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# ON THE LAGRANGIAN AVERAGED EULER EQUATIONS: LOCAL WELL-POSEDNESS AND BLOW-UP CRITERION

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Abstract. In this article we study local and global well-posedness of the Lagrangian Averaged Euler equations. We show local well-posedness in Triebel-Lizorkin spaces and further prove a Beale-Kato-Majda type necessary and sufficient condition for global existence involving the stream function. We also establish new sufficient conditions for global existence in terms of mixed Lebesgue norms of the generalized Clebsch variables.

1. **Introduction.** In [12, 13], Holm, Marsden and Ratiu introduced the 3D Lagrangian averaged Euler equations as follows:

$$\begin{cases} \partial_t u + (u_\alpha \cdot \nabla) u + (\nabla u_\alpha)^T \cdot u = -\nabla p, \\ \text{div} u = 0. \end{cases} \tag{1}$$

Here the $j-$th component of $\nabla v \cdot u$ is $(\nabla v \cdot u)_j = \sum_{k=1}^3 \partial_j v_k u_k$, and the relation between the velocity $u$ and the averaged velocity $u_\alpha$ is given by

$$u_\alpha = (1 - \alpha^2 \triangle)^{-1} u. \tag{2}$$

It is easy to see that when $\alpha = 0$ (1) reduces to the 3D incompressible Euler equations.

Similar to the 3D Euler equations, (1) also enjoys a "vorticity formulation" after taking curl of both sides and denoting $\omega = \nabla \times u$:

$$\begin{cases} \partial_t \omega + (u_\alpha \cdot \nabla)\omega = \nabla u_\alpha \cdot \omega, \\ \omega(0) = \omega_0. \end{cases} \tag{3}$$

Note that (3) has the same form as the vorticity formulation for the 3D Euler equations, except that the transporting velocity $u$ has been replaced by the "averaged" velocity $u_\alpha = (1 - \alpha^2 \triangle)^{-1} u$. One can further introduce the stream function $\psi$ through $-\triangle \psi = \omega$. For convenience of the readers, we summarize the relations between $u, \omega$, the averaged velocity $u_\alpha$ and the stream function $\psi$:

$$-\triangle \psi = \omega, \quad u = \nabla \times \psi = \nabla \times (-\triangle)^{-1} \omega, \quad u_\alpha = (1 - \alpha^2 \triangle)^{-1} \nabla \times \psi. \tag{4}$$

Equation (1) has both practical and theoretical significance. On one hand, it can be applied to the study of turbulence as a closure model ([5, 6, 10]); On the other hand, (1) enjoys similar geometrical and analytical structures as that of the

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3D Euler equations and thus can be studied as a regularized model of the latter. For the geometrical side of the theory, we refer the readers to [13, 21, 22]. In the following we will focus on the analytical side.

A long-standing open problem in mathematical fluid mechanics is the global well-posedness/finite time singularity of the 3D Navier-Stokes/Euler equations. It is thus natural to study the same problem for the Lagrangian averaged equations and hope for some insight. Interestingly, despite the fact that (1) is "regularized", a fact clearly seen from the vorticity formulation (3), its global well-posedness/finite time singularity still seems beyond current machinery of analysis. In this article we will derive some new necessary and sufficient conditions for the global well-posedness of (1). To put our results in context, we review some recent progress before introducing the main results of this article.

In [15], Hou and Li studied the local existence and blow-up criterion for solution in $H^m(\mathbb{R}^3)$. They proved that if $\omega_0 \in H^s(\mathbb{R}^3)$ with $s \geq 2$ and

$$\int_0^T \|\psi\|_{BMO} dt < \infty, \quad \text{for some} \quad T > 0,$$

then for any $\alpha > 0$, there exists a unique global solution in $H^s(\mathbb{R}^3)$ with

$$\|\omega(t)\|_{H^s(\mathbb{R}^3)} \leq C(T)\|\omega(0)\|_{H^s(\mathbb{R}^3)}, \quad \text{for} \quad 0 \leq t \leq T.$$

Liu, Wang and Zhang in [19] showed that if $\omega_0 \in W^{s,p}(\mathbb{R}^3)$ with $\frac{3}{2} < p < \infty$, and

$$\int_0^T \|\psi\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)} dt < \infty, \quad \text{for some} \quad T > 0,$$

then for any $\alpha > 0$, there exists a unique global solution in $W^{s,p}(\mathbb{R}^3)$ with

$$\|\omega(t)\|_{W^{s,p}(\mathbb{R}^3)} \leq C(T)\|\omega(0)\|_{W^{s,p}(\mathbb{R}^3)}, \quad \text{for} \quad 0 \leq t \leq T.$$

Recently, Liu and Jia in [20] proved that if $\omega_0 \in B^s_{p,q}(\mathbb{R}^3)$ with $s > \frac{3}{p}$, $1 < p < \infty$ and $1 \leq q \leq \infty$, or $s = \frac{3}{p}$, $1 < p < \infty$ and $q = 1$, and

$$\int_0^T \|\psi\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)} dt < \infty, \quad \text{for some} \quad T > 0,$$

then for any $\alpha > 0$, there exists a unique global solution in $B^s_{p,q}(\mathbb{R}^3)$ with

$$\|\omega(t)\|_{B^s_{p,q}(\mathbb{R}^3)} \leq C(T)\|\omega(0)\|_{B^s_{p,q}(\mathbb{R}^3)}, \quad \text{for} \quad 0 \leq t \leq T.$$

In this article, we will study the local existence and blow-up criterion of equation (3) in Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. More specifically, we show that

1. If $\omega_0 \in F^s_{p,q}(\mathbb{R}^3)$, for either $s > \frac{3}{p}$ with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, or $s = 3$ with $p = 1$ and $q \in [1, \infty]$, the Lagrangian averaged 3D Euler equations (3) is locally well-posed in $F^s_{p,q}(\mathbb{R}^3)$.
2. If

   $$\int_0^T \|\psi(t)\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^3)} dt < \infty \quad \text{for some} \quad T > 0,$$

   then the Lagrangian averaged 3D Euler equations have a unique global solution $\omega(t) \in F^s_{p,q}(\mathbb{R}^3)$, satisfying

   $$\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \leq C(T)\|\omega(0)\|_{F^s_{p,q}(\mathbb{R}^3)}, \quad \text{for} \quad 0 \leq t \leq T. \tag{5}$$

A novel approach to the global well-posedness problem is pioneered by Hou and Li in [15], with inspiration from the classical Clebsch representation of vorticity. They show that, if the initial vorticity $\omega$ can be written in terms of two level set functions as follows,

$$\omega(0,x) = \omega_0(\phi_0,\psi_0)\nabla\phi_0 \times \nabla\psi_0,$$

then this representation remains true for later times,

$$\omega(t,x) = \omega_0(\phi,\psi)\nabla\phi \times \nabla\psi,$$

as long as the level set functions $\phi, \psi$ evolve according to

$$\phi_t + (u_\alpha \cdot \nabla)\phi = 0, \quad \phi(0,x) = \phi_0(x),$$
$$\psi_t + (u_\alpha \cdot \nabla)\psi = 0, \quad \psi(0,x) = \psi_0(x).$$

In this case, they proved that if the initial data $\omega_0$, $\phi_0$ and $\psi_0$ are smooth and bounded, then the Lagrangian averaged 3D Euler equations have a unique smooth solution up to $T$ as long as either

$$\int_0^T \|\phi\|_{TV} dt < \infty \quad \text{or} \quad \int_0^T \|\psi\|_{TV} dt < \infty. \tag{6}$$

Moreover, the following estimate holds

$$\|\omega(t)\|_{H^m(\mathbb{R}^3)} \le C\|\omega(0)\|_{H^m(\mathbb{R}^3)}, \quad 0 \le t \le T$$

for $m > 5/2$. Here $\|\phi\|_{TV} = \sum_{i=1}^3 \|\phi\|_{TVx_i}$ for

$$\|\phi\|_{TVx_1} = \sup_{x_2,x_3} \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x_1}\phi(x_1,x_2,x_3)\right| dx_1$$

and $\|\phi\|_{TVx_2}$ and $\|\phi\|_{TVx_3}$ defined similarly. This global existence condition has been extended by Deng, Hou and Yu in [9] to the case of the vorticity represented by generalized Clebsch Variables.

In this article, we will generalize conditions (6). More specifically, we replace the TV norms by the following more general mixed norms:

$$\sum_{j=1}^3 \left\| \frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}, \quad \sum_{j=1}^3 \left\| \frac{\partial\phi}{\partial y_j}\right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}$$

of the level set functions $\psi$ and $\phi$ and $1 \le p_1, q_1 \le \infty$ with

$$1 - \frac{2}{p_1} - \frac{1}{q_1} \in [0,1],$$

for $i_j \in \{1,2,3\}$. Note that for each $j$, a different $i_j$ can be taken. Here $y'$ denotes the remaining 2D vector excluding $y_j$. It is easy to see that (6) corresponds to the special case $p_1 = \infty, q_1 = 1$ and $i_j = j$.

The rest of this paper is organized as follows. In Section 2, following some basic facts of the Littlewood-Paley theory, we prove the local existence and blow-up criterion for solution in Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. In Section 3, we first give a brief review of Clebsch variables as well as more general level set formulation of the Lagrangian averaged Euler equations, then establish the global existence conditions in terms of mixed norms.

## 2. Local existence and blow-up criterion in $F^s_{p,q}(\mathbb{R}^3)$.

### 2.1. Basics of Littlewood-Paley theory and Triebel-Lizorkin spaces.

The most intuitive definition of Triebel-Lizorkin spaces is based on the following Littlewood-Paley decomposition (c.f. [28, 29]).

Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. For a given $f \in \mathcal{S}$, its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

We consider $\phi \in \mathcal{S}$ with the properties $\mathrm{Supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$, and $\hat{\phi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Letting $\hat{\phi}_j = \hat{\phi}(2^{-j}\xi)$, we can adjust the normalization constant in front of $\hat{\phi}$ such that

$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}.$$

Given $k \in \mathbb{Z}$, we define $S_k \in \mathcal{S}$ through its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j\ge k+1} \hat{\phi}_j(\xi).$$

We observe that

$$\mathrm{Supp}\hat{\phi}_j \cap \mathrm{Supp}\hat{\phi}_{j'} = \emptyset \quad \text{if} \quad |j-j'| \ge 2.$$

Let $s \in \mathbb{R}$, $p, q \in [0, \infty]$. Given $f \in \mathcal{S}'$, denote $\triangle_j f = \phi_j * f$, and then the homogeneous Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \left\| \left( \sum_{j\in\mathbb{Z}} (2^{sj}|\triangle_j f(\cdot)|)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \quad 1 \le q < \infty$$

$$\|f\|_{\dot{F}^s_{p,\infty}(\mathbb{R}^n)} = \left\| \sup_{j\in\mathbb{Z}} (2^{sj}|\triangle_j f(\cdot)|) \right\|_{L^p(\mathbb{R}^n)}, \quad q = \infty,$$

where $L^p(\mathbb{R}^n)$ is the usual Lebesgue space on $\mathbb{R}^n$. The inhomogeneous Triebel-Lizorkin norm $\|f\|_{F^s_{p,q}}$ is defined by

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

which is equivalent to

$$\left\| \left( \sum_{j=0}^{\infty} (2^{sj}|\triangle_j f(\cdot)|)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \quad 1 \le q < \infty$$

with usual modification for $q = \infty$.

Triebel-Lizorkin spaces include the usual Sobolev space $W^{s,p}(\mathbb{R}^n)$ through the relation $W^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$. In particular, we have $F^s_{2,2} = W^{s,2} = H^s$.

The key to the application of Littlewood-Paley in nonlinear partial differential equations is a set of inequalities and estimates relating different function spaces and quantify the effect of common differential operators on them. We begin with Bernstein's inequality.

**Lemma 2.1** ([25, 28, 29]). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$.*

*(i) If* $Supp\hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \le r\}$*, then there is a constant* $C$ *such that, for* $1 \le p \le q \le \infty$,

$$\|f\|_{L^q(\mathbb{R}^n)} \le Cr^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^q(\mathbb{R}^n)},$$

$$\|\partial^\beta f\|_{L^p(\mathbb{R}^n)} \le Cr^{|\beta|}\|f\|_{L^p(\mathbb{R}^n)}.$$

*(ii) If* $Supp\hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \approx r\}$*, then there is a constant* $C$ *such that, for* $1 \le p \le q \le \infty$,

$$\|f\|_{L^q(\mathbb{R}^n)} \approx Cr^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^q(\mathbb{R}^n)},$$

$$\sup_{|\beta|=k}\|\partial^\beta f\|_{L^p(\mathbb{R}^n)} \approx Cr^k\|f\|_{L^p(\mathbb{R}^n)}.$$

Next we recall various embeddings for Triebel-Lizorkin spaces.

**Lemma 2.2** ([26])**.**
*(i) For* $s > \frac{n}{p}$ *with* $p, q \in [1,\infty]$*, or* $s = n$ *with* $p = 1$ *and* $q \in [1,\infty]$*, there holds*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

*(ii) For* $s \ge n\left(\frac{1}{p}-\frac{1}{r}\right)$ *with* $q \in [1,\infty]$ *and* $1 \le p < r < \infty$*, there holds*

$$\|f\|_{L^r(\mathbb{R}^n)} \le C\|f\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

Now we list the Commutator type estimates, Beale-Kato-Majda type inequalities and Moser type inequalities in Triebel-Lizorkin spaces, respectively. They are useful tools for the study of the local existence and blow-up criterion for some partial differential equations.

**Lemma 2.3** ([7])**.** *Let* $(p,q) \in (1,\infty)\times(1,\infty]$*, or* $p = q = \infty$*, and* $f$ *be a solenoidal vector field. Then, for* $s > -1$*, we have*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbb{Z}}(2^{ks}([f,\triangle_k]\cdot\nabla g))^q\right)^{1/q}\right\|_{L^{p_1}(\mathbb{R}^n)}\\ \le& C\left(\|\nabla f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)}\|\nabla f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\right), \qquad (7)\end{aligned}$$

*where* $[f,\triangle_k]\cdot\nabla g = (f\cdot\nabla)\triangle_k g - \triangle_k(f\cdot\nabla)g$.

**Lemma 2.4** ([3, 4])**.** *Let* $s > \frac{n}{p}$ *with* $p \in [1,\infty]$, $q \in [1,\infty)$*. Then, there exists a constant* $C$ *such that the following inequality holds*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C(1 + \|f\|_{\dot{F}^0_{\infty,\infty}}(\log^+\|f\|_{F^s_{p,q}(\mathbb{R}^n)} + 1)). \qquad (8)$$

**Lemma 2.5** ([3, 4])**.** *Let* $s > 0$, $(p,q) \in (1,\infty)\times(1,\infty]$*, or* $p = q = \infty$*. Then, there exists a constant* $C$ *such that the following inequality holds:*

$$\|fg\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \le C\left(\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{\dot{F}^s_{p_2,q}(\mathbb{R}^n)} + \|g\|_{L^{r_1}(\mathbb{R}^n)}\|f\|_{\dot{F}^s_{r_2,q}(\mathbb{R}^n)}\right) \qquad (9)$$

*for* $p_1, r_1 \in [1,\infty]$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Finally we need some understanding of how differential and pseudo-differential operators act on various Triebel-Lizorkin spaces.

A function $m$ defined in $\mathbb{R}^n\backslash\{0\}$ is called to satisfy a Hörmander condition of order $k$ if

$$|m(\xi)| \leq C, \quad \text{in} \quad \mathbb{R}^n\backslash\{0\},$$

and

$$L^{2|\beta|-n}\int_{L<|\xi|<2L} |D^\beta m(\xi)|^2 d\xi \leq C,$$

for all multi-indices $\beta$ with $|\beta| \leq k$ and $C$ independent of $L > 0$.

The Hörmander multiplier theorem (c.f. [27]) states that the multiplier operator $T$ associated with $m$, $\hat{Tf}(\xi) = m(\xi)\hat{f}(\xi), f \in \mathcal{S}(\mathbb{R}^n)$, is bounded in $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ if the multiplier $m$ satisfies a Hörmander condition of order $k > n/2$.

If we define $T_1 = (1-\alpha^2\triangle)^{-1}(-\triangle)$ by $\hat{T}_1 f(\xi) = m(\xi)\hat{f}(\xi)$ with $m(\xi) = \frac{|\xi|^2}{1+\alpha^2|\xi|^2}$, for any $f \in \mathcal{S}(\mathbb{R}^n)$. It is easy to see that $m$ satisfies a Hörmander condition. Therefore we have

$$\|T_1 f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{10}$$

Meanwhile, it is easy to prove that

$$\|T_1 f\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)}. \tag{11}$$

Furthermore, for any Riesz-type operator $R$, we have,

$$\|Rf\|_{F^s_{p,q}(\mathbb{R}^n)} \leq \|f\|_{F^s_{p,q}(\mathbb{R}^n)} \tag{12}$$

for $p, q \in [1,\infty], s \in \mathbb{R}$, and

$$\|Rf\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)}. \tag{13}$$

In particular, we have

$$\|Rf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{14}$$

for $1 < p < \infty$.

Since $\nabla u = \nabla\nabla \times (-\triangle)^{-1}\omega$, (14) implies

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C\|\omega\|_{L^p(\mathbb{R}^n)} \tag{15}$$

for $1 < p < \infty$. On the other hand, $\nabla u_\alpha = (1-\alpha^2\Delta)^{-1}\nabla\nabla\times\psi = T_1\nabla\nabla\times(-\triangle)^{-1}\psi$, consequently (11) and (13) imply

$$\|\nabla u_\alpha\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)} \leq C\|\psi\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^n)} \tag{16}$$

The following result can be deduced from Lemmas 2.1 and the Hörmander multiplier theorem.

**Lemma 2.6.** *Let $u_\alpha = (1-\alpha^2\triangle)^{-1}u$. Then for any $u \in L^p(\mathbb{R}^3)$, $1 < p < \infty$,*

$$\|u_\alpha\|_{W^{2,p}(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}, \tag{17}$$

*and for any $u \in F^s_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$,*

$$\|u_\alpha\|_{F^{s+2}_{p,q}(\mathbb{R}^n)} \leq C\|u\|_{F^s_{p,q}(\mathbb{R}^n)}. \tag{18}$$

According to (15) and (17), we have, for $1 < p < \infty$,

$$\|\nabla u_\alpha\|_{W^{2,p}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C\|\omega\|_{L^p(\mathbb{R}^n)}. \tag{19}$$

This can also be found in [19] and [20].

Since $u_\alpha = (1-\alpha^2\triangle)^{-1}\nabla\times(-\triangle)^{-1}\omega$, we get

$$\nabla u_\alpha = (1-\alpha^2\triangle)^{-1}R\omega$$

where $R = \nabla\nabla \times (-\triangle)^{-1}$ is a Riesz operator. Thus (12) and (18) imply

$$\|\nabla u_\alpha\|_{F^{s+2}_{p,q}(\mathbb{R}^n)} \le C\|\omega\|_{F^s_{p,q}(\mathbb{R}^n)}. \tag{20}$$

This estimate is very important in the proof of our main results.

2.2. **Local existence and blow-up criterion.** In this section, we get the following result about the local existence and blow-up criterion of solution in the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^3)$. Our criterion is sharper than the result of Hou and Li [15] in the sense that the $BMO(\mathbb{R}^3)$ norm of the stream function is replaced by the $\dot{F}^0_{\infty,\infty}(\mathbb{R}^3)$ norm, which is weaker than the $BMO(\mathbb{R}^3)$ norm (namely, $BMO(\mathbb{R}^n) \hookrightarrow \dot{F}^0_{\infty,\infty}(\mathbb{R}^n)$).

**Theorem 2.7.**
*(i) If $\omega_0 \in F^s_{p,q}(\mathbb{R}^3)$, for either $s > \frac{3}{p}$ with $1 \le p < \infty$ and $1 \le q \le \infty$, or $s = 3$ with $p = 1$ and $q \in [1, \infty]$, the Lagrangian averaged 3D Euler equations (3) is locally well-posed in $F^s_{p,q}(\mathbb{R}^3)$.*
*(ii) If*

$$\int_0^T \|\psi(t)\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^3)} dt < \infty,$$

*then the solution in (i) exists up to at least $T$, and satisfies*

$$\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \le C(T)\|\omega(0)\|_{F^s_{p,q}(\mathbb{R}^3)}, \quad \textit{for} \quad 0 \le t \le T. \tag{21}$$

*Proof.* First we clarify the source of the restriction on $p$. According to Lemma 2.6, we have

$$\|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla u_\alpha\|_{F^{s+2}_{p,q}(\mathbb{R}^3)} \le C\|\omega\|_{F^s_{p,q}(\mathbb{R}^3)} \tag{22}$$

and $\|\omega\|_{L^\infty(\mathbb{R}^3)} \le C\|\omega\|_{F^s_{p,q}(\mathbb{R}^n)}$ for $s > \frac{3}{p}$ with $1 \le p < \infty$ and $1 \le q \le \infty$, or $s = 3$ with $p = 1$ and $1 \le q \le \infty$.

Now we start the proof. Applying $\triangle_j$ to (3), we have

$$\partial_t \triangle_j \omega + (u_\alpha \cdot \nabla)\triangle_j \omega = (u_\alpha \cdot \nabla)\triangle_j \omega - \triangle_j (u_\alpha \cdot \nabla)\omega + \triangle_j(\nabla u_\alpha \cdot \omega).$$

Since $\mathrm{div} u_\alpha = 0$, we deduce from Lemmas 2.3 and 2.5 and inequality (20) that

$$\begin{aligned} &\frac{d}{dt}\|\omega(t)\|_{\dot{F}^s_{p,q}(\mathbb{R}^3)} \\ \le& C\left\{ \|\nabla u_\alpha(t) \cdot \omega(t)\|_{\dot{F}^s_{p,q}(\mathbb{R}^3)} + \left\| \left( \sum_{j\in\mathbb{Z}} 2^{jqs}[(u_\alpha \cdot \nabla)\triangle_j\omega - \triangle_j(u_\alpha \cdot \nabla)\omega]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^3)} \right\} \\ \le& C(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)}\|\omega(t)\|_{\dot{F}^s_{p,q}(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)}\|\nabla u_\alpha(t)\|_{\dot{F}^s_{p,q}(\mathbb{R}^3)}) \\ \le& C(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)}\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)}\|\nabla u_\alpha(t)\|_{F^s_{p,q}(\mathbb{R}^3)}) \\ \le& C(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)})\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)}). \end{aligned} \tag{23}$$

For any $r \in [1, \infty)$, multiplying (3) by $|\omega|^{r-2}\omega$ and integrating over $\mathbb{R}^3$, we have

$$\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^3} |\omega|^r dx + \int_{\mathbb{R}^3} (u_\alpha \cdot \nabla)\omega \cdot |\omega|^{r-2}\omega dx = \int_{\mathbb{R}^3} (\nabla u_\alpha \cdot \omega) \cdot |\omega|^{r-2}\omega dx.$$

Noting that $\mathrm{div} u_\alpha = 0$, we get $\int_{\mathbb{R}^3} (u_\alpha \cdot \nabla)\omega \cdot |\omega|^{r-2}\omega dx = 0$ according to integration by parts. Meanwhile, it is easy to see that

$$\int_{\mathbb{R}^3} (\nabla u_\alpha \cdot \omega) \cdot |\omega|^{r-2}\omega dx \le \|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)}\|\omega\|^r_{L^r(\mathbb{R}^3)}.$$

Thus, for $r \in [1, \infty)$,

$$\frac{d}{dt}\|\omega\|_{L^r(\mathbb{R}^3)} \leq \|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)}\|\omega\|_{L^r(\mathbb{R}^3)} \tag{24}$$

and

$$\frac{d}{dt}\|\omega(t)\|_{L^r(\mathbb{R}^3)} \leq C\left(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)}\right)\|\omega(t)\|_{L^r(\mathbb{R}^3)}. \tag{25}$$

The case $r = p$ of estimate (25) and the previous estimate (23) tell us

$$\frac{d}{dt}\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \leq C(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)})\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)}). \tag{26}$$

Thus (22) and (26) imply

$$\frac{d}{dt}\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \leq C\|\omega(t)\|^2_{F^s_{p,q}(\mathbb{R}^3)}.$$

This estimate and standard technique give us the local well-posedness in $F^s_{p,q}(\mathbb{R}^3)$. Thus finishes the proof of (i).

To prove (ii), it suffices to show that both $\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)}$ and $\|\omega(t)\|_{L^\infty(\mathbb{R}^3)}$ remains bounded up to $T$. For any $r > 3/2$, using (8), (16) and (19), we have

$$\begin{aligned}\|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)} &\leq C(1 + \|\nabla u_\alpha\|_{\dot{F}^0_{\infty,\infty}}(\log^+\|\nabla u_\alpha\|_{W^{2,r}(\mathbb{R}^3)} + 1)) \\ &\leq C(1 + \|\psi\|_{\dot{F}^0_{\infty,\infty}}(\log^+\|\omega\|_{L^r(\mathbb{R}^3)} + 1)).\end{aligned} \tag{27}$$

Next we estimate $\|\omega\|_{L^r}$ as well as $\|\omega\|_{L^\infty}$. We return to the vorticity equation (3) and obtain for any $r > 3/2$:

$$\begin{aligned}\frac{d}{dt}\|\omega\|_{L^r(\mathbb{R}^3)} &\leq \|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^n)}\|\omega\|_{L^r(\mathbb{R}^3)} \\ &\leq C(1 + \|\nabla u_\alpha\|_{\dot{F}^0_{\infty,\infty}}(\log^+\|\nabla u_\alpha\|_{W^{2,r}(\mathbb{R}^3)} + 1))\|\omega\|_{L^r(\mathbb{R}^3)} \\ &\leq C(1 + \|\psi\|_{\dot{F}^0_{\infty,\infty}}(\log^+\|\omega\|_{L^r(\mathbb{R}^3)} + 1))\|\omega\|_{L^r(\mathbb{R}^n)}.\end{aligned} \tag{28}$$

It now follows that when

$$\int_0^T \|\psi(t)\|_{\dot{F}^0_{\infty,\infty}(\mathbb{R}^3)} dt < \infty$$

holds, $\|\omega\|_{L^r}$ is bounded up to $T$ and furthermore the bound is independent of $r$. Letting $r \to \infty$ we obtain the boundedness of $\|\omega\|_{L^\infty(\mathbb{R}^n)}$. On the other hand, the boundedness of $\|\omega\|_{L^r(\mathbb{R}^n)}$ together with (27) immediately gives the boundedness of $\|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^n)}$. Thus ends the proof of (ii). □

## 3. Global existence conditions in terms of level set formulation.

### 3.1. Clebsch variables and level set formulations.

We recall some known facts about the classical Clebsch variables and its generalizations. We refer the readers to Hou and Li [15], Deng, Hou and Yu [9] and Graham and Henyey [11] for more information. For the 3D Euler equations in the vorticity form:

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \nabla u\cdot\omega, \\ \omega(0) = \omega_0. \end{cases}$$

the Lagrangian flow map $X(t, a)$ is defined as

$$\frac{d}{dt}X(t,a) = u(t, X(t,a)), \quad X(0,a) = a.$$

Since the flow is divergence-free, the Jacobian $\det(\nabla_a X) = 1$. Then, vorticity along the Lagrangian trajectory has the following analytical expression ([8]):

$$\omega(t, X(t,a)) = (\nabla_a X)\omega_0(a). \tag{29}$$

Now let $\theta(t,x)$ be the inverse map of $X(t,a)$, i.e. $X(t,\theta(t,x)) \equiv x$. Then, $\theta$ satisfies

$$\theta_t + (u \cdot \nabla)\theta = 0, \quad \theta(0,x) = x.$$

Denote $\theta = (\theta^1, \theta^2, \theta^3)$ and $\omega_0 = (\omega_0^1, \omega_0^2, \omega_0^3)$. Then (29) and the facts $(\nabla_a X)(\nabla_x \theta) = I$ and $\det(\nabla_x \theta) = 1$ imply that

$$\omega(t,x) = \omega_0^1(\theta)\nabla\theta_2 \times \nabla\theta_3 + \omega_0^2(\theta)\nabla\theta_3 \times \nabla\theta_1 + \omega_0^3(\theta)\nabla\theta_1 \times \nabla\theta_2.$$

Note that $\theta_j (j = 1,2,3)$ are level set functions convected by the flow velocity. In particular, if the initial vorticity can be written into the form $\omega(0,x) = \omega_0(\phi_0,\psi_0)\nabla\phi \times \psi_0$ and the level set functions $\phi$ and $\psi$ satisfy

$$\phi_t + (u \cdot \nabla)\phi = 0, \quad \phi(0,x) = \phi_0(x),$$
$$\psi_t + (u \cdot \nabla)\psi = 0, \quad \psi(0,x) = \psi_0(x),$$

then the vorticity at a later time can be expressed in terms of these two level set functions and their gradients

$$\omega(t,x) = \omega_0(\phi,\psi)\nabla\phi \times \nabla\psi.$$

In the case $\omega_0 = 1$, $\phi, \psi$ are known as the Clebsch variables. We review some properties of classical Clebsch variables.

- If $\omega = \nabla\phi \times \nabla\psi$ at one time $s$, then $\omega = \nabla\phi \times \psi$ for all $t > s$.
- If $\omega = \nabla\phi \times \nabla\psi$, then the helicity $\mathcal{H} \equiv \int u \cdot \omega = 0$.
- If $\omega = \nabla\phi \times \nabla\psi$ in a neighborhood of a point $x_0$ where the vorticity vanishes, then $\det[\nabla\omega(x_0)] = 0$.
- If $\omega \neq 0$ at some $x_0$, then $\omega = \nabla\phi \times \nabla\psi$ for some $\phi$ and $\psi$ in a neighborhood of $x_0$.

As pointed out in Hou-Li [15], the above Clebsch variables/level set formulation is a direct consequence of the Lagrangian structure of the flow and therefore also applies to the 3D Lagrangian averaged Euler equations. In this case, the level set functions satisfy

$$\phi_t + (u_\alpha \cdot \nabla)\phi = 0, \quad \phi(0,x) = \phi_0(x),$$
$$\psi_t + (u_\alpha \cdot \nabla)\psi = 0, \quad \psi(0,x) = \psi_0(x).$$

3.2. **Global existence conditions.** We establish the following new conditions for existence of global solutions.

**Theorem 3.1.** *Assume that the initial vorticity has the form*

$$\omega(0,x) = \omega_0(\phi_0,\psi_0)\nabla\phi_0 \times \nabla\psi_0$$

*with smooth and bounded $\omega_0$, $\phi_0$ and $\psi_0$. Then the Lagrangian averaged 3D Euler equations (3) have a unique smooth solution up to $T$ as long as, for each $j \in \{1,2,3\}$ there exists $i_j \in \{1,2,3\}$ such that one of the following two conditions is true:*
*(a)*

$$\int_0^T \sum_{j=1}^{3} \left\| \frac{\partial\psi}{\partial y_j} \right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} dt < \infty;$$

*(b)*

$$\int_0^T \sum_{j=1}^{3} \left\| \frac{\partial \phi}{\partial y_j} \right\|_{L_{y'}^{p_1} L_{y_{i_j}}^{q_1}(\mathbb{R}^3)} dt < \infty,$$

*for* $1 \le p_1, q_1 \le \infty$ *with*

$$1 - \frac{2}{p_1} - \frac{1}{q_1} \in [0, 1].$$

*Moreover, the following estimate holds*

$$\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \le C(T)\|\omega(0)\|_{F^s_{p,q}(\mathbb{R}^3)}, \quad 0 \le t \le T \tag{30}$$

*for* $s > \frac{3}{p}$ *with* $p \in [1, \infty]$, $q \in [1, \infty)$.

**Remark 1.** 1. Deng, Hou and Yu in [9] introduced the generalized Clebsch variables which are two triplets of real functions

$$\Phi = \{\phi_1, \phi_2, \phi_3\} \quad \text{and} \quad \mathbf{U} = \{U_1(\Phi), U_2(\Phi), U_3(\Phi)\},$$

such that the vorticity vector field $\omega$ can be represented in the following way

$$\omega = \sum_{k=1}^{3} \nabla U_k \times \nabla \phi_k. \tag{31}$$

Similar to Deng, Hou and Yu [9, Theorem 3.1], we can generalize Theorem 3.1 to the case where initial vorticity field is bounded and with compact support, and can be represented in the formula (31).

2. When $(p_1, q_1) = (\infty, 1)$ and $i_j = j$ for each $j \in \{1, 2, 3\}$, Theorem 3.1 covers Hou and Li's [15, Theorem 4]. Theorem 3.1 gives us more general blow-up criteria for more exponents. The region $A$ of these acceptable exponents are shown in the following figure.

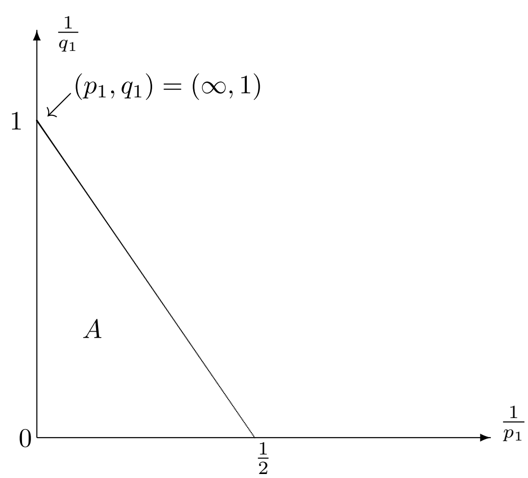


Figure 1: Region of acceptable exponents

Now, we give the proof of Theorem 3.1.

*Proof.* We only prove Theorem 3.1 under the assumption that that for each $j \in \{1,2,3\}$ there exists $i_j \in \{1,2,3\}$ such that

$$\int_0^T \sum_{j=1}^3 \left\| \frac{\partial \psi}{\partial y_j} \right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} dt < \infty.$$

We follow the proof of [15, Theorem 4]. We only need to consider the case $\omega_0 = 1$, that is, $\omega = \nabla\phi \times \nabla\psi$ for all times. We write $\omega = \nabla \times (\phi\nabla\psi)$. Like in [15], we define $B(y)$ be the integral kernel of the operator $(1-\alpha^2\triangle)^{-1}R$ in $\mathbb{R}^3$. Without loss of generality, we let $x = 0$ and omit the reference to time. Then

$$|\nabla u_\alpha(0)| = \left| \int_{\mathbb{R}^3} B(y)\omega(y)dy \right| = \left| \int_{\mathbb{R}^3} \nabla B(y) \times (\phi(y)\nabla\psi(y))dy \right|$$

with

$$|\nabla B(y)| \le \frac{C_\alpha}{|y|^2(1+|y|)}. \tag{32}$$

Estimate (32) can be deduced as follows. It follows from [17, p. 261-262] that the Green function associated with the operator $(1-\alpha^2\triangle)\triangle$ is $G_\alpha(|y|) = \frac{1-e^{-\frac{|y|}{\alpha}}}{4\pi|y|}$. Then,

$$u_\alpha(0) = \nabla \times \int_{\mathbb{R}^3} G_\alpha(|y|)\omega(y)dy = \int_{\mathbb{R}^3} f_\alpha(|y|)\frac{y}{|y|} \times \omega(y)dy, \tag{33}$$

where $f_\alpha(|y|) = \frac{1}{\alpha^2} f\left(\frac{|y|}{\alpha}\right)$ and $f(y) = \frac{(1+y)e^{-y}-1}{4\pi y^2}$ (also see [14] and [16]). Using (33), we can get

$$|\nabla B(y)| \le \frac{C_\alpha}{|y|^2(1+|y|)}$$

for $y \in \mathbb{R}^3$.

Let $q > 3$, $1 \le p_1, q_1 \le \infty$. In the following, $q'$ denote the dual index for $q$. Then, for $0 < \varepsilon < 1$, we can estimate $|\nabla u_\alpha(0)|$ in the following way:

$$\begin{aligned}
&|\nabla u_\alpha(0)| \\
=& \left| \int_{|y|<\varepsilon} + \int_{|y|\ge\varepsilon} \nabla B(y) \times (\phi(y)\nabla\psi(y))dy \right| \\
\le& C\|\phi\|_{L^\infty(\mathbb{R}^3)} \left( \varepsilon^{\frac{3-2q'}{q'}} \|\nabla\psi\|_{L^q(\mathbb{R}^3)} + \sum_{j=1}^3 \int_{|y|\ge\varepsilon} \frac{\left|\frac{\partial\psi}{\partial y_j}\right|}{|y|^2(1+|y|)} dy \right). \\
\le& C\|\phi\|_{L^\infty(\mathbb{R}^3)} \Bigg( \varepsilon^{\frac{3-2q'}{q'}} \|\nabla\psi\|_{L^q(\mathbb{R}^3)} \\
&+ \sum_{j=1}^3 \left( \int_{|y'|\ge\varepsilon/2} \left( \int_{|y_{i_j}|\ge\varepsilon/2} \left( \frac{1}{|y|^2(1+|y|)} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} dy' \right)^{\frac{1}{p_1'}} \left\| \frac{\partial\psi}{\partial y_j} \right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} \Bigg). \\
\le& C\|\phi\|_{L^\infty(\mathbb{R}^3)} \Bigg( \varepsilon^{\frac{3-2q'}{q'}} \|\nabla\psi\|_{L^q(\mathbb{R}^3)} \\
&+ \sum_{j=1}^3 \left( \int_{|y'|\ge\varepsilon/2} \left( \int_{|y_{i_j}|\ge\varepsilon/2} \left( \frac{1}{|y|^{2+c}} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} dy' \right)^{\frac{1}{p_1'}} \left\| \frac{\partial\psi}{\partial y_j} \right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} \Bigg),
\end{aligned}$$

for $c \in [0,1]$. In the previous inequalities, we used Hölder's inequality for mixed norms (see [1]) and the fact

$$\frac{1}{|y|^2(1+|y|)} \leq \frac{C}{|y|^{2+c}},$$

for $y \in \mathbb{R}^3$ and any $c \in [0,1]$. In fact, the cases $c=0$ and $c=1$ are obvious. For $c \in (0,1)$, we have

$$1+|y| \geq (1-c)1^{(1/c)'} + c(|y|^c)^{1/c} \geq |y|^c$$

according to Young's inequality.

Then, we have

$$|\nabla u_\alpha(0)| \leq C\|\phi\|_{L^\infty(\mathbb{R}^3)} \left( \varepsilon^{\frac{3-2q'}{q'}} \|\nabla\psi\|_{L^q(\mathbb{R}^3)} + \sum_{j=1}^{3} I_{i_j} \left\| \frac{\partial\psi}{\partial y_j} \right\|_{L^{p_1}_{y'} L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} \right),$$

for

$$\begin{aligned} I_{i_j} &:= \left( \int_{|y'|\geq\varepsilon/2} \left( \int_{|y_{i_j}|\geq\varepsilon/2} \left( \frac{1}{|y|^{2+c}} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} dy' \right)^{\frac{1}{p_1'}} \\ &= \left( \int_{|y'|\geq\varepsilon/2} \left( \int_{|y_{i_j}|\geq\varepsilon/2} \left( \frac{1}{(|y_{i_j}|^2+|y'|^2)^{\frac{2+c}{2}}} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} dy' \right)^{\frac{1}{p_1'}} . \end{aligned} \tag{34}$$

To estimate $I_{i_j}$, we first compute the inner integration as follows:

$$\begin{aligned} &\left( \int_{|y_{i_j}|\geq\varepsilon/2} \left( \frac{1}{(|y_{i_j}|^2+|y'|^2)^{\frac{2+c}{2}}} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} \\ = &\left( \int_{|y_{i_j}|\geq\varepsilon/2} |y'|^{-(2+c)q_1'} \left( \frac{1}{(|\frac{y_{i_j}}{y'}|^2+1)^{\frac{2+c}{2}}} \right)^{q_1'} dy_{i_j} \right)^{p_1'/q_1'} \\ = &\ |y'|^{-(2+c)p_1'} \left( \int_{|y_{i_j}|\geq\varepsilon/2} \frac{1}{\left(|\frac{y_{i_j}}{y'}|^2+1\right)^{\frac{(2+c)q_1'}{2}}} dy_{i_j} \right)^{p_1'/q_1'} \\ \leq &\ C|y'|^{-(2+c)p_1'} \left( \int_0^{\frac{\pi}{2}} \frac{|y'|\sec^2\theta}{(\tan^2\theta+1)^{\frac{(2+c)q_1'}{2}}} d\theta \right)^{p_1'/q_1'} \\ \leq &\ C|y'|^{-(2+c)p_1'+p_1'/q_1'} \left( \int_0^{\frac{\pi}{2}} (\sec\theta)^{2-2\frac{(2+c)q_1'}{2}} d\theta \right)^{p_1'/q_1'} \\ \leq &\ C|y'|^{-(2+c)p_1'+p_1'/q_1'} \left( \int_0^{\frac{\pi}{2}} (\cos\theta)^{(2+c)q_1'-2} d\theta \right)^{p_1'/q_1'} \\ \leq &\ C|y'|^{-(2+c)p_1'+p_1'/q_1'} \left( B\left( \frac{1}{2}, \frac{(2+c)q_1'-1}{2} \right) \right)^{p_1'/q_1'}, \end{aligned}$$

since $(2+c)q_1' - 2 > -1$ and $c \in [0,1]$, where

$$B\left(\frac{a+1}{2}, \frac{b+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^a (\cos\theta)^b d\theta, \quad a > -1, \quad b > -1,$$

is the classical Beta function. So

$$\begin{aligned}
I_{i_j} &= \left(\int_{|y'|\geq\varepsilon/2}\left(\int_{|y_{i_j}|\geq\varepsilon/2}\left(\frac{1}{(|y_{i_j}|^2+|y'|^2)^{\frac{2+c}{2}}}\right)^{q_1'} dy_{i_j}\right)^{p_1'/q_1'} dy'\right)^{\frac{1}{p_1'}} \\
&\leq C\left(\int_{|y'|\geq\varepsilon/2} |y'|^{-(2+c)p_1'+p_1'/q_1'}\left(B\left(\frac{1}{2}, \frac{(2+c)q_1'-1}{2}\right)\right)^{p_1'/q_1'} dy'\right)^{\frac{1}{p_1'}} \\
&\leq C\left(B\left(\frac{1}{2}, \frac{(2+c)q_1'-1}{2}\right)\right)^{1/q_1'}\left(\int_{\varepsilon/2}^{\infty} r^{1-(2+c)p_1'+p_1'/q_1'} dr\right)^{\frac{1}{p_1'}} \\
&\leq C\left(B\left(\frac{1}{2}, \frac{(2+c)q_1'-1}{2}\right)\right)^{1/q_1'}\left(\log\left(\frac{2}{\varepsilon}\right)\right)^{\frac{1}{p_1'}} \qquad (35)
\end{aligned}$$

if $c$ can be taken such that $1-(2+c)p_1' + p_1'/q_1' = -1$, which is equivalent to $c = 1 - \frac{2}{p_1} - \frac{1}{q_1}$. Consequently the existence of such $c \in [0,1]$ is the same as $1 - \frac{2}{p_1} - \frac{1}{q_1} \in [0,1]$ which is assumed to be true in the theorem.

Thus, we have

$$\begin{aligned}
&|\nabla u_\alpha(0)| \\
&\leq C\|\phi\|_{L^\infty(\mathbb{R}^3)}\left(\varepsilon^{\frac{3-2q'}{q'}}\|\nabla\psi\|_{L^q(\mathbb{R}^3)} + \left(\log\frac{2}{\varepsilon}\right)^{\frac{1}{p_1'}}\sum_{j=1}^{3}\left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}\right) \\
&\leq C\|\phi\|_{L^\infty(\mathbb{R}^3)}\left(2^{\frac{3-2q'}{q'}}\left(\frac{\varepsilon}{2}\right)^{\frac{3-2q'}{q'}}\|\nabla\psi\|_{L^q(\mathbb{R}^3)} + \left(\log\frac{2}{\varepsilon}\right)^{\frac{1}{p_1'}}\sum_{j=1}^{3}\left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}\right).
\end{aligned}$$

If we take $\varepsilon$ such that $\left(\frac{\varepsilon}{2}\right)^{\frac{3-2q'}{q'}}(e + \|\nabla\psi\|_{L^q(\mathbb{R}^3)}) = 1$, then

$$\begin{aligned}
\left(\log\frac{2}{\varepsilon}\right)^{\frac{1}{p_1'}} &= \left(\frac{q'}{3-2q'}\right)^{\frac{1}{q_1'}}\left[\log(e+\|\nabla\psi\|_{L^q(\mathbb{R}^3)})\right]^{\frac{1}{p_1'}} \\
&\leq \left(\frac{q'}{3-2q'}\right)^{\frac{1}{p_1'}}\log(e+\|\nabla\psi\|_{L^q(\mathbb{R}^3)})
\end{aligned}$$

since $\log(e+\|\nabla\psi\|_{L^q(\mathbb{R}^3)}) \geq 1$. Finally, we get

$$|\nabla u_\alpha(0)| \leq C\|\phi\|_{L^\infty(\mathbb{R}^3)}\left(2^{\frac{3-2q'}{q'}} + \sum_{j=1}^{3}\left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}\log\left(e+\|\nabla\psi\|_{L^q(\mathbb{R}^3)}\right)\right) \qquad (36)$$

for $1 \leq p_1, q_1 \leq \infty$ and $q > 3$. This immediately gives

$$\|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)} \leq C\left(2^{\frac{3-2q'}{q'}} + \sum_{j=1}^{3}\left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)}\log\left(e+\|\nabla\psi\|_{L^q(\mathbb{R}^3)}\right)\right)$$

for $1 \leq p_1, q_1 \leq \infty$ and $q > 3$, since $\|\phi\|_{L^\infty(\mathbb{R}^3)} \leq \|\phi_0\|_{L^\infty(\mathbb{R}^3)}$.

To close the estimates we return to the equation for $\psi$. Differentiating

$$\psi_t + u_\alpha \cdot \nabla\psi = 0$$

with respect to $x$, we get

$$(\nabla\psi)_t + (u_\alpha \cdot \nabla)(\nabla\psi) + \nabla u_\alpha \nabla\psi = 0. \tag{37}$$

Now applying standard energy estimate to (37), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\nabla\psi\|_{L^q(\mathbb{R}^3)} \\
\le&\|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)}\|\nabla\psi\|_{L^q(\mathbb{R}^3)} \\
\le&C\left(2^{\frac{3-2q'}{q'}} + \sum_{j=1}^{3}\left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} \log\left(e + \|\nabla\psi\|_{L^q(\mathbb{R}^3)}\right)\right)\|\nabla\psi\|_{L^q(\mathbb{R}^3)}.
\end{aligned}$$

If $\int_0^T \sum_{j=1}^3 \left\|\frac{\partial\psi}{\partial y_j}\right\|_{L^{p_1}_{y'}L^{q_1}_{y_{i_j}}(\mathbb{R}^3)} dt < \infty$, then the Gronwall inequality implies

$$\|\nabla\psi\|_{L^q(\mathbb{R}^3)} \le C(T)$$

which in turn gives us

$$\int_0^T \|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)}dt \le C\int_0^T \|\nabla\psi\|_{L^q(\mathbb{R}^3)}dt \le C(T).$$

The bound on $\int_0^T \|\nabla u_\alpha\|_{L^\infty(\mathbb{R}^3)}dt$ gives the $L^\infty$ bound on $\nabla\psi$ from (37). Similarly, we get the $L^\infty$ bound for $\nabla\phi$. Combining the $L^\infty$ estimates for $\nabla\psi$ and $\nabla\phi$, we have the $L^\infty$ bound for $\omega$. Now the energy estimate for $\omega$ in the $F^s_{p,q}(\mathbb{R}^3)$ can be proved by a standard argument and

$$\frac{d}{dt}\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)} \le C(\|\nabla u_\alpha(t)\|_{L^\infty(\mathbb{R}^3)} + \|\omega(t)\|_{L^\infty(\mathbb{R}^3)})\|\omega(t)\|_{F^s_{p,q}(\mathbb{R}^3)}).$$

□

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