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On global regularity of 2D generalized magnetohydrodynamic equations

Chuong V. Tran [a], Xinwei Yu [b,*], Zhichun Zhai [b]

[a] School of Mathematics and Statistics, University of St. Andrews, St. Andrews KY16 9SS, United Kingdom
[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada

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ABSTRACT

In this article we study the global regularity of 2D generalized magnetohydrodynamic equations (2D GMHD), in which the dissipation terms are $-\nu(-\triangle)^{\alpha}u$ and $-\kappa(-\triangle)^{\beta}b$. We show that smooth solutions are global in the following three cases: $\alpha \geqslant 1/2$, $\beta \geqslant 1$; $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$; $\alpha \geqslant 2$, $\beta = 0$. We also show that in the inviscid case $\nu = 0$, if $\beta > 1$, then smooth solutions are global as long as the direction of the magnetic field remains smooth enough.

1. Introduction

Recent mathematical studies of fluid mechanics have found it beneficial to replace the Laplace operator $\triangle$, representing molecular diffusion, by fractional powers of $-\triangle$. For the magnetohydrodynamic (MHD) equations, this practice results in the generalized MHD (GMHD) system

$$u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \nu \Lambda^{2\alpha} u, \tag{1}$$

$$b_t + u \cdot \nabla b = b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \tag{2}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \tag{3}$$

* Corresponding author.
E-mail addresses: chuong@mcs.st-and.ac.uk (C.V. Tran), xinweiyu@math.ualberta.ca (X. Yu), zhichun1@ualberta.ca (Z. Zhai).

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which is the subject of the present study. Here $\nu, \kappa, \alpha, \beta \geqslant 0$ and $\Lambda = (-\triangle)^{1/2}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi). \tag{4}$$

Eqs. (1)–(3) have been studied in some detail by Wu [28,29] and Cao and Wu [3], with an emphasis on the issue of solution regularity.

The generalization of diffusion in the above manner has been implemented to other fluid systems, including the Navier–Stokes, Boussinesq, and surface quasi-geostrophic equations (see e.g. [4,5,11,13–15,22]). Studying these generalized equations has enabled researchers to gain a deeper understanding of the strength and weaknesses of available mathematical methods and techniques, and, in some cases, motivated and inspired the invention of new methods. An illustrating example of the latter effect is the recent breakthroughs in the study of the surface quasi-geostrophic equations [1,7,17,18].

The problem of global well-posedness of the usual n-dimensional (nD) MHD (or GMHD with $\alpha, \beta \leqslant 2$) equations, where $n \geqslant 3$, is highly challenging for obvious reasons. One is that the MHD equations include the Navier–Stokes (or Euler when $\nu = 0$) system as a special case (obtained by setting the initial magnetic field to zero), for which the issue of regularity has not been resolved. Another is that the quadratic coupling between u and b can introduce additional technical difficulties, even though this coupling may actually have some regularizing effects (see below). For $n = 2$, this coupling invalidates the vorticity conservation, thereby becoming the main reason for the unavailability of a proof of global regularity for the ideal dynamics. Similar (but probably more manageable) situations arise when the 2D Euler equations are linearly coupled with the buoyancy equation in the Boussinesq system or have a linear forcing term [7].

So far the best result for the global regularity of the nD GMHD equations (1)–(3) has been derived in [30], where it has been proved that the system is globally regular as long as the following conditions

$$\alpha \geqslant \frac{1}{2} + \frac{n}{4}, \qquad \beta > 0, \qquad \alpha + \beta \geqslant 1 + \frac{n}{2}, \tag{5}$$

are satisfied. Note that for simplicity of presentation, the above conditions have been given in slightly stronger forms than the exact result in [30], where the dissipation terms are allowed to be logarithmically weaker than $-\Lambda^{2\alpha} u$ and $-\Lambda^{2\beta} b$. Note also that for the case $n = 3$, conditions similar to (5) have been obtained in [31], with $\beta > 0$ replaced by $\beta \geqslant 1$.

When $n \geqslant 3$, the result (5) is unlikely to be improved using current mathematical techniques. The reason is that the global regularity for the nD generalized Navier–Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p - \Lambda^{2\alpha} u, \qquad \nabla \cdot u = 0 \tag{6}$$

is still unavailable for $\alpha < \frac{1}{2} + \frac{n}{4}$ (see [25] for a proof of global regularity in the case of logarithmically weaker dissipation than $-\Lambda^{1+n/2} u$). On the other hand, when $n = 2$, the availability of global regularity for the generalized Navier–Stokes equations (6) for all $\alpha \geqslant 0$ suggests that the conditions in (5) could be excessive and may be weakened to some extent. In particular, it can be easily seen that the smoothness of either u or b guarantees that of the other and therefore of the system as a whole [26]. Hence, global regularity could intuitively be possible with either $\nu = 0$ or $\kappa = 0$ for suitable conditions on β or α.

In this article, we quantitatively confirm the above observations. More precisely, we show that when $n = 2$, the condition $\alpha \geqslant 1 = \frac{1}{2} + \frac{n}{4}$ is not needed for the global regularity of the system. In particular, we focus on the regime $\alpha < 1$ and show that the GMHD system is globally regular when $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$ or when $\alpha \geqslant 1/2$, $\beta \geqslant 1$. We also prove global regularity for the case $\alpha \geqslant 2$, $\kappa = 0$, thereby removing the technical condition $\beta > 0$. Furthermore, we study the inviscid case $\nu = 0$, $\kappa > 0$, and show that when $\beta > 1$, the GMHD system is globally regular as long as the magnetic lines are smooth enough. This result is consistent with numerical and experimental observations of the MHD dynamics, where the magnetic field appears to have the effect of "suppressing" the appearance of small scales in the fluid (see e.g. [20]), and as a consequence preventing the formation of singularities. Our finding is also consistent with a number of mathematical results exhibiting the regularizing effect on the streamlines and vortex lines in Navier–Stokes and Euler dynamics (see e.g. [6,9,10,27]).

The rest of this article is organized as follows. In Section 2 we summarize the main results and give a brief overview of the key ideas of their proofs. As these proofs use different methods for each case, we present them in separate sections. Section 3 features the proof for global regularity when $\alpha \geqslant 1/2$, $\beta \geqslant 1$. Sections 4 and 5 contain the proofs for the cases $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$ and $\alpha \geqslant 2$, $\beta = 0$, respectively. In Section 6 we prove global regularity under the assumption on the smoothness of magnetic lines.

Throughout this paper, we will set $\kappa = \nu = 1$ to simplify the presentation. It is a standard exercise to adjust various constants to accommodate other values of κ, ν, as long as both are positive. We also identify the cases $\alpha = 0$ and $\beta = 0$ with $\nu = 0$ and $\kappa = 0$, respectively.

2. Main results

Our first main result is the following global regularity theorem.

Theorem 1. *Consider the GMHD equations (1)–(3) in 2D. Assume $(u_0, b_0) \in H^k$ with $k > 2$. Then the system is globally regular for the following α, β:*

- $\alpha \geqslant 1/2$, $\beta \geqslant 1$;
- $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$;
- $\alpha \geqslant 2$, $\beta = 0$.

Remark 1. Combining the above theorem with the main result in [30], we see that the 2D GMHD system is globally regular for all $\alpha + \beta \geqslant 2$ except for $\alpha = 0$, $\beta = 2$. Thus we have removed almost all technical conditions on α and β.

The three cases will be proved using different methods, as different types of cancellation of the 2D GMHD system will be exploited. More specifically,

- for $\alpha \geqslant 1/2$, $\beta \geqslant 1$, we apply standard L^2-based energy method, taking advantage of the special cancellation that occurs for estimates in H^1;
- for $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$, we derive a new non-blow-up criterion in L^p norm of the vorticity $\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$ and then show that this criterion is indeed satisfied;
- for $\alpha \geqslant 2$, $\beta = 0$, we adapt the idea proposed in [21], carrying out a kind of "weakly nonlinear" energy estimate which takes advantage of the fact that in this case we have "almost" H^1 a priori bound.

Our second main result is the following theorem dealing with the case $\nu = 0$ (for our purpose this is the same as $\alpha = 0$ since we do not impose any restriction on the size of the initial data).

Theorem 2. *Consider the GMHD system (1)–(3) in 2D with $\alpha = 0$ and $\beta > 1$. Assume $(u_0, b_0) \in H^k$ with $k > 2$. Then the system is globally regular if $\hat{b} := \frac{b}{|b|} \in L^\infty(0, T; W^{2,\infty})$.*

Remark 2. The condition on $\hat{b}$ seems to be independent of the value of β, in the sense that there is no β_0 such that as soon as $\beta > \beta_0$, $\hat{b}$ automatically belongs to $L^\infty(0, T; W^{2,\infty})$.

Notation. In the following we will use the standard function spaces L^p, $W^{k,p}$, H^k whose norms are defined as

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^2} |f|^p \, dx\right)^{1/p}, \qquad \|f\|_{W^{k,p}} := \left(\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p}^p\right)^{1/p}, \qquad \|f\|_{H^k} := \|f\|_{W^{k,2}}$$

with standard modifications for the case $p = \infty$.

3. Proof of Theorem 1 Case I: $\alpha \geqslant 1/2$, $\beta \geqslant 1$

In this section we prove the first case of Theorem 1. We apply standard L^2-based energy estimates. The key idea here is to carry out the H^1, H^2, H^k estimates successively to explore possible cancellations at each stage. We would like to mention that the cancellation at the H^1 stage has been observed before by several authors in the case $\beta = 1$ [3,21]. The general case $\beta \geqslant 1$ is almost identical. However for completeness we still include detailed arguments.

3.1. H^1 estimates (L^2 estimates for ω, j)

Lemma 1 *(H^1 estimate). Consider the 2D GMHD equations (1)–(3), where $\alpha \geqslant 0$ and $\beta \geqslant 1$. Let $\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$ and $j = \nabla^\perp \cdot b$. Let $u_0, b_0 \in H^1$. For fixed $T > 0$ and $0 < t < T$, we have*

$$\|\omega\|_{L^2}^2(t) + \|j\|_{L^2}^2(t) + \int_0^t \left(\left\|\Lambda^\alpha \omega\right\|_{L^2}^2 + \left\|\Lambda^\beta j\right\|_{L^2}^2\right) d\tau \leqslant C(u_0, b_0, T). \tag{7}$$

Proof. We first apply $\nabla^\perp\cdot$ to the GMHD equations (1)–(3) to obtain the governing equations for the vorticity ω and the current j:

$$\omega_t + u \cdot \nabla \omega = b \cdot \nabla j - \Lambda^{2\alpha} \omega, \tag{8}$$

$$j_t + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b) - \Lambda^{2\beta} j. \tag{9}$$

Here

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1). \tag{10}$$

Note that T is bilinear in $\nabla u, \nabla b$ and therefore for any $k \geqslant 0$ we have

$$\left|\partial^k T(\nabla u, \nabla b)\right| \leqslant C \sum_{m=0}^k \left|\nabla^{m+1} u\right| \left|\nabla^{k-m+1} b\right| \tag{11}$$

for some constant C depending only on k.

Multiplying (8) and (9) by ω and j, respectively, integrating, and adding the resulting equations together we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \left(\omega^2 + j^2\right) dx = \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \, dx - \int_{\mathbb{R}^2} \left(\Lambda^\alpha \omega\right)^2 dx - \int_{\mathbb{R}^2} \left(\Lambda^\beta j\right)^2 dx, \tag{12}$$

where we have used the following consequences of $\nabla \cdot u = \nabla \cdot b = 0$:

$$\int_{\mathbb{R}^2} (u \cdot \nabla \omega)\omega \, \mathrm{d}x = 0; \tag{13}$$

$$\int_{\mathbb{R}^2} (u \cdot \nabla j) j \, \mathrm{d}x = 0; \tag{14}$$

$$\int_{\mathbb{R}^2} (b \cdot \nabla j)\omega \, \mathrm{d}x + \int_{\mathbb{R}^2} (b \cdot \nabla \omega) j \, \mathrm{d}x = 0. \tag{15}$$

Note that all the terms involving derivatives of ω and j – the "worst" terms from energy estimate point of view – disappear.

Now recall the standard energy conservation which can be obtained by multiplying (1) and (2) by u and b respectively, integrating, and applying the incompressibility condition (3):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \left(u^2 + b^2\right) \mathrm{d}x + \int_{\mathbb{R}^2} \left[\left(\Lambda^\alpha u\right)^2 + \left(\Lambda^\beta b\right)^2\right] \mathrm{d}x = 0. \tag{16}$$

This gives

$$u \in L^\infty\left(0, T; L^2\right) \cap L^2\left(0, T; H^\alpha\right), \qquad b \in L^\infty\left(0, T; L^2\right) \cap L^2\left(0, T; H^\beta\right). \tag{17}$$

As $\beta \geqslant 1$ by Sobolev embedding we easily get

$$b \in L^2\left(0, T; H^1\right) \quad \Rightarrow \quad j \in L^2\left(0, T; L^2\right). \tag{18}$$

On the other hand we have

$$\|\Lambda j\|_{L^2} \leqslant C\|b\|_{L^2}^a \left\|\Lambda^\beta j\right\|_{L^2}^{1-a} \tag{19}$$

for

$$a = \frac{\beta - 1}{\beta + 1}. \tag{20}$$

Using Young's inequality we obtain

$$\|\Lambda j\|_{L^2}^2 \leqslant a\|b\|_{L^2}^2 + (1-a)\left\|\Lambda^\beta j\right\|_{L^2}^2 \quad \Rightarrow \quad \left\|\Lambda^\beta j\right\|_{L^2}^2 \geqslant \frac{1}{1-a}\|\Lambda j\|_{L^2}^2 - \frac{a}{1-a}\|b\|_{L^2}. \tag{21}$$

It is worth emphasizing that the above calculation remains valid even when $a = 0$, that is $\beta = 1$.

This leads us to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) \leqslant{} & C\int_{\mathbb{R}^2} |\nabla u||\nabla b||j| \, \mathrm{d}x - \frac{1}{(1-a)}\|\Lambda j\|_{L^2} \\ & + \frac{a}{(1-a)}\|b\|_{L^2} - 2\left\|\Lambda^\alpha \omega\right\|_{L^2}^2 - \left\|\Lambda^\beta j\right\|_{L^2}^2. \end{aligned} \tag{22}$$

By Hölder's inequality, the trilinear term satisfies

$$\int_{\mathbb{R}^2} |\nabla u||\nabla b||j|\,\mathrm{d}x \leqslant \|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4}. \tag{23}$$

Owing to the relations

$$\nabla u = \nabla(-\triangle)^{-1}\nabla^{\perp}\omega \quad \text{and} \quad \nabla b = \nabla(-\triangle)^{-1}\nabla^{\perp} j \tag{24}$$

we have, following standard Fourier multiplier theory (see e.g. [24]),

$$\|\nabla u\|_{L^2} \leqslant C\|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b\|_{L^4} \leqslant C\|j\|_{L^4} \tag{25}$$

for some absolute constant C. It follows that

$$\int_{\mathbb{R}^2} |\nabla u||\nabla b||j|\,\mathrm{d}x \leqslant C\|\omega\|_{L^2}\|j\|_{L^4}^2. \tag{26}$$

Next, application of the Gagliardo–Nirenberg inequality

$$\|j\|_{L^4} \leqslant C\|j\|_{L^2}^{1/2}\|\Lambda j\|_{L^2}^{1/2} \tag{27}$$

yields

$$\int_{\mathbb{R}^2} |\nabla u||\nabla b||j|\,\mathrm{d}x \leqslant C\|\omega\|_{L^2}\|j\|_{L^2}\|\Lambda j\|_{L^2} \leqslant C(\varepsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^2 + \varepsilon\|\Lambda j\|_{L^2}^2, \tag{28}$$

where Young's inequality has been used. Here ε is a small positive number that will be chosen later.

Summarizing the above, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \left\|\Lambda^{\alpha}\omega\right\|_{L^2}^2 + \left\|\Lambda^{\beta} j\right\|_{L^2} \\ &\quad \leqslant C(\varepsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^2 + C\varepsilon\|\Lambda j\|_{L^2}^2 - \frac{1}{(1-a)}\|\Lambda j\|_{L^2}^2 + \frac{a}{(1-a)}\|b\|_{L^2}.\end{aligned} \tag{29}$$

Taking ε small enough so that $C\varepsilon < \frac{1}{1-a}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \left\|\Lambda^{\beta} j\right\|_{L^2}^2 + \left\|\Lambda^{\alpha}\omega\right\|_{L^2}^2 \leqslant C(\varepsilon)\|j\|_{L^2}^2\|\omega\|_{L^2}^2 + \frac{a}{1-a}\|b\|_{L^2}. \tag{30}$$

As $\|b\|_{L^2}$ is uniformly bounded in t, and $\|j\|_{L^2}^2 \in L^1(0,T)$ (17)–(18), the proof is completed. □

Remark 3. Note that the above proof can be shortened by skipping the steps

$$\|\Lambda j\|_{L^2} \leqslant \|b\|_{L^2}^{a}\left\|\Lambda^{\beta} j\right\|_{L^2}^{1-a} \tag{31}$$

and

$$\left\|\Lambda^\beta j\right\|_{L^2}^2 \geqslant \frac{1}{1-a}\|\Lambda j\|_{L^2}^2 - \frac{a}{1-a}\|b\|_{L^2} \tag{32}$$

and directly applying the Gagliardo–Nirenberg inequality

$$\|j\|_{L^4} \leqslant \|j\|_{L^2}^{a_1}\left\|\Lambda^\beta b\right\|_{L^2}^{a_2}\left\|\Lambda^\beta j\right\|_{L^2}^{a_3} \tag{33}$$

for appropriate a_1, a_2, a_3, and then use Young's inequality. However we choose to first reduce the general situation $\beta \geqslant 1$ to the particular one $\beta = 1$ to illustrate the following observation: For our problem, to prove regularity for $\alpha \geqslant \alpha_0$, $\beta \geqslant \beta_0$ using energy method, it suffices to do so for $\alpha = \alpha_0$, $\beta = \beta_0$. Such reduction significantly reduces the number of parameters in higher Sobolev norm estimates and makes the presentation much more transparent, as we will see in the following H^2 estimate.

3.2. H^2 estimates (H^1 estimates for ω, j)

With H^1 estimates at hand, we can move on to H^2 estimates. Differentiating (8)–(9) we reach

$$(\partial_i\omega)_t + u\cdot\nabla(\partial_i\omega) = -(\partial_i u)\cdot\nabla\omega + (\partial_i b)\cdot\nabla j + b\cdot\nabla(\partial_i j) - \Lambda^{2\alpha}(\partial_i\omega), \tag{34}$$

$$\begin{aligned}(\partial_i j)_t + u\cdot\nabla(\partial_i j) = {}& -(\partial_i u)\cdot\nabla j + (\partial_i b)\cdot\nabla\omega + b\cdot\nabla(\partial_i\omega)\\ &+ \partial_i\big(T(\nabla u, \nabla b)\big) - \Lambda^{2\beta}(\partial_i j).\end{aligned} \tag{35}$$

This gives the following integral relation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\frac{(\partial_i\omega)^2 + (\partial_i j)^2}{2}\,\mathrm{d}x = {}& -\int_{\mathbb{R}^2}\big[(\partial_i u)\cdot\nabla\omega\big](\partial_i\omega)\,\mathrm{d}x + \int_{\mathbb{R}^2}\big[(\partial_i b)\cdot\nabla j\big](\partial_i\omega)\,\mathrm{d}x\\ &- \int_{\mathbb{R}^2}\big[(\partial_i u)\cdot\nabla j\big](\partial_i j)\,\mathrm{d}x + \int_{\mathbb{R}^2}\big[(\partial_i b)\cdot\nabla\omega\big](\partial_i j)\,\mathrm{d}x\\ &+ \int_{\mathbb{R}^2}\big[\partial_i\big(T(\nabla u, \nabla b)\big)\big](\partial_i j)\,\mathrm{d}x\\ &- \int_{\mathbb{R}^2}\big(\Lambda^\alpha\partial_i\omega\big)^2\,\mathrm{d}x - \int_{\mathbb{R}^2}\big(\Lambda^\beta\partial_i j\big)^2\,\mathrm{d}x,\end{aligned} \tag{36}$$

after taking advantage of $\nabla\cdot u = \nabla\cdot b = 0$.

Summing up $i = 1, 2$, we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\big) \leqslant C(I_1 + I_2 + I_3 + I_4 + I_5) - 2\left\|\Lambda^\alpha\nabla\omega\right\|_{L^2}^2 - 2\left\|\Lambda^\beta\nabla j\right\|_{L^2}^2 \tag{37}$$

with C an absolute constant, and

$$I_1 = \int_{\mathbb{R}^2}|\nabla u||\nabla\omega|^2\,\mathrm{d}x; \tag{38}$$

$$I_2 = \int_{\mathbb{R}^2}|\nabla b||\nabla j||\nabla\omega|\,\mathrm{d}x; \tag{39}$$

$$I_3 = \int_{\mathbb{R}^2} |\nabla u||\nabla j|^2 \, \mathrm{d}x; \tag{40}$$

$$I_4 = \int_{\mathbb{R}^2} |\nabla b||\nabla \omega||\nabla j| \, \mathrm{d}x; \tag{41}$$

$$I_5 = \int_{\mathbb{R}^2} \big[\big|\nabla^2 u\big||\nabla b| + |\nabla u|\big|\nabla^2 b\big|\big]|\nabla j| \, \mathrm{d}x. \tag{42}$$

We estimate these quantities one by one. As discussed in Remark 3, we only need to carry out the estimates for the case $\alpha = 1/2$, $\beta = 1$.

There are four different cases (I_2 and I_4 are identical).

- Estimating $I_1 = \int_{\mathbb{R}^2} |\nabla u||\nabla \omega|^2 \, \mathrm{d}x$.

First, by Hölder's inequality we have

$$I_1 \leqslant \|\nabla u\|_{L^3} \|\nabla \omega\|_{L^3}^2 \leqslant C\|\omega\|_{L^3} \|\nabla \omega\|_{L^3}^2. \tag{43}$$

Consider the following Gagliardo–Nirenberg inequalities

$$\|\nabla \omega\|_{L^3} \leqslant C\big\|\Lambda^{1/2}\omega\big\|_{L^2}^{1/6}\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^{5/6}; \tag{44}$$

$$\|\nabla \omega\|_{L^3} \leqslant C\|\nabla \omega\|_{L^2}^{1/3}\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^{2/3}; \tag{45}$$

$$\|\omega\|_{L^3} \leqslant C\|\omega\|_{L^2}^{7/9}\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^{2/9}. \tag{46}$$

Eqs. (44) and (45) imply

$$\|\nabla \omega\|_{L^3} = \|\nabla \omega\|_{L^3}^{2/3}\|\nabla \omega\|_{L^3}^{1/3} \leqslant C\big\|\Lambda^{1/2}\omega\big\|_{L^2}^{1/9}\|\nabla \omega\|_{L^2}^{1/9}\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^{7/9}. \tag{47}$$

Now (46) and (47) give

$$I_1 \leqslant C\|\omega\|_{L^3}\|\nabla \omega\|_{L^3}^2 \leqslant C\|\omega\|_{L^2}^{7/9}\big\|\Lambda^{1/2}\omega\big\|_{L^2}^{2/9}\|\nabla \omega\|_{L^2}^{2/9}\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^{16/9}. \tag{48}$$

Applying Young's inequality we get

$$I_1 \leqslant C(\varepsilon)\|\omega\|_{L^2}^7\big\|\Lambda^{1/2}\omega\big\|_{L^2}^2\|\nabla \omega\|_{L^2}^2 + \varepsilon\big\|\Lambda^{1/2}\nabla \omega\big\|_{L^2}^2. \tag{49}$$

Here ε can be taken as small as we want and will be specified later.

- Estimating $I_2 = I_4 = \int_{\mathbb{R}^2} |\nabla b||\nabla j||\nabla \omega| \, \mathrm{d}x$.

Using Hölder's inequality we have

$$\int_{\mathbb{R}^2} |\nabla b||\nabla j||\nabla \omega| \, \mathrm{d}x \leqslant \|\nabla b\|_{L^4}\|\nabla j\|_{L^4}\|\nabla \omega\|_{L^2} \leqslant C\|j\|_{L^4}\|\nabla j\|_{L^4}\|\nabla \omega\|_{L^2}. \tag{50}$$

Applying the Gagliardo–Nirenberg inequalities

$$\|j\|_{L^4} \leqslant C\|j\|_{L^2}^{1/2}\|\nabla j\|_{L^2}^{1/2}; \qquad \|\nabla j\|_{L^4} \leqslant C\|\nabla j\|_{L^2}^{1/2}\|\Lambda\nabla j\|_{L^2}^{1/2} \tag{51}$$

yields

$$\int_{\mathbb{R}^2} |\nabla b||\nabla j||\nabla\omega|\,\mathrm{d}x \leqslant C\|j\|_{L^2}^{1/2}\|\nabla j\|_{L^2}\|\Lambda\nabla j\|_{L^2}^{1/2}\|\nabla\omega\|_{L^2}. \tag{52}$$

Applying Young's inequality further yields

$$\int_{\mathbb{R}^2} |\nabla b||\nabla j||\nabla\omega|\,\mathrm{d}x \leqslant C(\varepsilon)\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\|\nabla\omega\|_{L^2}^2 + \varepsilon\|\Lambda\nabla j\|_{L^2}^2. \tag{53}$$

- Estimating $I_3 = \int_{\mathbb{R}^2} |\nabla u||\nabla j|^2\,\mathrm{d}x$.

  Using Hölder's inequality we have

$$\int_{\mathbb{R}^2} |\nabla u||\nabla j|^2\,\mathrm{d}x \leqslant \|\nabla u\|_{L^2}\|\nabla j\|_{L^4}^2 \leqslant C\|\omega\|_{L^2}\|\nabla j\|_{L^4}^2. \tag{54}$$

Now using the second Gagliardo–Nirenberg inequality in (51) and Young's inequality we get

$$I_3 \leqslant C\|\omega\|_{L^2}\|\nabla j\|_{L^2}\|\Lambda\nabla j\|_{L^2} \leqslant C(\varepsilon)\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 + \varepsilon\|\Lambda\nabla j\|_{L^2}^2. \tag{55}$$

- Estimating $I_5 = \int_{\mathbb{R}^2} [|\nabla^2 u||\nabla b| + |\nabla u||\nabla^2 b|]|\nabla j|\,\mathrm{d}x$.

  We write

$$I_5 = I_{51} + I_{52} := \int_{\mathbb{R}^2} \left|\nabla^2 u\right||\nabla b||\nabla j|\,\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u|\left|\nabla^2 b\right||\nabla j|\,\mathrm{d}x. \tag{56}$$

It is clear that I_{51} can be estimated similar to I_2 while I_{52} can be estimated similar to I_3.

Remark 4. We would like to emphasize that the assumption $\alpha \geqslant 1/2$ is only needed for the estimation of I_1. The estimates I_2–I_5 only require $\alpha \geqslant 0$, $\beta \geqslant 1$.

Putting the above results together, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right) &\leqslant C(\varepsilon)\left[\|\omega\|_{L^2}^7\left\|\Lambda^{1/2}\omega\right\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + 1\right]\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\\ &\quad + C(\varepsilon)\|j\|_{L^2}^2 - 2\left\|\Lambda^{1/2}\nabla\omega\right\|_{L^2}^2 - 2\|\Lambda\nabla j\|_{L^2}^2\\ &\quad + C\varepsilon\left(\left\|\Lambda^{1/2}\nabla\omega\right\|_{L^2}^2 + \|\Lambda\nabla j\|_{L^2}^2\right).\end{aligned} \tag{57}$$

Taking ε small enough so that $C\varepsilon < 1$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right) &\leqslant C(\varepsilon)\left[\|\omega\|_{L^2}^7\left\|\Lambda^{1/2}\omega\right\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + 1\right]\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\\ &\quad + C(\varepsilon)\|j\|_{L^2}^2 - \left(\left\|\Lambda^{1/2}\nabla\omega\right\|_{L^2}^2 + \|\Lambda\nabla j\|_{L^2}^2\right).\end{aligned} \tag{58}$$

Recall that

$$\|\Lambda^{1/2}\omega\|_{L^2}, \|\nabla j\|_{L^2} \in L^2(0,T); \qquad \|\omega\|_{L^2}, \|j\|_{L^2} \in L^\infty(0,T) \tag{59}$$

thanks to the H^1 estimate. This, together with (58), implies

$$\nabla\omega, \nabla j \in L^\infty\big(0,T;L^2\big). \tag{60}$$

Combining with the H^1 estimate, we have the following H^2 estimate:

$$\|\omega\|_{H^1} + \|j\|_{H^1} \in L^\infty(0,T). \tag{61}$$

3.3. H^k estimates

An argument which by now is standard (see for example [21]) generalizes the classical BKM-type blow-up criterion [2] to

$$\text{The MHD system stays regular beyond } T \quad \text{if and only if} \quad \int_0^T \big(\|\omega\|_{\mathrm{BMO}} + \|j\|_{\mathrm{BMO}}\big)\,\mathrm{d}t < \infty. \tag{62}$$

Using the embedding

$$H^1 \hookrightarrow \mathrm{BMO} \tag{63}$$

in 2D, we see that

$$\|\omega\|_{H^1} + \|j\|_{H^1} \in L^\infty(0,T) \quad \Rightarrow \quad \|\omega\|_{\mathrm{BMO}} + \|j\|_{\mathrm{BMO}} \in L^\infty(0,T) \tag{64}$$

and consequently all H^k norms are bounded. This completes the proof of the first case.

4. Proof of Theorem 1 Case II: $0 \leqslant \alpha < 1/2$, $2\alpha + \beta > 2$

To prove global regularity in this case, we first derive a blow-up criterion in $\|\omega\|_{L^p}$ for appropriate p, then obtain a priori estimate for $\|\omega\|_{L^p}$. Note that in this case we have $\beta > 1$ and Lemma 1 together with the embedding

$$H^\beta \hookrightarrow L^\infty \tag{65}$$

in 2D already gives $j \in L^2(0,T;L^\infty) \hookrightarrow L^1(0,T;\mathrm{BMO})$.

Lemma 2. *Assume $0 < \alpha < 1/2$, $\beta > 1$. The GMHD system (1)–(3) is regular if $\omega \in L^p$ for any $p > \frac{1}{\alpha}$.*

Proof. As we have $\beta > 1$, we already have the following H^1 estimates thanks to Lemma 1:

$$\omega \in L^\infty\big(0,T;L^2\big) \cap L^2\big(0,T;H^\alpha\big); \qquad j \in L^\infty\big(0,T;L^2\big) \cap L^2\big(0,T;H^\beta\big). \tag{66}$$

Now arguing similarly as in Sections 3.2 and 3.3, we see that all we need to do is to bound I_1–I_5 as defined in (38)–(42). Furthermore, we note that the estimates for I_2–I_5 can be done similarly to that in Section 3.2, as explained in Remark 4. The only estimate that needs to be done differently is that of $I_1 = \int_{\mathbb{R}^2} |\omega||\nabla\omega|^2\,dx$.

For that purpose, we first apply Hölder's inequality to I_1 to obtain

$$\int_{\mathbb{R}^2} |\omega||\nabla\omega|^2\,dx \leqslant \|\omega\|_{L^{p_1}} \|\nabla\omega\|^2_{L^{2q_1}} \tag{67}$$

for p_1, q_1 satisfy

$$p_1 > \frac{1}{\alpha}, \qquad \frac{1}{p_1} + \frac{1}{q_1} = 1. \tag{68}$$

Next we use the following Gagliardo–Nirenberg type inequalities:

$$\|\nabla\omega\|_{L^{2q_1}} \leqslant C\|\Lambda^\alpha\omega\|^{\xi}_{L^2}\|\Lambda^\alpha\nabla\omega\|^{1-\xi}_{L^2} \quad \text{with } \xi = \alpha - \frac{1}{p_1} = \alpha\left(1 - \frac{1}{p_1\alpha}\right); \tag{69}$$

$$\|\nabla\omega\|_{L^{2q_1}} \leqslant C\|\nabla\omega\|^{\eta}_{L^2}\|\Lambda^\alpha\nabla\omega\|^{1-\eta}_{L^2} \quad \text{with } \eta = 1 - \frac{1}{p_1\alpha}. \tag{70}$$

Note that as long as $p_1 > \frac{1}{\alpha}$ both $\xi, \eta \in (0,1)$. Now setting

$$a = \frac{\alpha}{1+\alpha}\left(1 - \frac{1}{p_1\alpha}\right), \tag{71}$$

which satisfies $0 < a < 1/3$ owing to $0 < \alpha < 1/2$ and $p_1 > 1/\alpha$, we have

$$\|\nabla\omega\|_{L^{2q_1}} = \|\nabla\omega\|^{1/(1+\alpha)}_{L^{2q_1}}\|\nabla\omega\|^{\alpha/(1+\alpha)}_{L^{2q_1}} \leqslant C\|\Lambda^\alpha\omega\|^{a}_{L^2}\|\nabla\omega\|^{a}_{L^2}\|\Lambda^\alpha\nabla\omega\|^{1-2a}_{L^2}. \tag{72}$$

Next we apply the following Gagliardo–Nirenberg inequality

$$\|\omega\|_{L^{p_1}} \leqslant C\|\omega\|^{1-2a}_{L^p}\|\Lambda^\alpha\nabla\omega\|^{2a}_{L^2}, \tag{73}$$

where a is given by (71) and $p < p_1$. The exact value of p can be written down but what is important here is that $p > \frac{1}{\alpha}$, as can be seen from the following manipulation of the scaling relation:

$$-\frac{2}{p_1} = (1-2a)\left(-\frac{2}{p}\right) + 2a\alpha \quad \Rightarrow \quad -\frac{1}{p_1} = (1-2a)\left(-\frac{1}{p}\right) + a\alpha. \tag{74}$$

Writing (71) as $a = \frac{1}{1+\alpha}(\alpha - \frac{1}{p_1})$ and then adding α to both sides of (74), we reach

$$\alpha - \frac{1}{p} = \frac{1 - 3\alpha/(\alpha+1)}{1-2a}\left(\alpha - \frac{1}{p_1}\right). \tag{75}$$

Recalling $\alpha < 1/2$, we see that $\alpha - 1/p > 0$ if and only if $\alpha - 1/p_1 > 0$.

Combining the above, and applying Young's inequality, we see that I_1 can be bounded as

$$\begin{aligned} I_1 &\leqslant \|\omega\|_{L^{p_1}}\|\nabla\omega\|^2_{L^{2q_1}} \leqslant C\|\omega\|^{1-2a}_{L^p}\left(\|\Lambda^\alpha\omega\|^{a}_{L^2}\|\nabla\omega\|^{a}_{L^2}\|\Lambda^\alpha\nabla\omega\|^{1-a}_{L^2}\right)^2 \\ &\leqslant C(\varepsilon)\|\omega\|^{(1-2a)/a}_{L^p}\|\Lambda^\alpha\omega\|^2_{L^2}\|\nabla\omega\|^2_{L^2} + \varepsilon\|\Lambda^\alpha\nabla\omega\|^2_{L^2}. \end{aligned} \tag{76}$$

Now it is clear that once $\|\omega\|_{L^p} \in L^\infty(0,T)$, we can obtain H^2 estimate as in Section 3.2, and global regularity follows as in Section 3.3.

Finally, if $\|\omega\|_{L^q}$ is bounded for some $q > \frac{1}{\alpha} > 2$, then together with the H^1 estimate $\omega \in L^\infty(0,T;L^2)$ we see that

$$\|\omega\|_{L^r} \in L^\infty(0,T), \quad \forall r \in [2,q]. \tag{77}$$

Now we can simply take $p_1 = q$ in the above inequalities, then since $p < p_1$ we have the uniform boundedness of $\|\omega\|_{L^p}$ and global regularity follows. □

Remark 5. The case $\alpha = 0$ (which we identify with the case $\nu = 0$) is trivial. By our assumption $2\alpha + \beta > 2$ we have $\beta > 2$, which gives $\nabla j \in L^2(0,T;L^\infty)$. This result, together with the vorticity equation

$$\omega_t + u \cdot \nabla\omega = b \cdot \nabla j, \tag{78}$$

implies $\omega \in L^\infty(0,T;L^\infty)$. Global regularity then follows from the BKM type criterion in [2].

In light of Lemma 2, all we need to do is to show that when $2\alpha + \beta > 2$, there is indeed $p > \frac{1}{\alpha}$ such that $\|\omega\|_{L^p}$ remains uniformly bounded over $(0,T)$.

Recall the equation for ω:

$$\omega_t + u \cdot \nabla\omega = b \cdot \nabla j - \Lambda^\alpha \omega. \tag{79}$$

Multiply both sides by $p|\omega|^{p-2}\omega$ and integrate we reach

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |\omega|^p \,\mathrm{d}x \leqslant p\int_{\mathbb{R}^2} |b||\nabla j||\omega|^{p-1}\,\mathrm{d}x - p\int_{\mathbb{R}^2} \big(\Lambda^\alpha\omega\big)|\omega|^{p-2}\omega\,\mathrm{d}x, \tag{80}$$

after taking advantage of $\nabla \cdot u = 0$.

For the dissipation term, it is well-known that

$$\int_{\mathbb{R}^2} \big(\Lambda^\alpha\omega\big)|\omega|^{p-2}\omega\,\mathrm{d}x \geqslant 0. \tag{81}$$

This is originally proved in [23], and has later been refined in [8,16].

Taking into account the above "positivity" property and using Hölder's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p} \leqslant \|b\cdot\nabla j\|_{L^p} \leqslant \|b\|_{L^\infty}\|\nabla j\|_{L^p}. \tag{82}$$

Now as $\beta > 1$, we have H^1 estimate as in Section 3.1. In particular we have

$$j \in L^2\big(0,T;H^\beta\big). \tag{83}$$

Sobolev embedding then gives

$$j \in L^2\big(0,T;H^\beta\big) \quad\Rightarrow\quad \nabla j \in L^2\big(0,T;L^p\big) \quad \text{and} \quad b \in L^2\big(0,T;L^\infty\big), \tag{84}$$

with $p > \frac{1}{\alpha}$ satisfying

$$p \leqslant \frac{2}{2-\beta} \quad \text{when } \beta < 2, \quad \text{and} \quad p < \infty \quad \text{when } \beta \geqslant 2. \tag{85}$$

As $\alpha + \beta > 2$, such p exists. Now we have

$$\begin{aligned} \|\omega\|_{L^p} &\leqslant \|\omega_0\|_{L^p} + \int_0^t \|b\|_{L^\infty} \|\nabla j\|_{L^p} \, d\tau \leqslant \|\omega_0\|_{L^p} + \|b\|_{L^2(0,T;L^\infty)} \|\nabla j\|_{L^2(0,T;L^p)} \\ &\leqslant C(\omega_0, T). \end{aligned} \tag{86}$$

Therefore $\|\omega\|_{L^p} \in L^\infty(0,T)$ and global regularity follows from Lemma 2.

5. Proof of Theorem 1 Case III: $\alpha \geqslant 2$, $\beta = 0$

In this section we prove global regularity in the case $\alpha \geqslant 2$, $\beta = 0$. As we identify $\beta = 0$ with $\kappa = 0$, the GMHD equations now read

$$u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \Lambda^{2\alpha} u, \tag{87}$$

$$b_t + u \cdot \nabla b = b \cdot \nabla u, \tag{88}$$

$$\nabla \cdot u = \nabla \cdot b = 0. \tag{89}$$

In what follows we will only present the proof for the case $\alpha = 2$, $\beta = 0$. The case $\alpha > 2$ can be dealt with using the idea in Remark 3. In fact it can also be proved following standard energy estimates similar to that in Section 3, as when $\alpha > 2$ we immediately have $\omega \in L^2(0,T;L^\infty)$. This leads to a priori H^1 bounds which are sufficient to prove a priori H^2 bounds.

We will show that when $\alpha \geqslant 2$, the H^2 norms of ω and j must stay finite for any $T > 0$. Once this is proved, Sobolev embedding immediately gives the finiteness of $\|\omega\|_{L^\infty}$ and $\|j\|_{L^\infty}$ and regularity follows. The H^2 bound is proved by contradiction: Assume $\limsup_{t \nearrow T} \|\omega\|_{H^2} + \|j\|_{H^2} = \infty$ for some finite time $T > 0$. The idea is to start from a time T_0 close enough to T and show that under such assumption $\|\omega\|_{H^2} + \|j\|_{H^2}$ remains uniformly bounded for $T_0 < t < T$, thus reaching a contradiction.

First observe that in this case, energy conservation gives

$$\begin{aligned} &u, b \in L^\infty\big(0,T;L^2\big), \qquad \triangle u \in L^2\big(0,T;L^2\big) \\ &\quad \Rightarrow \quad \nabla u, \omega \in L^2(0,T;\mathrm{BMO}) \hookrightarrow L^1(0,T;\mathrm{BMO}). \end{aligned} \tag{90}$$

5.1. H^1 estimates

Similar to Section 3.1, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\big) + \|\triangle \omega\|_{L^2}^2 \leqslant \left| \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \, dx \right|. \tag{91}$$

Recalling (10)

$$T(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1) \tag{92}$$

and using

$$\|\nabla b\|_{L^2} \leqslant C\|j\|_{L^2}, \tag{93}$$

we have

$$\left|\int_{\mathbb{R}^2} jT(\nabla u, \nabla b)\,\mathrm{d}x\right| \leqslant C\|\nabla u\|_{L^\infty}\|j\|_{L^2}^2. \tag{94}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\big) + 2\|\triangle\omega\|_{L^2}^2 \leqslant C\|\nabla u\|_{L^\infty}\big(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\big). \tag{95}$$

Here we make use of the following Gronwall-type inequality, which is a variant of the standard Gronwall's inequality as presented in [12, Appendix B.j].

Lemma 3. *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the inequality*

$$\eta'(t) + \psi(t) \leqslant \phi(t)\eta(t), \tag{96}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) + \int_0^t \psi(\tau)\,\mathrm{d}\tau \leqslant \eta(0)\exp\left[\int_0^t \phi(\tau)\,\mathrm{d}\tau\right]. \tag{97}$$

Proof. The proof follows the same idea as that presented in [12] and is omitted. □

Taking $\eta := \|\omega\|_{L^2}^2 + \|j\|_{L^2}^2$ and $\psi := 2\|\triangle\omega\|_{L^2}^2$ in Lemma 3, then integrating from T_0 to t, we obtain

$$\int_{T_0}^t \|\triangle\omega\|_{L^2}^2\,\mathrm{d}\tau \leqslant \|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \int_{T_0}^t \|\triangle\omega\|_{L^2}^2\,\mathrm{d}\tau$$
$$\leqslant \big(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2\big)\exp\left[C\int_{T_0}^t \|\nabla u\|_{L^\infty}(\tau)\,\mathrm{d}\tau\right]. \tag{98}$$

Here $T_0 \in (0,T)$ will be fixed later and we denote $\omega_0 := \omega(\cdot, T_0)$, $j_0 := j(\cdot, T_0)$.

Now applying the logarithmic inequality (see e.g. [19])

$$\|\nabla u\|_{L^\infty} \leqslant C\big(1 + \|u\|_{L^2} + \|\omega\|_{\mathrm{BMO}}\big(1 + \log\big(1 + \|\omega\|_{H^2}^2 + \|j\|_{H^2}^2\big)\big)\big) \tag{99}$$

and setting

$$M(t) := \max_{\tau\in(T_0,t)}\big(\|\omega\|_{H^2}^2 + \|j\|_{H^2}^2\big)(\tau) \tag{100}$$

we reach

$$\int_{T_0}^{t} \|\triangle\omega\|_{L^2}^2 \, d\tau \leqslant \left(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2\right) \exp\left[C\left(1 + \|u\|_{L^2}\right)\right]$$
$$\times \exp\left[C\left(\int_{T_0}^{t} \|\omega\|_{\mathrm{BMO}} \, d\tau\right)\left(1 + \log\left(1 + M(t)\right)\right)\right]. \tag{101}$$

Note that thanks to the energy estimate $\|u\|_{L^2} \leqslant \|u(0)\|_{L^2}$ so $\exp(C(1 + \|u\|_{L^2}))$ is bounded by a constant independent of T_0.

As $\|\omega\|_{\mathrm{BMO}} \in L^1(T_0, T)$, we can take T_0 close enough to T so that

$$C\int_{T_0}^{t} \|\omega\|_{\mathrm{BMO}} \, d\tau \leqslant 2\delta \tag{102}$$

for some small positive number δ to be fixed later. With such choice of T_0 we have

$$\int_{T_0}^{t} \|\triangle\omega\|_{L^2}^2 \, d\tau \leqslant C(T_0)\left(1 + M(t)\right)^{2\delta}. \tag{103}$$

Now Hölder's inequality gives

$$\int_{T_0}^{t} \|\triangle\omega\|_{L^2} \, d\tau \leqslant C(T_0)\left(1 + M(t)\right)^{\delta}. \tag{104}$$

Before proceeding, we fix T_0 by the following requirements:

$$C\int_{T_0}^{t} \|\omega\|_{\mathrm{BMO}} \, d\tau \leqslant 2\delta, \qquad \log\left(1 + M(T_0)\right) > 1. \tag{105}$$

At the end of Section 5.2 we will show that δ can be taken as $1/24$.

5.2. H^3 estimate (H^2 estimate for ω, j)

In this subsection we prove the uniform boundedness of $M(t)$ for all $T_0 < t < T$, thus reaching contradiction.

Let ∂^2 denote any double partial derivative (such as ∂_{12}, ∂_{11} etc.). Taking ∂^2 of (8) and (9) and multiplying the resulting equations by $\partial^2\omega$ and $\partial^2 j$ respectively, we reach, after using $\nabla \cdot u = \nabla \cdot b = 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \left[\left(\partial^2\omega\right)^2 + \left(\partial^2 j\right)^2\right] dx \leqslant A + B + C + D + E - \int_{\mathbb{R}^2} \left(\triangle\partial^2\omega\right)^2 dx, \tag{106}$$

with

$$A = \left| \int_{\mathbb{R}^2} \left[\partial^2(u \cdot \nabla\omega) - u \cdot \nabla\partial^2\omega\right]\left(\partial^2\omega\right) \mathrm{d}x \right|$$

$$\leqslant \int_{\mathbb{R}^2} \left|\nabla^2 u\right| |\nabla\omega| \left|\nabla^2\omega\right| \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^2\omega\right|^2 \mathrm{d}x; \tag{107}$$

$$B = \left| \int_{\mathbb{R}^2} \left[\partial^2(b \cdot \nabla j) - b \cdot \nabla\partial^2 j\right]\left(\partial^2\omega\right) \mathrm{d}x \right|$$

$$\leqslant \int_{\mathbb{R}^2} \left|\nabla^2 b\right| |\nabla j| \left|\nabla^2\omega\right| \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla b| \left|\nabla^2 j\right| \left|\nabla^2\omega\right| \mathrm{d}x; \tag{108}$$

$$C = \left| \int_{\mathbb{R}^2} \left[\partial^2(u \cdot \nabla j) - u \cdot \nabla\left(\partial^2 j\right)\right]\left(\partial^2 j\right) \mathrm{d}x \right|$$

$$\leqslant \int_{\mathbb{R}^2} \left|\nabla^2 u\right| |\nabla j| \left|\nabla^2 j\right| \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^2 j\right|^2 \mathrm{d}x; \tag{109}$$

$$D = \left| \int_{\mathbb{R}^2} \left[\partial^2(b \cdot \nabla\omega) - b \cdot \nabla\left(\partial^2\omega\right)\right]\left(\partial^2 j\right) \mathrm{d}x \right|$$

$$\leqslant \int_{\mathbb{R}^2} \left|\nabla^2 b\right| |\nabla\omega| \left|\nabla^2 j\right| \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla b| \left|\nabla^2\omega\right| \left|\nabla^2 j\right| \mathrm{d}x; \tag{110}$$

$$E = \left| \int_{\mathbb{R}^2} \partial^2 T(\nabla u, \nabla b)\left(\partial^2 j\right) \mathrm{d}x \right|$$

$$\leqslant \int_{\mathbb{R}^2} \left|\nabla^3 u\right| |\nabla b| \left|\nabla^2 j\right| \mathrm{d}x + \int_{\mathbb{R}^2} \left|\nabla^2 u\right| \left|\nabla^2 b\right| \left|\nabla^2 j\right| \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^3 b\right| \left|\nabla^2 j\right| \mathrm{d}x. \tag{111}$$

Adding up all such partial derivatives, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^2\omega\right\|_{L^2}^2 + \left\|\nabla^2 j\right\|_{L^2}^2\right) \leqslant C(I_1 + I_2 + I_3 + I_4 + I_5 + I_6) - 2\left\|\nabla^4\omega\right\|_{L^2}^2, \tag{112}$$

with

$$I_1 = \int_{\mathbb{R}^2} \left|\nabla^2 u\right| |\nabla\omega| \left|\nabla^2\omega\right| \mathrm{d}x; \tag{113}$$

$$I_2 = \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^2\omega\right|^2 \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^2 j\right|^2 \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla u| \left|\nabla^3 b\right| \left|\nabla^2 j\right| \mathrm{d}x; \tag{114}$$

$$I_3 = \int_{\mathbb{R}^2} \left|\nabla^2 b\right| |\nabla j| \left|\nabla^2\omega\right| \mathrm{d}x; \tag{115}$$

$$I_4 = \int_{\mathbb{R}^2} |\nabla b|\left|\nabla^2 j\right|\left|\nabla^2 \omega\right| \mathrm{d}x + \int_{\mathbb{R}^2} \left|\nabla^3 u\right||\nabla b|\left|\nabla^2 j\right| \mathrm{d}x; \tag{116}$$

$$I_5 = \int_{\mathbb{R}^2} \left|\nabla^2 u\right||\nabla j|\left|\nabla^2 j\right| \mathrm{d}x; \tag{117}$$

$$I_6 = \int_{\mathbb{R}^2} \left|\nabla^2 b\right||\nabla \omega|\left|\nabla^2 j\right| \mathrm{d}x + \int_{\mathbb{R}^2} \left|\nabla^2 u\right|\left|\nabla^2 b\right|\left|\nabla^2 j\right| \mathrm{d}x. \tag{118}$$

We remark that the integrals in each I_k can be estimated similarly, therefore in the following we only show how to estimate the first integral in each I_k.

- I_1. For I_1 we write

$$\begin{aligned} I_1 &\leqslant \left\|\nabla^2 u\right\|_{L^4} \|\nabla \omega\|_{L^4} \left\|\nabla^2 \omega\right\|_{L^2} \\ &\leqslant C\|\nabla \omega\|_{L^4}^2 \left\|\nabla^2 \omega\right\|_{L^2} \\ &\leqslant C\|u\|_{L^2} \left\|\nabla^4 \omega\right\|_{L^2} \left\|\nabla^2 \omega\right\|_{L^2}, \end{aligned} \tag{119}$$

where we have used the following Gagliardo–Nirenberg inequality

$$\|\nabla \omega\|_{L^4} \leqslant C\|u\|_{L^2}^{1/2} \left\|\nabla^4 \omega\right\|_{L^2}^{1/2}. \tag{120}$$

Now by Young's inequality we have, after using $\|u\|_{L^2} \leqslant \|u_0\|_{L^2}$,

$$I_1 \leqslant C(\varepsilon)\|u\|_{L^2}^2 \left\|\nabla^2 \omega\right\|_{L^2}^2 + \varepsilon\left\|\nabla^4 \omega\right\|_{L^2}^2 \leqslant C(\varepsilon)\left\|\nabla^2 \omega\right\|_{L^2}^2 + \varepsilon\left\|\nabla^4 \omega\right\|_{L^2}, \tag{121}$$

with ε as small as necessary.

- I_2. We have

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u|\left|\nabla^2 \omega\right|^2 \mathrm{d}x &\leqslant \|\nabla u\|_{L^\infty} \left\|\nabla^2 \omega\right\|_{L^2}^2 \\ &\leqslant C\left(1 + \|u\|_{L^2} + \|\omega\|_{\mathrm{BMO}}\left(1 + \log\left(1 + \|\omega\|_{H^2}^2 + \|j\|_{H^2}^2\right)\right)\right)\left\|\nabla^2 \omega\right\|_{L^2}^2 \\ &\leqslant C\left(1 + \|\omega\|_{\mathrm{BMO}}\left(1 + \log\left(1 + \|\omega\|_{H^2}^2 + \|j\|_{H^2}^2\right)\right)\right)\left\|\nabla^2 \omega\right\|_{L^2}^2, \end{aligned} \tag{122}$$

where we have used the logarithmic inequality (99).

- I_3. We have

$$\begin{aligned} \int_{\mathbb{R}^2} \left|\nabla^2 b\right||\nabla j|\left|\nabla^2 \omega\right| \mathrm{d}x &\leqslant \left\|\nabla^2 b\right\|_{L^4} \|\nabla j\|_{L^4} \left\|\nabla^2 \omega\right\|_{L^2} \\ &\leqslant C\|\nabla j\|_{L^4}^2 \left\|\nabla^2 \omega\right\|_{L^2} \\ &\leqslant C\|b\|_{L^2}^{1/3} \left\|\nabla^2 j\right\|_{L^2}^{5/3} \left\|\nabla^2 \omega\right\|_{L^2}, \end{aligned} \tag{123}$$

where we have used the following Gagliardo–Nirenberg inequality

$$\|\nabla j\|_{L^4} \leqslant C\|b\|_{L^2}^{1/6}\|\nabla^2 j\|_{L^2}^{5/6}. \tag{124}$$

As a consequence (recall the definition of $M(t)$ in (100))

$$I_3 \leqslant C\|\nabla^2\omega\|_{L^2} M(t)^{5/6}. \tag{125}$$

Here we have used the energy conservation $\|b\|_{L^2} \leqslant \|b_0\|_{L^2} + \|u_0\|_{L^2}$.

- I_4. We have

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla b|\left|\nabla^2 j\right|\left|\nabla^2\omega\right| \mathrm{d}x &\leqslant \|\nabla b\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2} \\ &\leqslant C\|b\|_{L^2}^{1/3}\|\nabla^2 j\|_{L^2}^{5/3}\|\nabla^2\omega\|_{L^2},\end{aligned} \tag{126}$$

where we have used the following Gagliardo–Nirenberg inequality

$$\|\nabla b\|_{L^\infty} \leqslant C\|b\|_{L^2}^{1/3}\|\nabla^2 j\|_{L^2}^{2/3}. \tag{127}$$

Therefore

$$I_4 \leqslant C\|\nabla^2\omega\|_{L^2} M(t)^{5/6}. \tag{128}$$

- I_5. We have

$$\begin{aligned}I_5 &= \int_{\mathbb{R}^2} \left|\nabla^2 u\right||\nabla j|\left|\nabla^2 j\right| \mathrm{d}x \\ &\leqslant \|\nabla^2 u\|_{L^4}\|\nabla j\|_{L^4}\|\nabla^2 j\|_{L^2} \\ &\leqslant C\|u\|_{L^2}^{1/6}\|\nabla^2\omega\|_{L^2}^{5/6}\|b\|_{L^2}^{1/6}\|\nabla^2 j\|_{L^2}^{11/6},\end{aligned} \tag{129}$$

where we have used the following Gagliardo–Nirenberg inequalities

$$\|\nabla^2 u\|_{L^4} \leqslant C\|u\|_{L^2}^{1/6}\|\nabla^2\omega\|_{L^2}^{5/6}; \qquad \|\nabla j\|_{L^4} \leqslant C\|b\|_{L^2}^{1/6}\|\nabla^2 j\|_{L^2}^{5/6}. \tag{130}$$

Hence

$$I_5 \leqslant C\|\nabla^2\omega\|_{L^2}^{5/6} M(t)^{11/12} \leqslant C\left(1 + \|\nabla^2\omega\|_{L^2}\right)M(t)^{11/12}. \tag{131}$$

- I_6. We have

$$\begin{aligned}\int_{\mathbb{R}^2} \left|\nabla^2 b\right||\nabla\omega|\left|\nabla^2 j\right| \mathrm{d}x &\leqslant \|\nabla^2 b\|_{L^4}\|\nabla\omega\|_{L^4}\|\nabla^2 j\|_{L^2} \\ &\leqslant C\|\nabla j\|_{L^4}\|\nabla\omega\|_{L^4}\|\nabla^2 j\|_{L^2} \\ &\leqslant \|b\|_{L^2}^{1/6}\|u\|_{L^2}^{1/6}\|\nabla^2\omega\|_{L^2}^{5/6}\|\nabla^2 j\|_{L^2}^{11/6},\end{aligned} \tag{132}$$

where we have used the following Gagliardo–Nirenberg inequalities

$$\|\nabla\omega\|_{L^4} \leqslant C\|u\|_{L^2}^{1/6}\|\nabla^2\omega\|_{L^2}^{5/6}; \qquad \|\nabla j\|_{L^4} \leqslant C\|b\|_{L^2}^{1/6}\|\nabla^2 j\|_{L^2}^{5/6}. \tag{133}$$

Hence

$$I_6 \leqslant C\|\nabla^2\omega\|_{L^2}^{5/6} M(t)^{11/12} \leqslant C\big(1+\|\nabla^2\omega\|_{L^2}\big)M(t)^{11/12}. \tag{134}$$

Summarizing, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla^2\omega\|_{L^2}^2+\|\nabla^2 j\|_{L^2}^2\big) \leqslant C(T_0)\big[&M(t)+\big(1+\|\nabla^2\omega\|_{L^2}\big)M(t)^{11/12}\\ &+\|\omega\|_{\mathrm{BMO}}M(t)\log\big(1+M(t)\big)\big].\end{aligned} \tag{135}$$

Using our assumption on T_0 (105) and the monotonicity of $M(t)$, we have $\log(1+M(t))>1$ and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla^2\omega\|_{L^2}^2+\|\nabla^2 j\|_{L^2}^2\big) \leqslant C(T_0)\big[&\big(1+\|\nabla^2\omega\|_{L^2}\big)M(t)^{11/12}\\ &+\big(1+\|\omega\|_{\mathrm{BMO}}\big)M(t)\log\big(1+M(t)\big)\big].\end{aligned} \tag{136}$$

Integrating, we have

$$\begin{aligned}M(t) \leqslant C(T_0)\Bigg[&M_0+\bigg(\int_{T_0}^{t}\big(1+\|\nabla^2\omega\|_{L^2}\big)\,\mathrm{d}\tau\bigg)M(t)^{11/12}\\ &+\int_{T_0}^{t}\big[\big(1+\|\omega\|_{\mathrm{BMO}}\big)M(\tau)\log\big(1+M(\tau)\big)\big]\,\mathrm{d}\tau\Bigg],\end{aligned} \tag{137}$$

with $M_0 := \|\omega\|_{H^2}^2(T_0)+\|j\|_{H^2}^2(T_0)$.

Now taking $\delta = 1/24$, we have

$$\int_{T_0}^{t}\big(1+\|\nabla^2\omega\|_{L^2}\big)\,\mathrm{d}\tau \leqslant C(T_0)\big(1+M(t)\big)^{1/24}, \tag{138}$$

which leads to

$$\begin{aligned}M(t) \leqslant C(T_0)\Bigg[&M_0+M(t)^{11/12}\big(1+M(t)\big)^{1/24}\\ &+\int_{T_0}^{t}\big[\big(1+\|\omega\|_{\mathrm{BMO}}\big)M(\tau)\log\big(1+M(\tau)\big)\big]\,\mathrm{d}\tau\Bigg].\end{aligned} \tag{139}$$

This in turn gives

$$1 + M(t) \leqslant C(T_0)\Bigg[(1+M_0) + \big(1+M(t)\big)^{23/24} + \int_{T_0}^{t} \big[\big(1+\|\omega\|_{\mathrm{BMO}}\big)\big(1+M(\tau)\big)\log\big(1+M(\tau)\big)\big]\,\mathrm{d}\tau\Bigg]. \tag{140}$$

Now we set $N(t) := (1+M(t))^{1/24}$, $N_0 := (1+M_0)^{1/24}$ and divide both sides by $(1+M(t))^{23/24}$, using the monotonicity of $M(t)$ we reach

$$N(t) \leqslant C(T_0)\Bigg[(1+N_0) + \int_{T_0}^{t} \big(1+\|\omega\|_{\mathrm{BMO}}\big)N(\tau)\log\big(N(\tau)\big)\,\mathrm{d}\tau\Bigg]. \tag{141}$$

Application of the standard Gronwall's inequality now gives the following bound of N

$$N(t) \leqslant \big[C(T_0)(1+N_0)\big]^{\exp[C(T_0)\int_{T_0}^{t}(1+\|\omega\|_{\mathrm{BMO}})\,\mathrm{d}\tau]}, \tag{142}$$

which gives

$$M(t) \leqslant \big[C(T_0)(1+N_0)\big]^{24\exp[C(T_0)\int_{T_0}^{t}(1+\|\omega\|_{\mathrm{BMO}})\,\mathrm{d}\tau]}. \tag{143}$$

Since $\int_{T_0}^{t}\|\omega\|_{\mathrm{BMO}}(\tau)\,\mathrm{d}\tau$ remains bounded as $t \nearrow T$, (143) contradicts our assumption that $M(t) \nearrow \infty$ as $t \nearrow T$ and ends the proof.

5.3. H^k estimate

As we have already proved that the H^2 norms of ω and j have to remain bounded as $t \nearrow T$, thanks to the embedding $H^2 \hookrightarrow L^\infty$ in $\mathbb{R}^2$, we have

$$\omega, j \in L^\infty\big(0, T; L^\infty\big) \tag{144}$$

as a result of the argument in Sections 5.1 and 5.2. The H^k estimate and global regularity is now a simple consequence of the BKM-type criterion in [2].

6. Global regularity when the magnetic lines are smooth

This section proves Theorem 2, which states that the system

$$u_t + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \tag{145}$$

$$b_t + u\cdot\nabla b = b\cdot\nabla u - \Lambda^{2\beta} b, \tag{146}$$

$$\nabla\cdot u = \nabla\cdot b = 0, \tag{147}$$

with $\beta > 1$ and $(u_0, b_0) \in H^k$ for some $k > 2$, is globally regular if $\hat{b} := \frac{b}{|b|} \in L^\infty(0, T; W^{2,\infty})$.

Proof of Theorem 2. As $\beta > 1$, following Lemma 1 we already have H^1 estimate which in particular gives

$$j \in L^2\big(0,T;H^\beta\big) \hookrightarrow L^2\big(0,T;L^\infty\big) \tag{148}$$

since $H^\beta \hookrightarrow L^\infty$. Thanks to the BKM-type criteria in [2], all we need to prove is that $\omega \in L^1(0,T;L^\infty)$.

For a proof of $\omega \in L^1(0,T;L^\infty)$, let us examine the vorticity equation

$$\omega_t + u\cdot\nabla\omega = \nabla^\perp\cdot(b\cdot\nabla b), \tag{149}$$

where the "forcing" term has been given in its raw form instead of $b\cdot\nabla j$ for the very purpose of this proof. By writing

$$b = \hat{b}|b| \tag{150}$$

and using the divergence free condition $\nabla\cdot b = 0$, we have

$$\hat{b}\cdot\nabla|b| = -(\nabla\cdot\hat{b})|b|. \tag{151}$$

It follows that

$$b\cdot\nabla b = |b|\big[\hat{b}\cdot\nabla\big(\hat{b}|b|\big)\big] = \big[\hat{b}\cdot\nabla\hat{b} - (\nabla\cdot\hat{b})\hat{b}\big]|b|^2. \tag{152}$$

Therefore the vorticity equation can be written as

$$\omega_t + u\cdot\nabla\omega = \nabla^\perp\cdot\big\{\big[\hat{b}\cdot\nabla\hat{b} - (\nabla\cdot\hat{b})\hat{b}\big]|b|^2\big\} = A(x,t)|b|^2 + B(x,t)\cdot\big(b\cdot\nabla^\perp b\big), \tag{153}$$

where

$$A(x,t) = \nabla^\perp\cdot\big[\hat{b}\cdot\nabla\hat{b} - (\nabla\cdot\hat{b})\hat{b}\big], \qquad B(x,t) = \hat{b}\cdot\nabla\hat{b} - (\nabla\cdot\hat{b})\hat{b}. \tag{154}$$

As $\hat{b} \in W^{2,\infty}$ by our assumption, we readily deduce that

$$A(x,t), B(x,t) \in L^\infty\big(0,T;L^\infty\big). \tag{155}$$

Now since $\beta > 1$, the earlier estimates in H^1 mean

$$j \in L^2\big(0,T;H^\beta\big) \quad\Rightarrow\quad \nabla b \in L^2\big(0,T;H^\beta\big) \quad\Rightarrow\quad \nabla b \in L^2\big(0,T;L^\infty\big). \tag{156}$$

It follows that

$$|b|^2, b\cdot\nabla^\perp b \in L^1\big(0,T;L^\infty\big). \tag{157}$$

Putting things together, we see that

$$\omega_t + u\cdot\nabla\omega = F(x,t) := A(x,t)|b|^2 + B(x,t)\cdot\big(b\cdot\nabla^\perp b\big), \tag{158}$$

with $F(x,t) \in L^1(0,T;L^\infty)$. Since we are dealing with smooth solutions here, this immediately leads to

$$\omega \in L^\infty\big(0,T;L^\infty\big) \hookrightarrow L^1\big(0,T;L^\infty\big) \tag{159}$$

and the proof is completed. □

Remark 6. For solutions not smooth enough, we can argue as follows. First note that $j \in L^2(0,T;H^\beta)$ implies $\nabla b \in L^2(0,T;L^q)$ for any q, and furthermore $\|\nabla b\|_{L^2(0,T;L^q)}$ is uniformly bounded in q. Consequently $F \in L^1(0,T;L^q)$ for any q with uniformly bounded norms. Now we multiply the equation by $|\omega|^{p-2}\omega$ and integrate. After simplification we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p \leqslant p\left|\int_{\mathbb{R}^2} F(x,t)|\omega|^{p-2}\omega\,\mathrm{d}x\right| \leqslant p\|F\|_{L^p}\|\omega\|_{L^p}^{p-1}, \tag{160}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p} \leqslant \|F\|_{L^p}. \tag{161}$$

This gives a uniform bound on $\|\omega\|_{L^p}$ and consequently a bound on $\|\omega\|_{L^\infty}$.

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References

[1] Luis A. Caffarelli, Alexis Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. (2) 171 (3) (2010) 1903–1930.
[2] R.E. Caflisch, I. Klapper, G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Comm. Math. Phys. 184 (1997) 443–455.
[3] Chongsheng Cao, Jiahong Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math 226 (2) (2011) 1803–1822.
[4] Dongho Chae, Peter Constantin, Jiahong Wu, Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations, Arch. Ration. Mech. Anal. 202 (1) (2011) 35–62.
[5] Dongho Chae, Peter Constantin, Diego Córdoba, Francisco Gancedo, Jiahong Wu, Generalized surface quasi-geostrophic equations with singular velocities, Comm. Pure Appl. Math. 65 (8) (2012) 1037–1066.
[6] Peter Constantin, Charles Fefferman, Direction of vorticity and the problem of global regularity for the Navier–Stokes equations, Indiana Univ. Math. J. 42 (3) (1993) 775–789.
[7] Peter Constantin, Vlad Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, arXiv:1110.0179, 2011.
[8] Antonio Cordoba, Diego Cordoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (3) (2004) 511–528.
[9] Jian Deng, Thomas Y. Hou, Xinwei Yu, Geometric properties and nonblowup of 3D incompressible Euler flow, Comm. Partial Differential Equations 30 (1–2) (2005) 225–243.
[10] Jian Deng, Thomas Y. Hou, Xinwei Yu, Improved geometric conditions for non-blowup of the 3D incompressible Euler equation, Comm. Partial Differential Equations 31 (1–3) (2006) 293–306.
[11] Hongjie Dong, Dong Li, On the 2D critical and supercritical dissipative quasi-geostrophic equation in Besov spaces, J. Differential Equations 248 (11) (2010) 2684–2702.
[12] Lawrence C. Evans, Partial Differential Equations, Amer. Math. Soc., 1998.
[13] Taoufik Hmidi, Sahbi Keraani, F. Rousset, Global well-posedness for Euler–Boussinesq system with critical dissipation, Comm. Partial Differential Equations 36 (3) (2011) 420–445.
[14] Taoufik Hmidi, Sahbi Keraani, Frederic Rousset, Global well-posedness for a Boussinesq–Navier–Stokes system with critical dissipation, J. Differential Equations 249 (9) (2010) 2147–2174.
[15] Taoufik Hmidi, M. Zerguine, On the global well-posedness of the Euler–Boussinesq system with fractional dissipation, Phys. D 239 (15) (2010) 1387–1401.
[16] Ning Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys. 255 (1) (2005) 161–181.
[17] Alexander Kiselev, Fedor Nazarov, A variation on a theme of Caffarelli and Vasseur, J. Math. Sci. 166 (1) (2010) 31–39.
[18] Alexander Kiselev, Fedor Nazarov, Alexander Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (3) (2007) 445–453.
[19] Hideo Kozono, Yasushi Taniuchi, Limiting case of the Sobolev inequality in BMO, with application to the Euler equations, Comm. Math. Phys. 214 (1) (2000) 191–200.

[20] R.H. Kraichnan, Inertial-range spectrum of hydromagnetic turbulence, Phys. Fluids 8 (1965) 1385–1387.
[21] Zhen Lei, Yi Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, Discrete Contin. Dyn. Syst. 25 (2) (2009) 575–583.
[22] Pengtao Li, Zhichun Zhai, Well-posedness and regularity of generalized Navier–Stokes equations in some critical q-spaces, J. Funct. Anal. 259 (10) (2010) 2457–2519.
[23] S. Resnick, Dynamical problems in non-linear advective partial differential equations, PhD thesis, University of Chicago, 1995.
[24] Elias M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[25] Terence Tao, Global regularity for a logarithmically supercritical hyperdissipative Navier–Stokes equation, Anal. Partial Differ. Equ. 2 (3) (2009) 361–366.
[26] Chuong V. Tran, Xinwei Yu, Bounds for the number of degrees of freedom of magnetohydrodynamic turbulence in two and three dimensions, Phys. Rev. E 85 (2012) 066323.
[27] Alexis Vasseur, Regularity criterion for 3D Navier–Stokes equations in terms of the direction of the velocity, Appl. Math. 54 (1) (2009) 47–52.
[28] Jiahong Wu, Generalized MHD equations, J. Differential Equations 195 (2) (2003) 284–312.
[29] Jiahong Wu, The generalized incompressible Navier–Stokes equations in Besov spaces, Dyn. Partial Differ. Equ. 1 (4) (2004) 381–400.
[30] Jiahong Wu, Global regularity for a class of generalized magnetohydrodynamic equations, J. Math. Fluid Mech. 13 (2011) 295–305.
[31] Yong Zhou, Regularity criteria for the generalized viscous MHD equations, Ann. Inst. H. Poincaré Anal. Non Lineaire 24 (2007) 491–505.