Math 527 A1 Homework 5 (Due Nov. 26 in Class)

Exercise 1. (15 pts) (Evans 4.7.7) Consider the viscous conservation law

$$u_t + F(u)_x - a\, u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) \tag{1}$$

where $a > 0$ and F is uniformly convex.

i. **(5 pts)** Show u solves (1) if $u(x, t) = v(x - \sigma\, t)$ and v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma\, z + b} \mathrm{d}z \qquad (s \in \mathbb{R}), \tag{2}$$

where b and c are constants.

ii. **(5 pts)** Demonstrate that we can find a traveling wave satisfying

$$\lim_{s \to -\infty} v(s) = u_l, \qquad \lim_{s \to \infty} v(s) = u_r \tag{3}$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}. \tag{4}$$

iii. **(5 pts)** Let u^ε denote the above traveling wave solution of (1) for $a = \varepsilon$, with $u^\varepsilon(0,\ 0) = \frac{u_l + u_r}{2}$. Compute $\lim_{\varepsilon \to 0} u^\varepsilon$ and explain your answer.

Proof.

i. Set $u = v(x - \sigma\, t)$, the equation becomes

$$-\sigma\, v' + F(v)' - a\, v'' = 0. \tag{5}$$

Integrating, we have

$$-\sigma\, v + F(v) - a\, v' = -b. \tag{6}$$

This gives

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{F(v) - \sigma\, v + b}{a} \Longrightarrow \frac{\mathrm{d}s}{\mathrm{d}v} = \frac{a}{F(v) - \sigma\, v + b}. \tag{7}$$

ii. "Only if". Let $s \to \pm\infty$, naturally we require $v' \to 0$. thus

$$-\sigma\, u_l + F(u_l) = -b, \qquad -\sigma\, u_r + F(u_r) = -b. \tag{8}$$

The conclusion then follows.

"If". It is clear that for $\lim v = u_{l,r}$ as $s \longrightarrow \mp\infty$ to be possible, F has to have two roots. This is possible when $\sigma = \frac{u_l + u_r}{2}$. Since F is uniformly convex, we have $F > 0$ for $v < u_r$ and $v > u_l$, $F < 0$ for $v \in (u_r, u_l)$. Now let $s \nearrow +\infty$

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma\, z + b} \mathrm{d}z \qquad (s \in \mathbb{R}), \tag{9}$$

using argument similar to that in iii, we see that v takes limit u_r when $c \in (u_r,\ u_l)$. Similar argument works for $s \longrightarrow -\infty$.

iii. First note that, $v(s)$ cannot "cross" u_l or u_r. In other words, either $v(s) \geqslant u_l$, or $v(s) \leqslant u_r$, or $v(s) \in [u_r,\ u_l]$. To see this, assume the contrary. Wlog assume v has values above and below u_l. Then as $v \to u_l$ as $s \to -\infty$, there is s_0 such that v reaches maximum $v_{\max} > u_l$. At this point, we have $v' = 0$ and therefore $v_{\max}$ solves

$$-\sigma\, v + F(v) = -b. \tag{10}$$

But as $-\sigma\, v + F(v)$ is uniformly convex (thus strictly convex), there can be at most two solutions. As u_l and u_r already solve it, we obtain contradiction.[1]

Next we show that, in fact for all s finite, $v(s) \in (u_r, u_l)$. To see this, study the formula

$$s = \int_c^{v(s)} \frac{\varepsilon}{F(z) - \sigma\, z + b} \mathrm{d}z. \tag{11}$$

1. From this it is clearly see that in the condition $u^\varepsilon(0, 0) = \frac{u_l + u_r}{2}$, the RHS can be replaced by any value in (u_r, u_l), as the purpose is just to restrict all v in $[u_l, u_r]$.

As F is strictly convex, the behavior of the denominator close to u_l and u_r is like $(z - z_0)^{-1}$. And thus if $v(s) = u_l$ or u_r, necessarily $s = \mp\infty$.

Finally, fix any $s \neq 0$ finite, we have

$$\frac{s}{\varepsilon} = \int_c^{v(s)} \frac{1}{F(z) - \sigma z + b} \, \mathrm{d}z. \tag{12}$$

When $\varepsilon \searrow 0$, the LHS $\to \mp\infty$, consequently $v(s) \to u_l$ or u_r. Thus we see that as $\varepsilon \searrow 0$, v^ε converges to v at every $s \neq 0$. □

Exercise 2. (5 pts) (5.10.3) Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & x_1 > 0, |x_2| < x_1 \\ 1 + x_1 & x_1 < 0, |x_2| < -x_1 \\ 1 - x_2 & x_2 > 0, |x_1| < x_2 \\ 1 + x_2 & x_2 < 0, |x_1| < -x_2 \end{cases}. \tag{13}$$

For which $1 \leqslant p \leqslant \infty$ does u belong to $W^{1,p}(U)$?

Solution. First we can easily check that $u \in C(\bar{U})$ and is smooth inside each triangle. If we define $\boldsymbol{v}$ piecewisely such that $\boldsymbol{v} = Du$ in each triangle, it is clear that $\boldsymbol{v} \in W^{1,p}$ for any p. Now what is left to show is that $\boldsymbol{v} = Du$ in U.

Take any $\phi \in C_0^1(U)$. Denote by U_i, $i = 1, \ldots, 4$, the triangles. We have

$$\int_U \boldsymbol{v}\,\phi = \sum_i \int_{U_i} Du\,\phi = -\sum_i \int u\,D\phi + \sum_i \int_{\partial U_i} \boldsymbol{n}_i\,u\,\phi. \tag{14}$$

As u is continuous across any common boundary of any two U_i's, and $\phi = 0$ on ∂U, the boundary terms sum up to 0.

Exercise 3. (10 pts) (5.10.14) Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

Proof. Compute

$$Du = \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \frac{1}{1 + \frac{1}{|x|}} \left(-\frac{x}{|x|^3}\right) \tag{15}$$

Thus we have

$$|Du| \leqslant C \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \frac{1}{|x|}. \tag{16}$$

Now we have

$$\int_{B(0,1)} |Du|^n \, \mathrm{d}x \ \leqslant \ C \int_0^1 \frac{1}{\log\left(1 + \frac{1}{r}\right)^n} \frac{1}{r} \, \mathrm{d}r \tag{17}$$

Setting $z = \log\left(1 + \frac{1}{r}\right)$, we have

$$\int_{B(0,1)} |Du|^n \, \mathrm{d}x \leqslant C \int_{\log 2}^{\infty} \frac{1}{z^n} \, \mathrm{d}z < +\infty \tag{18}$$

when $n > 1$. □