NAME: ______________________

QUIZ 3

Problem 1. Consider the Hamilton-Jacobi equation in 1D

$$u_t + \frac{u_x^2}{2} = 0, \qquad u(x,0) = g(x) = \begin{cases} -x & x<0 \\ x & x>0 \end{cases}. \tag{1}$$

a) **(9 pts)** Solve the problem using Hopf-Lax formula.

b) **(1 pts)** For a fixed $t>0$, plot the graph of your solution.

Solution. Using the Hopf-Lax formula we have (since $H(p) = p^2/2, L(q) = q^2/2$)

$$u(x,t) = \min_{y\in\mathbb{R}} f(y). \tag{2}$$

where

$$f(y) := \begin{cases} \frac{(x-y)^2}{2t} - y & y<0 \\ \frac{(x-y)^2}{2t} + y & y>0 \end{cases}$$

Thus all we need to do is to find

$$\min_{y<0} \frac{(x-y)^2}{2t} - y, \qquad \min_{y>0} \frac{(x-y)^2}{2t} + y \tag{3}$$

and compare. Note that both functions are quadratic functions in y and tends to $+\infty$ as $y \to \pm\infty$. Therefore the minimums are either the global minimum (that is, forget about the constraints $y < (>) 0$) or at 0.

First consider the $y<0$ case. We compute

$$0 = \frac{\partial}{\partial y} f(y) = \frac{y-x}{t} - 1 \Longrightarrow y_{\min} = x+t. \tag{4}$$

Thus when $x < -t$, $y_{\min} < 0$ and the minimum is

$$-x - \frac{t}{2}; \tag{5}$$

When $x > -t$, $y_{\min} > 0$ so the minimum over $y<0$ occurs at $y=0$, which gives

$$\frac{x^2}{2t}. \tag{6}$$

Summarizing,

$$\min_{y<0} \frac{(x-y)^2}{2t} - y = \begin{cases} -x - t/2 & x<-t \\ x^2/2t & x>-t \end{cases}. \tag{7}$$

Similarly we have

$$\min_{y>0} \frac{(x-y)^2}{2t} + y = \begin{cases} x - t/2 & x>t \\ x^2/2t & x<t \end{cases}. \tag{8}$$

Thus

$$u(x,t) = \begin{cases} \min\left\{ -x - \frac{t}{2}, \frac{x^2}{2t} \right\} & x<-t \\ \frac{x^2}{2t} & -t<x<t \\ \min\left\{ x - \frac{t}{2}, \frac{x^2}{2t} \right\} & x>t \end{cases}. \tag{9}$$

it is easy to see that

$$u(x,t) = \begin{cases} -x - \frac{t}{2} & x<-t \\ \frac{x^2}{2t} & -t<x<t \\ x - \frac{t}{2} & x>t \end{cases}. \tag{10}$$