Math 527 B1 Homework 1 (Due Sep. 24 in Class)

Sep. 17, 2010

Exercise 1. (5 pts) (1.5.5) Assume that $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is smooth. Prove

$$f(x) = \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha + O\Big(|x|^{k+1}\Big) \qquad \text{as } x \to 0$$

for each $k = 1, 2, \ldots$. This is *Taylor's formula* in multiindex notation.

(Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(t\, x)$.)

Notation: For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1, \ldots, \alpha_n \geqslant 0$, $x = (x_1, \ldots, x_n)$,

$$\begin{aligned}
|\alpha| &:= \alpha_1 + \cdots + \alpha_n; \\
\alpha! &:= \alpha_1!\, \alpha_2! \cdots \alpha_n! \\
D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}; \\
x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \\
|x| &:= \Big(x_1^2 + \cdots + x_n^2\Big)^{1/2}.
\end{aligned}$$

Exercise 2. (15 pts) (Well-posedness for ODE) We develop a complete theory of well-posedness for the initial value problem of ODE. Consider an ODE of the form

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0. \tag{1}$$

where f is defined on $D \subseteq \mathbb{R} \times \mathbb{R}^d$ and $(t_0, u_0) \in D$. Naturally, we say u is a classical solution if $u \in C^1$.

a) **(3 pts)** Existence I: Prove the following theorem.

   Theorem. *Assume that f is continuous in t and uniformly Lipschitz in u, then there exists an interval $\big(t^-, t^+\big) \ni t_0$, such that at least one classical solution $u \in C^1\big(t^-, t^+\big)$ exists.*

   Remark. The proof still works when $\mathbb{R}^d$ is replaced by any Banach space. Thus it can be applied to many PDEs.

b) **(Optional)** Existence II: Prove the following theorem.

   Theorem. *The "uniform Lipschitz" condition on f in the above theorem can be replaced by $f \in C(D)$.*

   Hint: On any compact subset of D, approximate f uniformly by Lipschitz functions f_n, let u_n be a solution of the corresponding ODE, then use Ascoli-Arzela Theorem (a uniformly bounded, equicontinuous sequence has a subsequence which converges uniformly).

c) Uniqueness:

   i. **(3 pts)** Show that the solution obtained in a) is in fact the only solution for the initial value problem.

   ii. **(3 pts)** Construct an example to show that under the condition of the theorem in b), uniqueness may fail.

   iii. **(Optional)** Show that uniqueness still holds when the "uniform Lipschitz" condition on f in a) is replaced by the following weaker "Osgood" condition:

   $$|(f(t, u) - f(t, v)) \cdot (u - v)| \leqslant g(|u - v|) \tag{2}$$

   where the modulus g satisfies

   $$\int_0^\delta \frac{1}{g(r)}\, \mathrm{d}r = \infty \tag{3}$$

   for any $\delta > 0$.

d) **(3 pts)** Continuous dependence on initial value:

   Prove that the unique solution obtained in a) depends continuously on (t_0, u_0). Note that continuous dependence on data automatically fails when the solution is not unique.

e) **(3 pts)** Different definitions of solution, regularity:

   One can integrate and obtain the following "mild" formulation

   $$u(t) = u_0 + \int_{t_0}^t f(s, u(s))\, \mathrm{d}s. \tag{4}$$

   We say $u \in C(I)$ is a "weak solution" of the ODE if it satisfies this integral formulation. Prove that, $u \in C^m$ if $f \in C^{m-1}$ (as a function of (t, u)) for $m \geqslant 1$. Thus any weak solution is automatically classical and even smooth.

Remark 1. This problem shows how much more complicated PDE theory is compared with ODE theory.

Exercise 3. (10 pts) (2.5.6) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_U |u| \leqslant C \left(\max_{\partial U} |g| + \max_U |f| \right) \tag{5}$$

whenever u is a smooth solution of

$$-\triangle u = f \text{ in } U; \qquad u = g \text{ on } \partial U. \tag{6}$$

(Hint: $-\triangle\left(u + \frac{|x|^2}{2n}\lambda\right) \leqslant 0$ for $\lambda := \max_{\bar{U}} |f|$)

Exercise 4. (Optional) Consider the eikonal equation

$$u_{x_1}^2 + \cdots + u_{x_n}^2 = 1 \qquad x \in B := \left\{x_1^2 + \cdots + x_n^2 < 1\right\},$$
$$u = 0 \qquad x \in \partial B := \left\{x_1^2 + \cdots + x_n^2 = 1\right\}.$$

Clearly, the natural class of functions for the solution is $C(\bar{B}) \cap C^1(B)$, that is, functions that are continuously differentiable in B, while continuous up to the boundary. We call such solutions "classical".

a) Show that no classical solution exists. Thus the equation is not well-posed if we consider only classical solutions.

b) One way to define "weak solutions" is through "testing" by smooth functions. For example, suppose we try to define "weak solutions" for the equation $u_{x_1} = f$ in B, $u = 0$ on ∂B, then we can multiply the equation by a smooth function φ with $\varphi = 0$ on ∂B and (formally) integrate by parts and obtain

$$\int u \varphi_{x_1} = -\int f \varphi.$$

and use this integral relation (which we require to hold for all smooth φ) as the definition. We see that as a consequence u need not be in C^1 anymore, in fact u being integrable is enough for the definition to make sense.

Try to define "weak solutions" for the eikonal equation this way. What difficulty do you meet?

c) Another way to relax the regularity requirement is to require $u \in C(\bar{B})$ but not $C^1(B)$, only differentiable almost everywhere. Consider the case $n = 1$. By this definition $u = 1 - |x|$ solves the eikonal equation. Can you establish well-posedness for such kind of "weak solutions" in the $n = 1$ case? If not, why?