Math 527 A1 Homework 6 (Due Dec. 8 in Class)

Exercise 1. (10 pts) (5.10.9) Integrate by parts to prove the interpolation inequality

$$\int_U |Du|^2 \,\mathrm{d}x \leqslant C \left(\int_U u^2 \,\mathrm{d}x \right)^{1/2} \left(\int_U |D^2 u|^2 \,\mathrm{d}x \right)^{1/2} \tag{1}$$

for all $u \in C_c^\infty(U)$. Assume ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$. (Hint: Take $\{v_k\} \subset C_c^\infty$ converging to u in $H_0^1(U)$, and $\{w_k\} \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$.)

Proof. It is clear that the inequality holds for $u \in C_c^\infty(U)$. Now pick

$$v_k \in C_c^\infty \longrightarrow u \text{ in } H_0^1; \qquad w_k \in C^\infty(\bar{U}) \longrightarrow u \text{ in } H^2. \tag{2}$$

Note that we cannot use one sequence as if $v_k \in C_c^\infty \longrightarrow u$ in H^2, u would be in H_0^2.

Now compute

$$\left| \int D v_k \cdot D w_k \right| = \left| \int v_k \,\triangle w_k \right| \leqslant \left(\int v_k^2 \right)^{1/2} \left(\int (\triangle w_k)^2 \right)^{1/2} \leqslant \left(\int v_k^2 \right)^{1/2} \left(\int |D w_k|^2 \right)^{1/2}. \tag{3}$$

Letting $k \nearrow \infty$ finishes the proof. □

Exercise 2. (10 pts) (6.6.2)

Let

$$L u = - \sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} + c\, u \tag{4}$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\,,\,]$ satisfies the hypothesis of the Lax – Milgram theorem, provided

$$c(x) \geqslant -\mu \qquad (x \in U). \tag{5}$$

Proof. The space is $H_0^1(U)$. Recall that the Lax-Milgram theorem has two conditions, boundedness and coerciveness. Boundedness follows immediately from the boundedness of the coefficients.

For the coerciveness, we compute

$$B[u, u] = \int \sum a^{ij} u_{x_i} u_{x_j} + c\, u^2 \geqslant \theta \int |Du|^2 + \int c\, u^2 \geqslant \frac{\theta}{2} \int |Du|^2 + \frac{\theta}{2k} \int u^2 + \int c\, u^2 = \frac{\theta}{2} \int |Du|^2 + \int \left(\frac{\theta}{2k} + c \right) u^2. \tag{6}$$

Here the last step is due to Poincaré inequality. We see that the conclusion follows. □

Exercise 3. (10 pts) (6.6.8)

Let u be a smooth solution of $L u = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = 0$ in U. Set $v := |Du|^2 + \lambda u^2$. Show that

$$L v \leqslant 0 \quad \text{in } U, \text{ if } \lambda \text{ is large enough.} \tag{7}$$

Deduce

$$\|Du\|_{L^\infty(U)} \leqslant C \left(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)} \right). \tag{8}$$

Proof. As $L u = 0$, u satisfies the maximum principle. Therefore as soon as we have shown $L v \leqslant 0$, the conclusion follows. We compute

$$\begin{aligned}
L v &= -\sum a^{ij} (Du \cdot Du + \lambda u^2)_{x_i x_j} \\
&= -2 \sum a^{ij} \left[D u_{x_i x_j} \cdot Du + D u_{x_i} \cdot D u_{x_j} + \lambda\, u\, u_{x_i x_j} + \lambda\, u_{x_i} u_{x_j} \right] \\
&= -2 \left[D\left(\sum a^{ij} u_{x_i x_j} \right) \cdot Du - \sum (D a^{ij})\, u_{x_i x_j} \cdot Du + \sum a^{ij} D u_{x_i} \cdot D u_{x_j} + \lambda\, u \sum a^{ij} u_{x_i x_j} + \lambda \sum a^{ij} u_{x_i} u_{x_j} \right] \\
&\quad (\text{Using } L u = 0) \\
&\leqslant -2 \sum (D a^{ij})\, u_{x_i x_j} \cdot Du - \theta \left| D^2 u \right|^2 - \lambda\, \theta\, |Du|^2.
\end{aligned} \tag{9}$$

Now it is clear that $L v \leqslant 0$ when λ is large enough (assuming $D a^{ij} \in L^\infty$) □