X\cdot\xi(0)\,|\omega(0)|\\
&= \xi(t)\cdot X_\beta\,|\omega(0)|\\
&= (\xi(t)\cdot X_s)\,|\omega(0)|\,s_\beta\\
&= (\xi(t)\cdot\xi(t))\,|\omega(0)|\,s_\beta\\
&= |\omega(0)|\,s_\beta.
\end{aligned}$$

Thus ends the proof. ∎

The next step is to develop the evolution equation of s_β. By (3.1) it is clear that $\frac{D_t s_\beta}{s_\beta} = \frac{D_t|\omega|}{|\omega|}$. Therefore we have

$$D_t s_\beta = \alpha s_\beta$$

where α is the same stretching factor defined in (1.2). Instead of treating it as a singular integral operator on ω, as the conventional approach does, we take a detour and give α an alternative formulation, namely

$$\begin{aligned} \alpha &= (\xi \cdot \nabla u) \cdot \xi \\ &= (\xi \cdot \nabla)(u \cdot \xi) - u \cdot (\xi \cdot \nabla) \xi \\ &= (u \cdot \xi)_s - \kappa (u \cdot n). \end{aligned}$$

Thus we obtain an alternative evolution equation for s_β:

$$D_t s_\beta = (u \cdot \xi)_\beta - (u \cdot n) \kappa s_\beta. \tag{3.2}$$

The next step is crucial in our analysis. We integrate (3.2), first along any vortex line segment l_τ for $\tau \in [t_0, t]$, and then from t_0 to t. Let $l(t)$ denote the arclength of the vortex line segment in between $X(\beta_1, t_0, t)$ and $X(\beta_2, t_0, t)$ with β_1 and β_2 being the end points of the vortex line segment l_0 at $t = t_0$. We obtain the following estimate for the stretching of l_t:

$$l(t) \le l(t_0) + \int_{t_0}^{t} \left[U_\xi(\tau) + U_n(\tau)\, l(\tau) \max_{l_t} |\kappa| \right] d\tau, \tag{3.3}$$

where $U_\xi(\tau) \equiv |(u \cdot \xi)(X(\beta_2, t_0, \tau)) - (u \cdot \xi)(X(\beta_1, t_0, \tau))|$ and $U_n(\tau) \equiv \max_{l_t} |u \cdot n|(\tau)$. Note that in (3.3), no term of order ∇u appears.

To take advantage of this property of (3.3), we need to estimate the growth of vorticity by $l(t)$. This is fulfilled by the following lemma.

Lemma 3.2. *Let l_t be a vortex line segment that is carried by the flow. Denote its length by $l(t)$. Also denote $\Omega_l(t) \equiv \max_{l_t} |\omega|$. Then we have*

$$e^{-\int_{l_t} |\nabla \cdot \xi|\, ds} \frac{\Omega_l(t)}{\Omega_l(t_0)} \le \frac{l(t)}{l(t_0)} \le e^{\int_{l_t} |\nabla \cdot \xi|\, ds} \frac{\Omega_l(t)}{\Omega_l(t_0)}, \tag{3.4}$$

where the integration is along l_t.

Proof. The proof relies on Lemma 3.1 as well as the following property of the vorticity field: For any two points $x, y \in l_t$ such that $\omega(x), \omega(y) \neq 0$,

$$e^{-\left|\int_x^y \nabla \cdot \xi\, ds\right|} \le \frac{|\omega(x)|}{|\omega(y)|} \le e^{\left|\int_x^y \nabla \cdot \xi\, ds\right|}. \tag{3.5}$$

We first prove (3.5). Since by definition ω is divergence free, we have

$$\begin{aligned} 0 &= \nabla \cdot \omega = \nabla \cdot (|\omega| \xi) \\ &= (\xi \cdot \nabla)|\omega| + (\nabla \cdot \xi)|\omega| \\ &= \frac{\partial |\omega|}{\partial s} + (\nabla \cdot \xi)|\omega|. \end{aligned}$$

Integrating along the vortex line segment between x and y gives (3.5).

Now we are ready to prove (3.4). We have

$$\begin{aligned} l(t) &= \int_{\beta_1}^{\beta_2} s_\beta \; d\beta \\ &= \int_{\beta_1}^{\beta_2} \frac{|\omega(X(\alpha,t_0,t),t)|}{|\omega(\alpha,t_0)|} \; d\beta \\ &\le \int_{\beta_1}^{\beta_2} \frac{\Omega_l(t)}{e^{-\int_{l_{t_0}} |\nabla\cdot\xi| \; ds}\Omega_l(t_0)} \\ &= e^{\int_{l_{t_0}} |\nabla\cdot\xi| \; ds} \frac{\Omega_l(t)}{\Omega_l(t_0)} l(t_0), \end{aligned}$$

where we have used Lemma 3.1 and (3.5). □

Now dividing both sides of (3.3) and applying Lemma 3.2, we obtain the following estimate for the growth of maximum vorticity along the vortex line segment l_t:

$$\Omega_l(t) \le e^{\int_{l_t} |\nabla\cdot\xi| d\beta} \left[1 + \frac{1}{l(t_0)} \int_{t_0}^{t} \left[U_\xi(\tau) + \max_{l_t} |\kappa| \cdot l(\tau) U_n(\tau)\right] \; d\tau\right] \Omega_l(t_0) \quad (3.6)$$

where $\Omega_l(t) \equiv \max_{l_t} |\omega|$. (3.6) will be the key to our proof of Theorems 2.1 and 2.3.

Now we are ready to sketch the proof of Theorem 2.1.

We prove by contradiction. Assume that the solution blows up at time T^*, then according to the BKM criterion, we must have

$$\int_{t_0}^{T^*} \Omega(t) = \infty, \quad (3.7)$$

for any $t_0 < T^*$. Therefore fixing t_0, we can divide $[t_0, T^*)$ into an infinite number of intervals $[t_k, t_{k+1})$ such that $\Omega(t_{k+1}) = r\Omega(t_k)$ for some constant r to be fixed later. It is easy to see that $\Omega(t_{k+1}) = r^{k+1}\Omega(t_0)$. Now by the monotonicity of $\Omega(t)$ we have

$$\int_{t_0}^{T^*} \Omega(t) \; dt \le \sum_{k=0}^{\infty} \Omega(t_{k+1})(t_{k+1} - t_k) \le r\Omega(t_0) \sum_{k=0}^{\infty} r^k (T^* - t_k).$$

We will show that there exists one particular r such that $r^k(T^* - t_k) < a^k$ for some constant $a < 1$. This would lead to a contradiction.

To this end, we fix k, and take $l_{t_{k+1}} = L_{t_{k+1}}$. For any $t \in [t_k, t_{k+1})$ we define l_t to be the image of $l_{t_{k+1}}$ under the inverse flow map, i.e., $l_{t_{k+1}} = X(l_t, t, t_{k+1})$. We can show that $l_t \subseteq L_t$ and its length $l(t)$ is bounded from below by $\tilde{c}l(t_{k+1}) = \tilde{c}L(t_{k+1})$ for some constant $\tilde{c}$.

Now the key estimate (3.6) with the assumption $\Omega_l(t_{k+1}) = \Omega_L(t_{k+1}) \ge c_0\Omega(t_{k+1})$ yields

$$\Omega(t_{k+1}) \le \frac{e^{C_0}}{c_0} \left[1 + C(T - t_{k+1})^{-B} \int_{t_k}^{t_{k+1}} (T-t)^{-A} \; dt\right] \Omega(t_k) \quad (3.8)$$

where we have used the assumptions of Theorem 2.1, i.e.,

$$
\begin{aligned}
&U_\xi(t) + C_0 U_n(t) \le C_U (T^* - t)^{-A}, \\
&\int_{L_t} |\nabla \cdot \xi|, \max_{L_t} |\kappa| \cdot L(t) \le C_0, \\
&L(t) \ge c_L (T^* - t)^B .
\end{aligned}
$$

Take $r = e^{C_0}/c_0 + 1$. Recall that $\Omega(t_{k+1}) = r\Omega(t_k)$. After some calculations we obtain the following estimate

$$
(T^* - t_{k+1}) \quad \le \quad C (T^* - t_k)^\delta (T^* - t_k)
$$

where $\delta = (1 - A)/B - 1 > 0$. This implies

$$
r^{k+1} (T^* - t_{k+1}) \le \left[Cr (T^* - t_k)^\delta \right] r^k (T^* - t_k) .
$$

Finally, since $t_k \nearrow T^*$ as $k \to \infty$, we are guaranteed to have $Cr (T^* - t_k)^\delta < a$ for some constant $a < 1$ when k is large. Therefore $r^k (T - t_k) < a^k$ for k large, and $\int_{t_0}^{T^*} \Omega(t)dt$ is bounded from above by a convergent geometric series. This contradicts with the assumption that $\int_{t_0}^{T^*} \Omega(t)dt = \infty$. This proves the theorem.

4. Localized Non-blowup Conditions for the 2D Quasigeostrophic Equation

In the above, we have shown that subtle cancellations in the 3D Euler flow can be revealed by taking a Lagrangian point of view and focusing on curves that are tangent to the vorticity vectors (i.e., vortex lines). It turns out that this new approach is effective for other incompressible fluid equations too. One example is the following 2D quasigeostrophic equation.

The 2D quasigeostrophic equation reads

$$
\begin{aligned}
D_t \theta &\equiv \theta_t + u \cdot \nabla \theta = 0 \\
u &= (-\partial_2, \partial_1) (-\triangle)^{-1/2} \theta
\end{aligned}
\tag{4.1}
$$

where θ is a scalar function defined on $\mathbb{R}^2$.

It has been shown in Constantin-Majda-Tabak [CMT94] that (4.1) is analogous to the 3D Euler equations. In particular, the possible blowup of (4.1) is controlled by

$$
\int_0^{T^*} \left\| \nabla^\perp \theta \right\|_\infty \, dt,
$$

similar to the BKM criterion for the 3D Euler equations, and furthermore the level sets of θ act similarly to the vortex lines in the 3D Euler dynamics in the sense that they are tangent to $\nabla^\perp \theta$ and are carried by the flow.

In Deng-Hou-Yu [DHY04b], it is proved that as long as $\int_{L_t} |\nabla \cdot \xi|$ and $L(t) \max_{L_t} |\kappa|$ are both bounded for an $O(1)$ segment of the level set passing through the maximum $\nabla^\perp \theta$, the growth of $\nabla^\perp \theta$ is bounded from above by double exponential and consequently no blowup can occur. This result is sharp in the sense that so far the most singular behavior observed in numerical computations of the 2D quasi-geostrophic equation exhibits just double exponential growth of $\nabla^\perp \theta$. Furthermore, the existence of such an $O(1)$ segment can be verified in this most singular situation under mild assumptions. For details, see Deng-Hou-Yu [DHY04b].

References

[BMN01] A. Babin, A. Mahalov, B. Nicolaenko. 3D Navier-Stokes and Euler equations with initial data charaterized by uniformly large vorticity. *Indiana University Mathematical Journal*, 50(1): 1–35, 2001.

[BKM84] J. T. Beale, T. Kato, A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Commun. Math. Phys.*, 94: 61–66, 1984.

[Caf93] R. E. Caflisch. Singularity formation for complex solutions of the 3D incompressible Euler equations. *Physica D*, 67: 1–18, 1993.

[Cha02] D. Chae. On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. *Comm. Pure Appl. Math.*, 40: 654–678, 2002.

[CM93] A. J. Chorin, J. E. Marsden. A Mathematical Introduction to Fluid Mechanics. 3rd ed., Springer-Verlag, New York, 1993.

[Con94] P. Constantin. Geometric statistics in turbulence. *SIAM Review*, 36(1): 73–98, 1994.

[CFM96] P. Constantin, C. Fefferman, A. Majda. Geometric constraints on potentially singular solutions for the 3-D Euler equations. *Commun. PDE*, 21: 559–571, 1996.

[CMT94] P. Constantin, A. Majda, E. G. Tabak. Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. *Nonlinearity*, 7: 1495–1533, 1994.

[DHY05] J. Deng, T. Y. Hou, X. Yu. Geometric properties and non-blowup of 3D incompressible Euler equations. *Commun. PDE*, 30: 225–243, 2005.

[DHY04a] J. Deng, T. Y. Hou, X. Yu. Improved geometric conditions for non-blowup of the 3D incompressible Euler equations. submitted to *Commun. PDE*, 2004.

[DHY04b] J. Deng, T. Y. Hou, x. Yu. Geometric properties and non-blowup of the 2D Quasi-Geostrophic equation. Preprint, 2004.

[EFM70] D. G. Ebin, A. E. Fischer, J. E. Marsden. Diffeomorphism groups, hydrodynamics and relativity. In *Proc. of the 13th Biennial Seminar of Canadian Mathematical Congress*, ed. J. Vanstone, Montreal, 1970.

[EM70] D. G. Ebin, J. E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math.*, 92: 102–163, 1970.

[GMG98] R. Grauer, C. Marliani, K. Germaschewski. Adaptive mesh refinement for singular solutions of the incompressible Euler equations. *Physical Review Letters*, 80(19), 1998.

[Ker93] R. M. Kerr. Evidence for a singularity of the three-dimensional incompressible Euler equations. *Phys. fluids A*, 5: 1725–1746, 1993.

[Ker95] R. M. Kerr. The role of singularities in Euler. In *Small-scale structure in hydro and magnetohydrodynamic turbulence*. Eds. A. Pouquet, P. L. Sulem, Lecture Notes, Springer-Verlag, 1995.

[Ker96] R. M. Kerr. Cover illustration: vortex structure of Euler collapse. *Nonlinearity*, 9: 271-272, 1996.

[Ker97] R. M. Kerr. Euler singularities and turbulence. In *19th ICTAM Kyoto' 96*, Eds. T. Tatsumi, E. Watanabe, T. Kambe, Elsevier Science, 1997.

[Ker98] R. M. Kerr. The outer regions in singular Euler. In *Fundamental Problematic Issues in Turbulence*, Eds. Tsinober and Gyr, Birkhäuser, 1998.

[KT00] H. Konzono, Y. Taniuchi. Limiting case of the Sobolev inequality in BMO, with application to the Euler equations. *Comm. Math. Phys.*, 214: 191-200, 2000.

[Pel01] R. B. Pelz. Symmetry and the hydrodynamic blow-up problem. *J. Fluid Mech.*, 444: 299-320, 2001.

[Tad01] E. Tadmor. On a new scale of regularity spaces with applications to Euler's equations. *Nonlinearity*, 14: 513-532, 2001.