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Math 527 A1 Fall 2010

NAME: ______________________

Quiz 1

Problem 1. Let u be harmonic in the n-dimensional ball B_r and continuous on $\bar{B}_r$. Denote the fundamental solution in $\mathbb{R}^n$ by Φ, that is

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases}. \tag{1}$$

From

$$u(0) = \int_{B_r} \delta(x)\, u(x)\, \mathrm{d}x = -\int_{B_r} (\triangle \Phi)\, u \tag{2}$$

derive the mean value formula

$$u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, \mathrm{d}S_y. \tag{3}$$

Solution. We use Gauss' formula:

$$\begin{aligned} u(0) &= -\int_{B_r} (\triangle \Phi)\, u \\ &= -\int_{B_r} \nabla \cdot [u \nabla \Phi] + \int_{B_r} \nabla u \cdot \nabla \Phi \\ &= -\int_{\partial B_r} u \frac{\partial \Phi}{\partial n} + \int_{B_r} \nabla \cdot [\Phi \nabla u] - \int_{B_r} \Phi \triangle u \\ &= -\int_{\partial B_r} u \frac{\partial \Phi}{\partial n} + \int_{\partial B_r} \Phi \frac{\partial u}{\partial n} - 0. \end{aligned} \tag{4}$$

Now the proof ends through the following two observations:

- $\Phi = \Phi(r)$ along ∂B_r, therefore

$$\int_{\partial B_r} \Phi \frac{\partial u}{\partial n} = \Phi(r) \int_{\partial B_r} \frac{\partial u}{\partial n} = \Phi(r) \int \triangle u = 0. \tag{5}$$

- Using the formulas for Φ, we easily obtain

$$\frac{\partial \Phi}{\partial n} |_{\partial B_r} = -\frac{1}{|\partial B_r|}. \tag{6}$$