Math 527 A1 Homework 6 (Due Dec. 8 in Class)

Exercise 1. (10 pts) (5.10.9) Integrate by parts to prove the interpolation inequality

$$\int_U |Du|^2 \,\mathrm{d}x \leqslant C \left(\int_U u^2 \,\mathrm{d}x \right)^{1/2} \left(\int_U |D^2 u|^2 \,\mathrm{d}x \right)^{1/2} \tag{1}$$

for all $u \in C_c^\infty(U)$. Assume ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$. (Hint: Take $\{v_k\} \subset C_c^\infty$ converging to u in $H_0^1(U)$, and $\{w_k\} \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$.)

Exercise 2. (10 pts) (6.6.2)

Let

$$L u = - \sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + c u \tag{2}$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\,,\,]$ satisfies the hypothesis of the Lax – Milgram theorem, provided

$$c(x) \geqslant -\mu \qquad (x \in U). \tag{3}$$

Exercise 3. (10 pts) (6.6.8)

Let u be a smooth solution of $L u = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = 0$ in U. Set $v := |Du|^2 + \lambda u^2$. Show that

$$L v \leqslant 0 \quad \text{in } U, \text{ if } \lambda \text{ is large enough.} \tag{4}$$

Deduce

$$\|Du\|_{L^\infty(U)} \leqslant C \left(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)} \right). \tag{5}$$