Robust Matching for Teams

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Motivation

Commodity market

- Price that equate demand to supply
- **Choose what you want if you can afford**
- E.g. Grocery shopping

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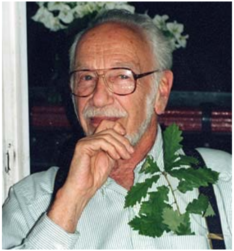
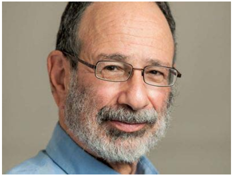
Lloyd Sharpley



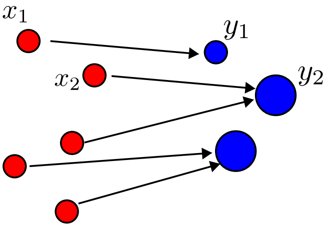
David Gale



Alvin Roth

Outline

Basic matching model



- Let $X = \{x_1, \ldots, x_k\}$ be the **set of types of consumers** and $Y = \{y_1, \ldots, y_m\}$ be the **set of types of producers**, where $|x_i| = a_i$ and $|y_j| = b_j$.

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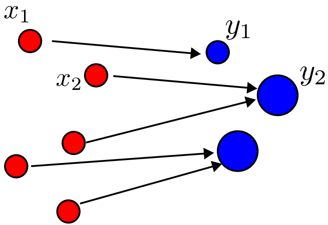

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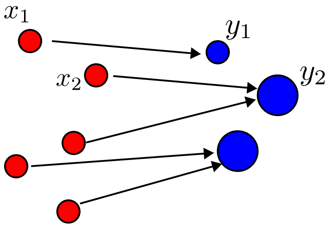

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- If consumer of type x_i trade with producer of type y_j, **they receive a joint satisfaction** $s(x_i, y_j)$.

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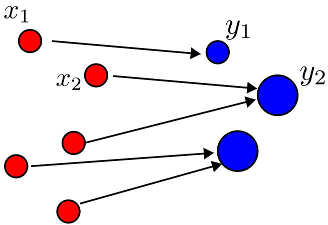

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- **Portion of the satisfaction to the consumer of type x_i will be $\phi(x_i)$** .

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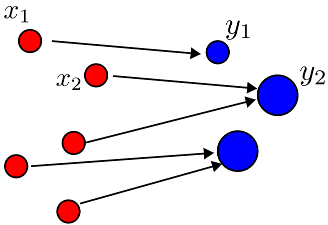

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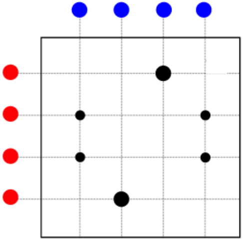
- **portion of the satisfaction to the producer of type y_j will be $\psi(y_j)$.**

More on basic matching model



- γ_{ij} number of consumers of type x_i that engaged in trade with producers of type y_j.

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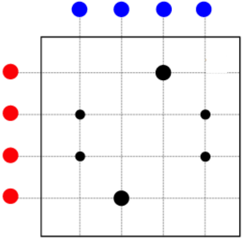

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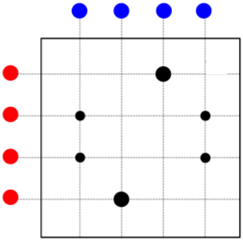

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- $a = (a_1, \ldots, a_k) \in \mathbb{R}_+^k$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}_+^m$, consider the matching set

$$\Pi(a,b) := \{\gamma \in \mathbb{R}_+^{k\times m} : \gamma \mathbb{1}_m = a \textbf{ and } \gamma^{\mathrm{T}} \mathbb{1}_k = b\}.$$

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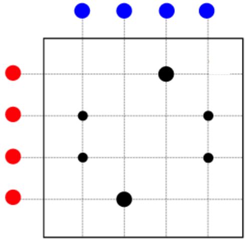

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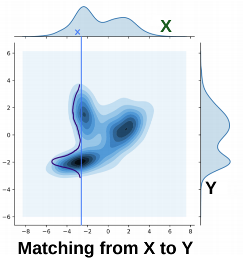
Definition (Stable matching (Discrete case))

A matching $\gamma \in \Pi(a,b)$ is stable **if there exist functions** $\phi(\cdot)$ **and** $\psi(\cdot)$ satisfies $\phi(x_i) + \psi(y_j) = s(x_i, y_j)$, whenever $\gamma_{ij} \neq 0$. We call $(\phi(\cdot), \psi(\cdot), \gamma)$ matching equilibrium.

The continuum case



- $X, Y \subset \mathbb{R}^d$ be the set of **continuum of consumers and producers**, respectively, distributed according to the given measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$.

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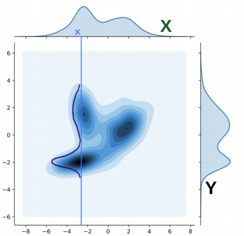

Matching from X to Y

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- $\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(X \times Y) : \gamma \circ \pi_X^{-1} = \mu \text{ and } \gamma \circ \pi_Y^{-1} = \nu\}$, where $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$ for all $(x, y) \in X \times Y$.

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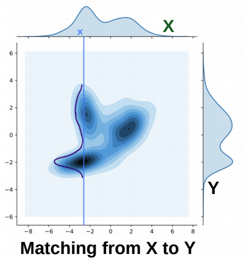

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- Given $s(\cdot, \cdot)$ and the measures μ and ν, **our aim is to find** $(\phi(\cdot), \psi(\cdot), \gamma)$ such that $\gamma \in \Pi(\mu, \nu)$ and

$$\phi(x)+\psi(y) = s(x, y), \quad \text{for all } (x, y) - \gamma \text{ a.e.}$$

Optimization problem for stable matching

Theorem (N. E. Gertsky, J. M. Ostroy, W. R. Zame 1992)

The problem of finding a stable matching can be recasted in linear programming (LP) terms:

- *Find* $\gamma \in \Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(X \times Y) : \gamma \circ \pi_X^{-1} = \mu$ *and* $\gamma \circ \pi_Y^{-1} = \nu\}$ *so as to achieve*

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This maximization problem is known in the literature as **Kantorovich optimal transport problem** and it admits a corresponding dual problem

$$D_s(\mu, \nu) := \inf_{(\phi,\psi) \in \Phi_s} \int_X \phi(x) d\mu + \int_Y \psi(y) d\nu,$$

where $\Phi_s := \{(\phi, \psi) : \phi(x) + \psi(y) \geq s(x, y)$ for all $(x, y) \in X \times Y\}$.

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Theorem (Fundamental theorem of optimal transport 1)

If $s(\cdot, \cdot)$ *is LSC, then* $P_s(\mu, \nu) = D_s(\mu, \nu)$ *and* $P_s(\mu, \nu)$ *admits a maximizer.*

More on optimization problem

Theorem (Fundamental theorem of optimal transport 2)

- *If* $s(\cdot,\cdot)$ **is continuous**, *then existence of minimizers for* $D_s(\mu,\nu)$ *holds;*

$$\max_{\gamma\in\Pi(\mu,\nu)}\int_{X\times Y} s(x,y)d\gamma = \min_{\phi\in s-\mathrm{conc}(X;\mathbb{R})}\int_X \phi(x)d\mu + \int_Y \phi^c(y)d\nu,$$

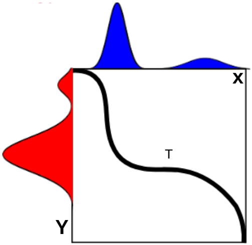
where $\phi^s(y) := \max_{x\in X} s(x,y) - \phi(x)$, *and* $y \in Y$,
$s - \mathrm{conc}(X;\mathbb{R}) := \{\phi : X \to \mathbb{R} : \exists v : Y \to \mathbb{R}$ *such that* $v^s = \phi\}$.

- *The optimal matching* γ *satisfies*

$$s(x,y) = \phi(x) + \phi^s(y) \quad \textit{for all } (x,y) - \gamma \textit{ a.e.}$$

- The discrete case corresponds to the discrete versions of the LP problem (Shapley and Shibik (1971)).

More on optimal transport theory



Matching concentrated on a graph of T

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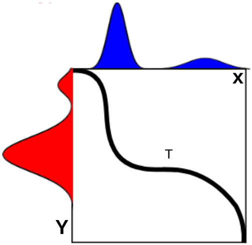


Matching concentrated on a graph of T

Theorem

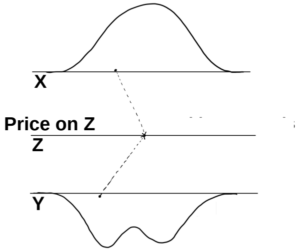
- *$\mu \in \mathcal{P}(X)$ is absolutely continuous with respect to Lebesgue.*
- *$c(\cdot, y)$ is differentiable on $\operatorname{int}(X)$, for all $y \in Y$ and satisfies if $(x, y_1, y_2) \in X \times Y^2$ and*

$$\nabla_x c(x, y_1) = \nabla_x c(x, y_2) \textit{ then } y_1 = y_2.$$

Then the optimal matching of the form $\gamma = (\mathrm{Id} \times T)_{\#}\mu$.

Outline

A type of matching model: Hedonic model



Matching X to Y through Z

Structure:

- X, Y and Z model **continuum of consumers, producers and goods**, where the consumers and the producers are distributed according to μ and ν, respectively.

Behavior: Given $p(\cdot)$,

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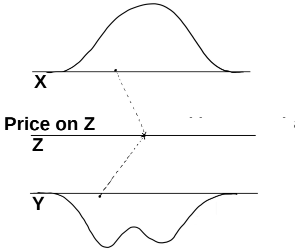

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Behavior: Given $p(\cdot)$,

- consumer of type $x \in X$ solves

$$U(x) = \max_{z \in Z}(u(x, z) - p(z)),$$

  where $u(\cdot, \cdot)$ is her direct utility function.

- producer of type $y \in Y$ solves

$$C(y) = \max_{z \in Z}(p(z) - c(y, z)),$$

  where $c(\cdot, \cdot)$ is his cost.

More on hedonic model

- Given $u(\cdot,\cdot)$ and $c(\cdot,\cdot)$ and $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, **we want to find a pair** $(p(\cdot), \alpha)$, where $\alpha \in \mathcal{P}(X \times Y \times Z)$ such that

$$\alpha \circ \pi_X^{-1} = \mu \quad \text{and} \quad \alpha \circ \pi_Y^{-1} = \nu$$

and $p(\cdot)$ is the price function such that

$$U(x) = u(x,z) - p(z) \quad \text{and} \quad C(y) = p(z) - c(y,z)$$

for all $(x,y,z) - \alpha$ a.e in $X \times Y \times Z$.

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- $p(\cdot)$ matches a consumer of type x to a producer of type y through their most preferred good $z \in Z$.
- The pair $(p(\cdot), \alpha)$ is called hedonic equilibrium.

Correspondence between hedonic and matching model

P.-A, Chiappori, R. J. McCann, and L.P. Nesheim (2010): There is a correspondence between the hedonic model and the matching model.

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$$\sup_{\gamma \in \Pi(\mu,\nu)} \int_{X \times Y} s(x,y) d\gamma$$

where $s(x,y) = \max_{z \in Z} u(x,z) - c(y,z)$ to obtain the payoff functions (ϕ, ψ).

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- There exists a price $p(\cdot)$ satisfying

$$\min_{y \in Y} c(x, z) - \psi(y) \geq p(z) \geq \max_{x \in X} u(x, z) - \phi(x).$$

- $(p(\cdot), \alpha)$, where $\alpha := (\mathrm{Id}_X \times \mathrm{Id}_Y \times z)_{\#}\gamma$, is a hedonic equilibrium pair.

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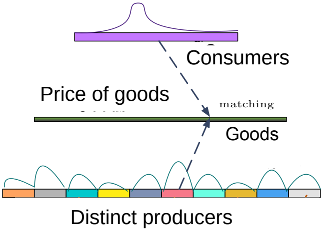
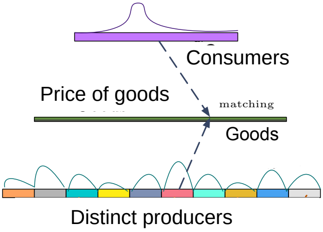
Matching and hedonic equilibriums are optimizers for optimal transport problems

Outline



- (X_0, μ_0) parametrize the **continuum of consumers**.
- (X_i, μ_i) parametrize the **continuum of producers**, where $i \in \{1, \ldots, N\}$.
- Z the set of all different types of a good in the market.



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Behavior of producers:

- Given wages $\psi_i(\cdot)$ producer of type $x_i \in X_i$ solves
$$\min_{z \in Z} c_i(x_i, z) - \psi_i(z),$$
where $c_i(\cdot, \cdot)$ is cost for producer in category $i \in \{1, \ldots, N\}$.
- Assume $p(z) = \sum_{i=1}^{N} \psi_i(z)$.

Matching for teams problem

Given $c_i(\cdot,\cdot)$ and $\mu_i \in \mathcal{P}(X_i)$, **our aim is to find** a family of functions $\psi_i \in C(Z;\mathbb{R})$, probability measures $\gamma_i \in \mathcal{P}(X_i \times Z)$, and $\nu \in \mathcal{P}(Z)$ such that

- $$\sum_{i=0}^{N} \psi_i(z) = 0, \text{ **for any** } z \in Z,$$

- $\gamma_i \in \Pi(\mu_i, \nu)$ such that

  $$c_i(x_i, z) = \psi_i(z) + \psi_i^{c_i}(x_i), \quad \text{for all } (x_i, z) - \gamma_i \text{ a.e.},$$

  where $\psi_i^{c_i}(x_i) := \inf_{z \in Z} c_i(x_i, z) - \psi_i(z)$, for all $x_i \in X_i$.

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- $\gamma_i \in \Pi(\mu_i, \nu)$ such that

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- $(\psi_i(\cdot), \nu, \gamma_i)$, where $i \in \{0, \ldots, N\}$, is called matching equilibrium.

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- Matching equilibrium is **deterministic** if $\gamma_i = (\mathrm{Id} \times T_i)_{\#}\mu_i$, where $T_i : X_i \to Z$ is a measurable map.

Optimization problem for matching for teams

Theorem (G. Carlier and I. Ekeland, 2010)

The problem of finding a matching equilibrium can be formulated as

- *find $\nu \in \mathcal{P}(Z)$ that solves the primal problem*

$$\mathrm{P} := \inf_{\nu \in \mathcal{P}(Z)} \sum_{i=0}^{N} W_{c_i}(\mu_i, \nu),$$

where γ_i **solves** $W_{c_i}(\mu_i, \nu) := \inf_{\gamma_i \in \Pi(\mu_i, \nu)} \int_{X_i \times Z} c_i(x_i, z) d\gamma_i$.

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- ψ_i*'s solves the dual problem*

$$\mathrm{P}^* := \sup \Big\{ \sum_{i=0}^{N} \int_{X_i} \psi_i^{c_i}(x_i) d\mu_i : \sum_{i=0}^{N} \psi_i(z) = 0, \quad \textit{for all } z \in Z \Big\}$$

where $\psi_i^{c_i}(x_i) := \inf_{z \in Z} c_i(x_i, z) - \psi_i(z)$, *for all* $x_i \in X_i$.

Matching for team: main result

Theorem (G. Carlier and I. Ekeland, 2010)

If $c_i(\cdot,\cdot)$ *is LSC, then* $\mathrm{P} = \mathrm{P}^*$ *and minimizers for* P *exists.*

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- *If* $\mu_i \in \mathcal{P}(X_i)$ *is* **absolutely continuous with respect to Lebesgue** *and* $c_i(\cdot, z)$ *is differentiable on* $\mathrm{int}(X_i)$, *for all* $z \in Z$ *and satisfies if* $(x_i, z_1, z_2) \in X \times Z^2$ *and*

$$\nabla_{x_i} c_i(x_i, z_1) = \nabla_{x_i} c_i(x_i, z_2) \text{ then } z_1 = z_2.$$

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  then, matching equilibrium is uniquely deterministic.

Outline

Literature review

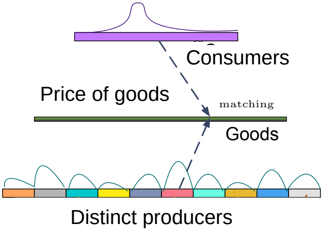
Matching for teams, without uncertainty, has been investigated extensively. We only provide a few related references here

- G. Carlier and I. Ekeland, 2010, Matching for teams, J. Economic Theory.
- I. Ekeland, 2005, An optimal matching problem, J. ESAIM: Control, Optimisation and Calculus of Variations.
- P. A. Chiappori, 2017, Matching with transfers, J. Princeton University Press.
- **B . Pass**, 2012, Multi-marginal optimal transport and multi-agent matching problems: uniqueness and structure of solutions, arXiv.
- P. A, Chiappori, R. J. McCann, and **B . Pass**, 2016, Multidimensional matching, arXiv.

Formulation of robust matching for teams problem

For the producer part:

- **Uncertainty in cost of production**



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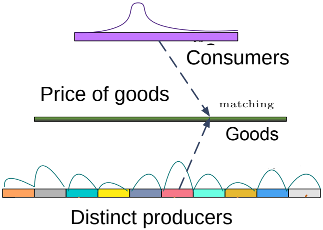

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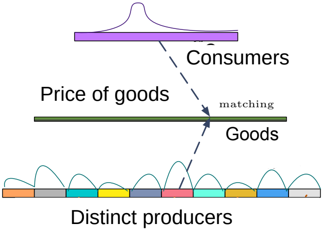

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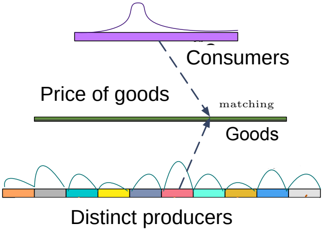

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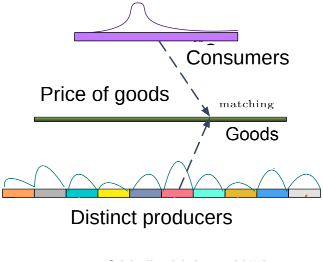

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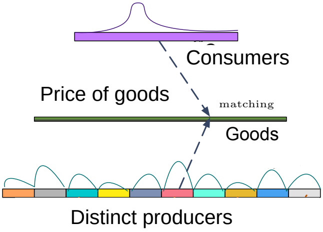

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- we require

$$\sum_{i=0}^{N} \psi_i(z) = 0.$$

Robust matching

Proposition (D. O. Adu and B. Gharesifard (2022))

- *If* $\gamma_i \in \Pi(\mu_i, \nu)$ *in spite of the uncertainty in variable cost, then* $\gamma_i|_{\mathcal{W}_i} = 0$**, that is for all** $\omega_i \in \mathcal{W}_i$ **we have that** $\int_{X_i \times Z} \omega_i(x_i, z) d\gamma_i = 0$.

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- Existence of robust matching $\Rightarrow \nu \in \mathcal{P}(Z)$ and $\gamma_i \in \Pi_{\mathcal{W}_i}(\mu_i, \nu)$, for all $i \in \{0, \ldots, N\}$.

Robust matching equilibrium

Given $c_i(\cdot,\cdot)$, $\mathcal{W}_i$ and $\mu_i \in \mathcal{P}(X_i)$, **our aim is to find** a family of functions $\psi_i \in C(Z;\mathbb{R})$, probability measures $\gamma_i \in \mathcal{P}(X_i \times Z)$ and $\nu \in \mathcal{P}(Z)$ such that

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where $\psi_i^{(c_i+\omega_i)}(x_i) := \min_{z\in Z} \big(c_i(x_i,z) + \omega_i(x_i,z) - \psi_i(z)\big)$, for some $\omega_i \in \mathcal{W}_i$.

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- We call $(\psi_i(\cdot), \gamma_i, \nu)$, where $i \in \{0, \ldots, N\}$, a Robust matching equilibrium (RME).

Beyond classical matching and hedonic model

- D. A. Zaev (2015): Studied Kantorovich problem with additional linear constraint.

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Problem statement:

- Given $c(\cdot,\cdot) \in C(X \times Z; \mathbb{R})$, $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Z)$ and a subspace $\mathcal{W} \subset C(X \times Z; \mathbb{R})$

$$\mathrm{K}_{c,\mathcal{W}}(\mu,\nu) := \inf_{\gamma \in \Pi_{\mathcal{W}}(\mu,\nu)} \int_{X \times Z} c(x,z) d\gamma.$$

- The dual problem is

$$D_{c,\mathcal{W}}(\mu,\nu) := \sup_{\phi+\psi+\omega \leq c} \int_X \phi(x) d\mu + \int_Z \psi(z) d\nu.$$

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Theorem (D. A. Zaev (2015))

- *We have that $\mathrm{K}_{c,\mathcal{W}}(\mu,\nu) = D_{c,\mathcal{W}}(\mu,\nu)$ and existence of $\mathrm{K}_{c,\mathcal{W}}(\mu,\nu)$ holds if and only if $\Pi_{\mathcal{W}}(\mu,\nu) \neq \emptyset$. In general existence of solution for $D_{c,\mathcal{W}}(\mu,\nu)$ may fail.*

Martingale optimal transport on a line

Martingale matchings: $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Z)$

$$\mathcal{M}(\mu, \nu) := \{\gamma \in \Pi(\mu, \nu) : \mathbb{E}_\gamma[\pi_Z | \pi_X] = \pi_X\},$$

where $\pi_X(x, z) = x$ and $\pi_Z(x, z) = z$, for all $(x, z) \in X \times Z$.

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i.e. $\mathbb{E}_\gamma[\pi_Z | \pi_X] = \pi_X \iff \int_{X \times Z} h(x)(z - x) d\gamma = 0$**, for all** $h \in C(X; \mathbb{R})$.

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- V. Strassen (1965) : $\mathcal{M}(\mu,\nu) \neq \emptyset \iff \mu \preceq_c \nu$:

$$\int_X f(x)d\mu \leq \int_Z f(z)d\nu, \quad \text{for all convex functions } f(\cdot) \text{ over } \mathbb{R}.$$

More on martingale optimal transport

Problem statement: $\mathrm{P}_c(\mu, \nu) := \inf_{\gamma \in \mathcal{M}(\mu,\nu)} \int_{X \times Z} c(x, z) d\gamma.$

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Theorem (M. Beiglböck and P. Henry-Labordere and F. Penkner (2013))

If $c(\cdot,\cdot)$ LSC and bounded below and $\mu \preceq_c \nu$, *then* $\mathrm{P}_c(\mu,\nu)$ *admits a minimizer and* $\mathrm{P}_c(\mu,\nu) = \mathrm{D}_c(\mu,\nu)$.

- There exist examples where maximizer for $\mathrm{D}_c(\mu,\nu)$ may fail.

More on martingale optimal transport

Theorem (M. Beiglböck, T. Lim and J. Obloj (2019))

If $\mu \preceq_c \nu$, then existence for $\mathrm{D}_c(\mu, \nu)$ holds when $c(\cdot, \cdot)$ **is Lipschitz and there exists** $u(\cdot)$ **Lipschitz function over** Z **such that** $c(x, \cdot) - u(\cdot)$ **is convex over** Z.

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Theorem (M. Beiglböck and N. Juillet (2016))

- $\mu \in \mathcal{P}(X)$ *is absolutely continuous with respect to Lebesgue measure*
- $c(x, z) = q(x - z)$*, where* $q(\cdot)$ *is differentiable whose derivative is strictly convex. There exists* $S \subset X$ *such that* $\gamma(\mathrm{Graph}(T_1) \cup \mathrm{Graph}(T_2)) = 1$ *on* S.

Robust matching for team: Main result

Theorem (D. O. Adu and B. Gharesifard (2022))

- *Let* $\mu_i \in \mathcal{P}(X_i)$*, and* $c_i(\cdot,\cdot) \in C(X_i \times Z; \mathbb{R})$*, and* $\mathcal{W}_i$ *be such that*

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where $\mu := (\mu_0, \dots, \mu_N) \in \mathcal{P}(X_0) \times \cdots \times \mathcal{P}(X_N)$ *and* $\mathcal{W} := \mathcal{W}_0 \times \cdots \times \mathcal{W}_N$*, is non-empty.*

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- *Given* $\psi_i(\cdot) \in C(Z; \mathbb{R})$ *the problem*

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  admits a maximizer for $i \in \{0, \ldots, N\}$. *Then there exists an RME.*

Special case for our main result

- Consider the set

$$\mathcal{W}_i := \{\omega_i \in \mathcal{F}(X_i \times Z; \mathbb{R}) : \omega_i(x_i, z) := h_i(x_i)(z - x_i), \text{ where } h_i \in C(X_i; \mathbb{R})\}.$$

- Then

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Special case for our main result

- Consider the set

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- Uncertainty in $\omega_i(\cdot, \cdot)$ caused by exogenous factors (prices of fuel, oil, natural gas, hydro etc.) **independent of** $z \in Z$.
- Given $\nu \in \mathcal{P}(Z)$, the robust matching $\gamma_i \in \Pi_{\mathcal{W}_i}(\mu_i, \nu)$ is a martingale.

Optimization Problems for RME

Theorem (D. O. Adu and B. Gharesifard (2022))

The problem of finding an RME can be recasted as

- *finding* $\nu \in \mathcal{M}_{\mathcal{W}}(\mu) := \{\nu \in \mathcal{P}(Z) : \Pi_{\mathcal{W}_i}(\mu_i, \nu) \neq \emptyset, \quad \text{for all } i \in \{0, \dots, N\}\}$ *that*

$$\mathrm{P}_{\mathcal{W}}(\mu) := \inf_{\rho \in \mathcal{M}_{\mathcal{W}}(\mu)} \sum_{i=0}^{N} \mathrm{K}_{c_i, \mathcal{W}_i}(\mu_i, \rho),$$

where $\mathrm{K}_{c_i, \mathcal{W}_i}(\mu_i, \rho) := \inf_{\gamma_i \in \Pi_{\mathcal{W}_i}(\mu_i, \rho)} \int_{X_i \times Z} c_i(x_i, z) d\gamma_i,$

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- *and finding* $\psi_i(\cdot)$ *and* $\omega_i(\cdot, \cdot)$, *where* $i \in \{0, \ldots, N\}$ *that solves*

$$\mathrm{P}^*_{\mathcal{W}}(\mu) := \sup_{(\omega_0, \ldots, \omega_N) \in \mathcal{W}} \sup_{(\varphi_0, \ldots, \varphi_N) \in \mathcal{T}} \sum_{i=0}^{N} \int_{X_i} \varphi_i^{(c_i + \omega_i)}(x_i) d\mu_i,$$

  where $\mathcal{T} = \{(\varphi_0(\cdot), \ldots, \varphi_N(\cdot)) \subset C(Z; \mathbb{R}) \mid \sum_{i=0}^{N} \varphi_i(z) = 0, \textit{ for all } z \in Z\}.$

RME are optimizers

Theorem (D. O. Adu and B. Gharesifard (2022))

- *If* $c_i(\cdot,\cdot)$ *is LSC and* $\mathcal{W}$ *is such that* $\mathcal{M}_{\mathcal{W}}(\mu) \neq \emptyset$, *then* $\mathrm{P}_{\mathcal{W}}(\mu) = \mathrm{P}^*_{\mathcal{W}}(\mu)$ *and the minimizer for* $\mathrm{P}_{\mathcal{W}}(\mu)$ *exists.*

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Proposition (D. O. Adu and B. Gharesifard (2022))

We have that $(\psi_i(\cdot), \gamma_i, \nu)$ *is an RME for* $i \in \{0, \dots, N\}$*, if and only if* ν *solves* $\mathrm{P}_{\mathcal{W}}(\mu)$ *and* $(\psi_0(\cdot), \dots, \psi_N(\cdot))$ *and* $(\omega_0^*(\cdot), \dots, \omega_N^*(\cdot))$*, with* $\omega_i^* \in \mathcal{W}_i$*, solves* $\mathrm{P}^*_{\mathcal{W}}(\mu)$.

Special case: Martingale matching for teams

Theorem (D. O. Adu and B. Gharesifard (2022))

Assume $c_i(\cdot,\cdot)$ *is Lipschitz on* $X_i \times Z$ *and there exists a Lipschitz function* $u_i(\cdot)$ *over* Z *such that* $c_i(x_i,\cdot) - u_i(\cdot)$ *is convex over* Z*. Then there exists an RME* $(\psi_i(\cdot), \gamma_i, \nu)$*, for all* $i \in \{0, \ldots, N\}$*.*

Idea:

- Solve $\mathrm{P}_{\mathcal{W}}(\mu)$ to obtain $\nu \in \mathcal{M}_{\mathcal{W}}(\mu)$ such that $\gamma_i \in \Pi_{\mathcal{W}_i}(\mu_i, \nu)$

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Idea:

- Solve $\mathrm{P}_{\mathcal{W}}(\mu)$ to obtain $\nu \in \mathcal{M}_{\mathcal{W}}(\mu)$ such that $\gamma_i \in \Pi_{\mathcal{W}_i}(\mu_i,\nu)$
- For $\mathrm{P}^*_{\mathcal{W}}(\mu)$, solve

$$\sup_{(\omega_1,\ldots,\omega_N)\in\mathcal{W}} \sup_{(\varphi_1,\ldots,\varphi_N)\in\mathcal{T}} \sum_{i=1}^{N} \int_{X_i} \varphi_i^{(c_i+\omega_i)}(x_i)d\mu_i + \int_Z \varphi_i(z)d\nu$$

  to obtain $(\psi_1,\ldots,\psi_N)$ and then set

- $\psi_0(z) = -\sum_{i=1}^{N} \psi_i(z)$, for all $z \in Z$.

Some comments on purity

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Theorem (D. O. Adu and B. Gharesifard (2022))

- $(\psi_i(\cdot), \gamma_i, \nu)$, *for* $i \in \{0, \ldots, N\}$ *be an RME.*
- $\mu_i \in \mathcal{P}(X_i)$ *is absolutely continuous with respect to Lebesgue measure*
- $c_i(x_i, z) = q_i(x_i - z)$, *where* $q_i(\cdot)$ *is a differentiable whose derivative is strictly convex.*
- *There exists* $S_i \subset X_i$ *such that* $\gamma_i(\mathrm{Graph}(T_{i1}) \cup \mathrm{Graph}(T_{i2})) = 1$ *on* S_i.

Outline

Concluding Remarks

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Future work:

- Matching problems with capacity constraints.
- Matching problems with coordination among individuals.