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Well-posedness for fractional Navier–Stokes equations in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$

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This note studies the well-posedness of the fractional Navier–Stokes equations in some supercritical Besov spaces as well as in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ for $\beta \in (1/2, 1)$. Meanwhile, the well-posedness for fractional magnetohydrodynamics equations in these Besov spaces is also studied. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

In this note, we will study the well-posedness of mild solutions to the fractional Navier–Stokes equations (also called generalized Navier–Stokes equations) on the half-space $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$, $n \geq 2$:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + (u\cdot\nabla)u - \nabla p = 0, & \text{in } \mathbb{R}^{1+n}_+; \\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^{1+n}_+; \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^n \end{cases} \tag{1.1}$$

with $\beta \in (1/2, 1)$ in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. The mild solution to (1.1) is defined as the fixed point of the operator

$$(Au)(t,x) = e^{-t(-\triangle)^\beta}u_0(x) - \int_0^t e^{-(t-s)(-\triangle)^\beta}P\nabla(u\otimes u)(s,x)\mathrm{d}s := e^{-t(-\triangle)^\beta}u_0(x) - B(u,u). \tag{1.2}$$

Here,

$$e^{-t(-\triangle)^\beta}f(x) := K_t^\beta(x) * f(x) \quad \text{with} \quad \widehat{K_t^\beta}(\xi) = e^{-t|\xi|^{2\beta}}$$

and P is the Helmholtz–Weyl projection onto divergence free vector fields:

$$P = \{P_{j,k}\}_{j,k=1,\cdots,n} = \{\delta_{j,k} + R_jR_k\}_{j,k=1,\cdots,n}$$

where $\delta_{j,k}$ is the Kronecker symbol and $R_j = \partial_j(-\triangle)^{-1/2}$ are the Riesz transforms.

An important property of the fractional Navier–Stokes equations is its invariance under the following time and space scaling:

$$u_\lambda(t,x) = \lambda^{2\beta-1}u(\lambda^{2\beta}t,\lambda x), \quad p_\lambda(t,x) = \lambda^{4\beta-2}p(\lambda^{2\beta}t,\lambda x), \quad (u_0)_\lambda(x) = \lambda^{2\beta-1}u_0(\lambda x). \tag{1.3}$$

This scaling invariance naturally leads to the following notion of critical spaces. A function space is critical for (1.1) if it is invariant under the same spatial scaling

$$f(x) \longrightarrow \lambda^{2\beta-1}f(\lambda x).$$

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In other words, a function space is critical for (1.1) when it is homogeneous and of degree $-(2\beta-1)$. Examples include $L^{n/(2\beta-1)}$ ($\beta > 1/2$), $BMO^{-(2\beta-1)}$, and $\dot{B}^{-(2\beta-1)}_{\infty,\infty}$, the space that will be discussed in this note. Non-critical spaces are further classified into supercritical and subcritical. A space is said to be supercritical for (1.1) if it is of degree $\alpha > -(2\beta-1)$ and subcritical if it is of degree $\alpha < -(2\beta-1)$. A general understanding in the study of nonlinear PDEs is that one can expect local well-posedness in supercritical spaces. However, the issue is much more complicated for critical spaces, as we will see in the following discussion.

When $\beta = 1$, Equation (1.1) becomes the classical incompressible Navier–Stokes equations. In this case, the theory of mild solutions is pioneered by Kato and Fujita [1, 2] where they transformed the classical incompressible Navier–Stokes equations into an integral equation and proved the local existences in some Lebesgue and Sobolev spaces. These works inspired extensive study in the following years on the well-posedness of Navier–Stokes equations in various critical spaces. Kato in [3] proved the existence of mild solutions in $L^p(\mathbb{R}^n)$ for $p \geq n$. Existence of solutions for initial values in L^p spaces had also been studied by Fabes *et al.* [4] and by Giga [5]. The local well-posedness in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ was studied by Cannone in [6]. Giga and Miyakawa [7], Kato [8], and Taylor [9] studied the well-posedness in certain Morrey spaces. Koch and Tataru in [10] proved the well-posedness of classical incompressible Navier–Stokes equations in the space $BMO^{-1}(\mathbb{R}^n) = \nabla\cdot(BMO(\mathbb{R}^n))^n$. Xiao in [11] generalized the result of Koch and Tataru [10] to $Q^{-1}_{\alpha;\infty}(\mathbb{R}^n)$ for $\alpha \in (0,1)$. Chen and Xin in [12] studied the classical incompressible Navier–Stokes equations in several critical spaces. All these works naturally lead to the question of whether the 3D classical incompressible Navier–Stokes equations is well posed in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, whose significance is due to the fact that it is the largest critical space in the sense that all other critical spaces are continuously embedded in it (see, e.g., Cannone [13], Frazier *et al*. [14], and Meyer [15]). In fact, this question was proposed as conjectures in Cannone [13] and Meyer [15]. The answer to this question is most likely negative in light of several recent results. In [16], Montgomery-Smith constructed a 1D model equation with the same scaling invariance as the Navier–Stokes equations, but is ill posed in its largest critical space, which is of course $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, the same as that of the 3D Navier–Stokes equations. Later, Bourgain and Pavlović studied in [17] the Navier–Stokes equations itself and constructed a class of initial values whose $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ norm can grow arbitrarily fast, a.k.a. the phenomenon of 'norm inflation'. This result is later generalized by Yoneda [18] to a generalized Besov space, which is smaller than $\dot{B}^{-1}_{\infty,q}$ for all $q > 2$. Recently, Cheskidov and Shvydkoy in [19] showed that there are initial values for which the solution map of Leray–Hopf solutions in $B^{-1}_{\infty,\infty}$ to the 3D Navier–Stokes equations is not continuous at $t = 0$.

For the general case (1.1), Lions [20] proved the global existence of the classical solutions when $\beta \geq \frac{5}{4}$ in a 3D case. Wu in [21] obtained a similar result for $\beta \geq \frac{1}{2} + \frac{n}{4}$ in dimension n. For the important case $\beta < \frac{1}{2} + \frac{n}{4}$, Wu in [22, 23] considered the existence of solution to Equation (1.1) in $\dot{B}^{1+\frac{n}{p}-2\beta}_{p,q}(\mathbb{R}^n)$. Dong and Li in [24] established the optimal local smoothing estimates of solutions to (1.1) in Lebesgue spaces. In Li and Zhai [25, 26], inspired by Koch and Tataru [10] and Xiao [11], they studied (1.1) in critical space $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n) = \nabla\cdot(Q^{\beta}_{\alpha}(\mathbb{R}^n))^n$ for $\beta \in (1/2,1)$ and $\alpha \in (0,\beta)$. Here, $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ for $\alpha \in (-\infty,\beta)$ is the set of all measurable functions with

$$\sup_I (l(I))^{2(\alpha+\beta-1)-n}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}}\mathrm{d}x\mathrm{d}y < \infty,$$

where the supremum is taken over all cubes I with edge length $l(I)$ and edges parallel to the coordinate axes in $\mathbb{R}^n$. $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is a generalization of $Q_{\alpha}(\mathbb{R}^n)$ studied in Essen *et al*. [27], Xiao [28], and Dafni and Xiao [29]. Recently, in [30], Zhai proved the well-posedness for Equation (1.1) in critical spaces $BMO^{-(2\beta-1)}(\mathbb{R}^n) = (-\triangle)^{\frac{2\beta-1}{2}}BMO(\mathbb{R}^n)$ and $G^{-(2\beta-1)}_{n}(\mathbb{R}^n)$, which are all close to $\dot{B}^{-(2\beta-1)}_{\infty,\infty}$ for $\beta \in (1/2,1)$. Here, for $s > 0$,

$$G^{-s}_{p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : |f| \in \mathcal{S}'(\mathbb{R}^n), \|f\|_{G^{-s}_p(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{sn}{2p\beta}}\|e^{-t(-\triangle)^{\beta}}|f|\|_{L^{\infty}(\mathbb{R}^n)} < \infty \right\},$$

which is a subclass of Besov spaces and also contains the Morrey-type space of measures. These well-posedness results extend that of Chen and Xin [12] and Koch and Tataru [10]. For the regularity of mild solutions to Equation (1.1), we refer the readers to Dong and Li [24], Katz and Pavlović [31], Li and Zhai [25], Wu [32], and Zhai [33].

For the fractional Navier–Stokes equation (1.1), it has also been shown that all critical spaces are continuously embedded in the largest space $\dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$ (see, e.g., Li-Zhai [25]). Given that all the well-posedness results for the Navier–Stokes equations have been developed in the fractional case, a natural conjecture would be that Bourgain and Pavlović's and Cheskidov and Shvydkoy's ill-posedness results for Navier–Stokes equations can also be transplanted. It is difficult to answer this problem directly. In this note, we will prove that the fractional Navier–Stokes equations is globally well-posed in its largest critical space $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ with $\beta \in (1/2,1)$ for small initial data. Thus, our result leads us to expect that it is unlikely to extend the above-mentioned ill-posedness result to the fractional case.

Our proof of the global existence depends on the contraction principle in a suitable space. In fact, for adapted value space $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ with $\beta \in (1/2,1)$, we can find an admissible path space Y defined by

$$Y = \left\{ u : (0,\infty) \longrightarrow L^{\infty}(\mathbb{R}^n) : \nabla\cdot u = 0 \quad \text{and} \quad \sup_{t>0} t^{\frac{2\beta-1}{2\beta}}\|u(t)\|_{L^{\infty}(\mathbb{R}^n)} < \infty \right\}$$

such that the bilinear operator $B(\cdot,\cdot) : Y \times Y \longrightarrow Y$ is continuous. Then, we can apply the contraction principle in Y to find a solution u. Finally, we prove $u \in \dot{B}^{1-2\beta}_{\infty,\infty}(\mathbb{R}^n)$. Note that our method breaks down for the Navier–Stokes equations ($\beta = 1$), consistent with the ill-posedness of the latter in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$.

We will also show local well-posedness for the fractional Navier–Stokes equations in supercritical spaces $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\frac{n}{p} > \alpha > 1 - 2\beta + \frac{n}{p}, 1 \le q \le \infty, 2\alpha + \min\{0, 1 - \frac{2}{p}\} > 0$ and $\beta \in \left(\frac{1}{2}, \frac{1}{2} + \frac{n}{4}\right)$.

Finally, our method can be applied without difficulty to the fractional magnetohydrodynamics (MHD) equations:

$$\begin{cases} \partial_t u + (-\triangle)^{\beta} u + u \cdot \nabla u + \nabla p - b \cdot \nabla b = 0, & \text{in } \mathbb{R}^{1+n}_+; \\ \partial_t b + (-\triangle)^{\beta} b + u \cdot \nabla b - b \cdot \nabla u = 0, & \text{in } \mathbb{R}^{1+n}_+; \\ \nabla \cdot u = \nabla \cdot b = 0, & \text{in } \mathbb{R}^{1+n}_+; \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.4}$$

with $\beta \in (1/2, 1)$ and establish the global well-posedness for small initial data in its largest critical space $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. We refer the readers to Cao and Wu [34], Wu [21, 32, 35], Yuan [36], Zhou [37], and the references therein for more information about the MHD system.

The remainder of this note is organized as follows: In Section 2, we will review the classical Littlewood–Paley theory as well as the general framework for mild solutions. Then, in Section 3, we will state and prove our main results.

2. Preliminaries

2.1. Littlewood–Paley theory

We recall the definition of the homogeneous Besov spaces. For more details, see Berg and Lofstrom [38], Runst and Sickel [39], and Triebel [40, 41]. We start with the Fourier transform. The Fourier transform $\widehat{f}$ of $f \in \mathcal{S}$ is defined as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.$$

Here, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The fractional power of the Laplacian can be defined by the Fourier transform. For $\theta \in \mathbb{R}$,

$$\widehat{(-\triangle)^{\theta/2} f}(\xi) = |\xi|^{\theta} \widehat{f}(\xi).$$

Then, we introduce the Littlewood–Paley decomposition by means of $\{\varphi_j\}_{j=-\infty}^{\infty}$. Take a function $\phi \in C_0^{\infty}$ with

$$\text{supp}(\phi) = \{\xi \in \mathbb{R}^n : 1/2 < |\xi| \le 2\}$$

such that $\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1$ for all $\xi \neq 0$. Then, we define functions $\varphi_j (j = 0, \pm 1, \pm 2, \cdots)$ as

$$\widehat{\varphi_j}(\xi) = \phi(2^{-j}\xi).$$

Let $\triangle_j f = \varphi_j * f$, for $j = 0, \pm 1, \pm 2, \pm 3, \cdots$. Then, for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=-\infty}^{\infty} (2^{sj} \|\triangle_j f\|_{L^p(\mathbb{R}^n)})^q \right)^{1/q}, \quad 1 \le q < \infty$$

$$\|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} = \sup_{-\infty<j<\infty} (2^{sj} \|\triangle_j f\|_{L^p(\mathbb{R}^n)}), \quad q = \infty,$$

where $L^p(\mathbb{R}^n)$ means the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$. The homogeneous Bosev space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty \right\}.$$

Moreover, for negative s, the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ can be defined equivalently as follows.

Lemma 2.1

[42] Let $1 \le p, q \le \infty$, $s < 0$ and $0 < \beta < \infty$. Then, $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\left(\int_0^{\infty} (t^{-\frac{s}{2\beta}} \|e^{-t(-\triangle)^{\beta}} f\|_{L^p(\mathbb{R}^n)})^q \frac{dt}{t} \right)^{1/q} < \infty, \quad 1 \le q < \infty,$$

$$\sup_{t>0} t^{-\frac{s}{2\beta}} \|e^{-t(-\triangle)^{\beta}} f\|_{L^p(\mathbb{R}^n)} < \infty, \quad q = \infty.$$

We will use the $L^p - L^q$ type estimates for $e^{-t(-\Delta)^\theta}$ in Lebesgue and Besov spaces and product in Besov spaces. See Kozono *et al*. [43], Miao *et al*. [42], Runst and Sickel [39], and Zhai [44] for the proof of the following lemma.

Lemma 2.2
Let $\theta > 0$ and $1 \le p, q \le \infty$. Then, the following statements hold:

(i)

$$\|e^{-t(-\Delta)^\theta} f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)},$$

$$\|\nabla e^{-t(-\Delta)^\theta} f\|_{L^p(\mathbb{R}^n)} \le Ct^{-\frac{1}{2\theta}}\|f\|_{L^p(\mathbb{R}^n)},$$

$$\|P\nabla e^{-t(-\Delta)^\theta} f\|_{L^p(\mathbb{R}^n)} \le Ct^{-\frac{1}{2\theta}}\|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) If $s_1 \le s_2$, then

$$\|e^{-t(-\Delta)^\theta} f\|_{\dot{B}^{s_2}_{p,q}(\mathbb{R}^n)} \le Ct^{-\frac{s_2-s_1}{2\theta}}\|f\|_{\dot{B}^{s_1}_{p,q}(\mathbb{R}^n)},$$

$$\|\nabla e^{-t(-\Delta)^\theta} f\|_{\dot{B}^{s_2}_{p,q}(\mathbb{R}^n)} \le Ct^{-\frac{s_2-s_1+1}{2\theta}}\|f\|_{\dot{B}^{s_1}_{p,q}(\mathbb{R}^n)}.$$

(iii) If $s_1, s_2 < \frac{n}{p}$ and $s_1 + s_2 + n\min\left\{0, 1 - \frac{2}{p}\right\} > 0$, then there exists a positive constant C such that

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{p}}_{p,q}(\mathbb{R}^n)} \le C\|u\|_{\dot{B}^{s_1}_{p,q}(\mathbb{R}^n)}\|v\|_{\dot{B}^{s_2}_{p,q}(\mathbb{R}^n)}.$$

2.2. An abstract lemma

We need the following abstract result, which can be proved by Banach fixed point theorem (see Lemarié-Rieusset [45] and Meyer [15]).

Lemma 2.3
Let $(Z, \|\cdot\|_Z)$ be a Banach space and $H : Z \times Z \to Z$ a bounded bilinear form satisfying

$$\|H(x_1, x_2)\|_Z \le C_0\|x_1\|_Z\|x_2\|_Z \tag{2.1}$$

for all $x_1, x_2 \in Z$ and a constant $C_0 > 0$. Then, if $0 < \varepsilon < \frac{1}{4C_0}$ and if $y \in Z$ with $\|y\|_Z < \varepsilon$, then the equation $u = y + H(u, u)$ has a solution in Z such that $\|u\|_Z \le 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution u depends continuously on y in the sense that if $\|y'\|_Z \le \varepsilon$, $u' = y' + H(u', u')$ and $\|u'\|_Z \le 2\varepsilon$, then

$$\|u - u'\|_Z \le \frac{1}{1 - 4\varepsilon C_0}\|y - y'\|_Z.$$

Recalling the definition of mild solutions to Equation (1.1), we easily see that to establish local or global well-posedness in a space X; roughly speaking, all we need is the bilinear estimate (2.1) with $C_0 = CT^a$ for $a > 0$ and $Z = L^\infty((0, T); X)$ or an absolute constant C_0 and $Z = L^\infty((0, \infty); X)$.

3. Main results

We divide this section into two parts in which we study fractional Navier–Stokes equations and fractional MHD equations, respectively.

3.1. Fractional Navier–Stokes equations

Theorem 3.1
(i) (Global existence in critical spaces) For all $\beta \in (1/2, 1)$ there exists $\varepsilon_\beta > 0$ such that for all $u_0 \in \dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$ and

$$\|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} \le \varepsilon_\beta,$$

for (1.1), there exists a solution $u \in L^\infty((0, \infty), \dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n))$ such that

$$\sup_{t>0} t^{\frac{2\beta-1}{2\beta}}\|u(t)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

(ii) (Local existence in supercritical spaces) Let $1 < p < \infty, 1 \le q \le \infty, \beta \in (\frac{1}{2}, \frac{1}{2} + \frac{n}{4}), \frac{n}{p} > \alpha > 1-2\beta+\frac{n}{p}$ and $2\alpha+\min\{0, 1-\frac{2}{p}\} > 0$. Then, for any $u_0 \in \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$, for (1.1), there exists $T = T(\|u_0\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)})$ and a unique solution $u \in L^{\infty}([0, T], \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n))$.

Proof

(i) We will use the method of integral equations and the contraction mapping principle to prove the existence. To do this, we define

$$Y = \left\{ u : (0, \infty) \longrightarrow L^{\infty}(\mathbb{R}^n) : \nabla \cdot u = 0 \quad \text{and} \quad \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^{\infty}(\mathbb{R}^n)} < \infty \right\},$$

which is a Banach space with the norm

$$\|u\|_Y = \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^{\infty}(\mathbb{R}^n)}.$$

We will find a fixed point of operator Au in Y. To do this, we first estimate $\|Au(t)\|_{L^{\infty}(\mathbb{R}^n)}$ for $t > 0$. By (1.2) and Lemma 2.2, we have

$$\|Au(t)\|_{L^{\infty}(\mathbb{R}^n)} \le \|e^{-t(-\triangle)^{\beta}} u_0\|_{L^{\infty}(\mathbb{R}^n)} + \int_0^t (t-s)^{-\frac{1}{2\beta}} \|u(s)\|^2_{L^{\infty}(\mathbb{R}^n)} ds.$$

This estimate and the definition of X imply that

$$\begin{aligned}
\|Au(t)\|_Y &\le \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \|e^{-t(-\triangle)^{\beta}} u_0\|_{L^{\infty}(\mathbb{R}^n)} + \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} \|u(s)\|_{L^{\infty}(\mathbb{R}^n)} \|u(s)\|_{L^{\infty}(\mathbb{R}^n)} ds \\
&\le C_1 \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + \|u\|^2_Y \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} s^{-\frac{2\beta-1}{\beta}} ds \\
&\le C_1 \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + C_2 \|u\|^2_Y
\end{aligned}$$

because

$$\int_0^t (t-s)^{-\frac{1}{2\beta}} s^{-\frac{2\beta-1}{\beta}} ds = B\left(\frac{1}{\beta} - 1, 1 - \frac{1}{2\beta}\right) t^{\frac{1-2\beta}{2\beta}}$$

for $\beta \in (1/2, 1)$. Here, $B(a, b)$ is the classical beta function for positive a and b. Thus, we have

$$\|Au - A\overline{u}\|_Y \le C_3 \max\{\|u\|_Y, \|\overline{u}\|_Y\} \|u - \overline{u}\|_Y,$$

for any $u, \overline{u} \in Y$.

Then, we will prove that A is a contraction mapping on $B_Y(0, r)$ with a suitable r. In fact, if there exists $\varepsilon_{\beta} > 0$ small enough such that

$$\|e^{-t(-\triangle)^{\beta}} u_0\|_Y \le C_1 \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} < \varepsilon_{\beta} \quad \text{and} \quad 6C_3C_1 \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} < 1/2,$$

then, we can prove that for any $u, v \in B_Y(0, r)$ with $r = 2C_1 \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)}$,

$$\begin{aligned}
\|Au - Av\|_Y &\le C_3 \max\{\|u\|_Y, \|v\|_Y\} \|u - v\|_Y \\
&\le C_3 r \|u - v\|_Y \\
&\le \frac{1}{2} \|u - v\|_Y.
\end{aligned}$$

Thus, A is a contraction mapping in $B_Y(0, r)$ and has a unique fixed point u.

Now, we prove $\sup_{t>0} \|u(t)\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} < \infty$. In fact, by (1.2) and Lemma 2.2,

$$\begin{aligned}
\|e^{-l(-\triangle)^{\beta}} Au(t)\|_{L^{\infty}(\mathbb{R}^n)} &\le \|e^{-(l+t)(-\triangle)^{\beta}} u_0\|_{L^{\infty}(\mathbb{R}^n)} + \left\| \int_0^t e^{-(l+t-s)(-\triangle)^{\beta}} P\nabla(u \otimes u) ds \right\|_{L^{\infty}(\mathbb{R}^n)} \\
&\le \|e^{-l(-\triangle)^{\beta}} u_0\|_{L^{\infty}(\mathbb{R}^n)} + \int_0^{t+l} (l+t-s)^{-\frac{1}{2\beta}} s^{-\frac{2\beta-1}{\beta}} ds \|u\|^2_Y.
\end{aligned}$$

So, we have

$$\begin{aligned}\|Au(t)\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} &= \sup_{l>0} l^{\frac{2\beta-1}{2\beta}} \|e^{-l(-\triangle)^\beta} Au(t)\|_{L^\infty(\mathbb{R}^n)} \\ &\le C\|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + \sup_{l>0} l^{\frac{2\beta-1}{2\beta}} \int_0^{t+l} (l+t-s)^{-\frac{1}{2\beta}} s^{-\frac{2\beta-1}{\beta}} ds \|u\|_Y^2 \\ &\le C\|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + \sup_{l>0} l^{\frac{2\beta-1}{2\beta}} (l+t)^{-\frac{2\beta-1}{2\beta}} B\left(\frac{1}{\beta}-1, 1-\frac{1}{2\beta}\right) \|u\|_Y^2 \\ &\le C\|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + C\|u\|_Y^2\end{aligned}$$

for all $t>0$. Thus, $\sup_{t>0} \|Au(t)\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} < \infty$.

(ii) For $T>0$, we define

$$X_T = \left\{ u : (0,T) \longrightarrow \dot{B}^\alpha_{p,q}(\mathbb{R}^n) : \nabla \cdot u = 0 \quad \text{and} \quad \sup_{0<t<T} \|u(t)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty \right\},$$

which is a Banach space with the norm

$$\|u\|_{X_T} = \sup_{0<t<T} \|u(t)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)}.$$

For any $u_0 \in \dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|Au(t)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} &\le \|e^{-t(-\triangle)^\beta} u_0\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} + \int_0^t \|e^{-(t-s)(-\triangle)^\beta} P\nabla(u\otimes u)(s,x)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} ds \\ &\le \|e^{-t(-\triangle)^\beta} u_0\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} + \int_0^t (t-s)^{-\frac{\alpha-(2\alpha-\frac{n}{p})+1}{2\beta}} \|(u\otimes u)(s,x)\|_{\dot{B}^{2\alpha-\frac{n}{p}}_{p,q}(\mathbb{R}^n)} ds \\ &\le \|e^{-t(-\triangle)^\beta} u_0\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} + \int_0^t (t-s)^{-\frac{\alpha-(2\alpha-\frac{n}{p})+1}{2\beta}} \|u(s,x)\|_{\dot{B}^\alpha_{p,\infty}(\mathbb{R}^n)} \|u(s,x)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} ds \\ &\le \|e^{-t(-\triangle)^\beta} u_0\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} + \int_0^t (t-s)^{-\frac{\alpha-(2\alpha-\frac{n}{p})+1}{2\beta}} ds \|u\|_{X_T}^2\end{aligned}$$

by applying Lemma 2.2. Thus, we have

$$\|Au(t)\|_{X_T} \le \|e^{-t(-\triangle)^\beta} u_0\|_X + CT^{\frac{2\beta+\alpha-\frac{n}{p}-1}{2\beta}} \|u\|_{X_T}^2$$

as $2\beta+\alpha-\frac{n}{p}-1>0$. Therefore, Lemma 2.3 finishes the proof.

□

3.2. *Fractional MHD equations*

We prove the existence of solutions to the fractional MHD equations by similar methods used in the proof of Theorem 3.1.

Proposition 3.2

(i) (Global existence in critical spaces) For all $\beta \in (1/2, 1)$, there exists $\varepsilon_\beta > 0$ such that for all $(u_0, b_0) \in \dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ with

$$\|b_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} + \|u_0\|_{\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)} \le \varepsilon_\beta,$$

for (1.4), there exists a solution $(u,b) \in L^\infty((0,\infty), \dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n))$ such that

$$\sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} \|b(t)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

(ii) (Local existence in supercritical spaces) Let $1<p<\infty$, $\beta \in \left(\frac{1}{2}, \frac{1}{2}+\frac{n}{4}\right)$, $\frac{n}{p} > \alpha > 1-2\beta+\frac{n}{p}$, $1 \le q \le \infty$ and $2\alpha + \min\{0, 1-\frac{2}{p}\} > 0$. Then, for any $(u_0, b_0) \in \dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, for (1.4), there exists $T = T(\|(u_0,b_0)\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)})$ and a unique solution $(u,b) \in L^\infty([0,T], \dot{B}^\alpha_{p,q}(\mathbb{R}^n))$.

Proof

Here, we only prove (i) because (ii) follows from a similar argument and the proof of Theorem 3.1 (ii). The solution (u,b) to Equation (1.4) can be written as

$$u(t,x) = e^{-t(-\triangle)^\beta} u_0(x) - B(u,u) + B(b,b) := F_1(u,b),$$

$$b(t,x) = e^{-(-\triangle)^{\beta}} b_0(x) - B(u,b) + B(b,u) := F_2(u,b),$$

with

$$B(u,v) = \int_0^t e^{-(t-s)(-\triangle)^{\beta}} P\nabla \cdot (u \otimes v)(s) ds.$$

Define

$$Y = \left\{(u,b) : (0,\infty) \longrightarrow L^{\infty}(\mathbb{R}^n) | \nabla \cdot u = \nabla \cdot b = 0, \quad \|(u,b)\|_Y < \infty\right\}$$

with

$$\|(u,b)\|_Y = \|u\|_Y + \|b\|_Y = \sup_{t>0} t^{\frac{2\beta-1}{2\beta}} (\|u(t)\|_{L^{\infty}(\mathbb{R}^n)} + \|b(t)\|_{L^{\infty}(\mathbb{R}^n)}).$$

We want to show that F_1 and F_2 are contraction mappings from a ball of Y to itself. We rewrite the solution (u,b) as

$$\begin{pmatrix} u \\ b \end{pmatrix} = \begin{pmatrix} F_1(u,b) \\ F_2(u,b) \end{pmatrix} := F(u,b).$$

By similar argument as Theorem 3.1, we obtain

$$\|F_1(u,b)\| \leq C\|u_0|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + C\|(u,b)\|_Y^2$$

and

$$\|F_1(u,b)(t) - F_1(u',b')\|_Y \leq C\|(u-u',b-b')\|_Y(\|(u,b)\|_Y + \|(u',b')\|_Y).$$

Similarly,

$$\|F_2(u,b)\| \leq C\|b_0|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + C\|(u,b)\|_Y^2$$

and

$$\|F_2(u,b)(t) - F_2(u',b')\|_Y \leq C\|(u-u',b-b')\|_Y(\|(u,b)\|_Y + \|(u',b')\|_Y).$$

Like in the proof of Theorem (3.1), we find a fixed point (u,b) of the operator F. Finally, we can prove similarly that

$$\sup_{t>0} \|u(t)\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} + \sup_{t>0} \|b(t)\|_{\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)} < \infty.$$

□

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