SECOND-ORDER HYPERBOLIC EQUATIONS

We consider the initial/boundary-value problem

$$\left\{\begin{array}{rcll} u_{tt}+Lu &=& f & \text{in } U_T \\ u &=& 0 & \text{on } \partial U\times[0,T] \\ u &=& g & \text{on } U\times\{t=0\} \\ u_t &=& h & \text{on } U\times\{t=0\} \end{array}\right. \tag{1}$$

Here L is again the second-order elliptic operator in the divergence or nondivergence forms.

1. Weak solutions.

1.1. Definition.

We assume

$$a^{ij}, b^i, c\in C^1(\bar{U}_T), \qquad f\in L^2(U_T), \quad g\in H_0^1(U), \quad h\in L^2(U). \tag{2}$$

Define the bilinear form

$$B[u,v;t] := \int_U \left[\sum_{i,j=1}^n a^{ij}(\,\cdot\,,t)\, u_{x_i} v_{x_i} + \sum_{i=1}^n b^i(\,\cdot\,,t)\, u_{x_i} v + c(\,\cdot\,,t)\, u\, v\right] \mathrm{d}x. \tag{3}$$

Definition 1. (Weak solutions) *We say a function*

$$u\in L^2(0,T;H_0^1(U)), \quad \textit{with} \quad u'\in L^2(0,T;L^2(U)), \quad u''\in L^2(0,T;H^{-1}(U)) \tag{4}$$

is a weak solution of the hyperbolic initial/boundary-value problem provided that it satisfies the initial conditions, and

$$\langle u'', v\rangle + B[u,v;t] = (f,v) \tag{5}$$

for each $v\in H_0^1(U)$ and a.e. time $0\leqslant t\leqslant T$.

1.2. Existence.

As in the parabolic case, we use Galerkin's method. The procedure is very similar to the parabolic case. See Evans pp.380–385 for details.

1.3. Uniqueness.

The proof for uniqueness is different from the parabolic case. The main reason is the following. Ideally, we would like to take u' to be the test function and have

$$(u'', u') + B[u, u'; t] = 0 \tag{6}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u'\|_{L^2(U)}^2 + A[u,u;t]\big) \leqslant C\left(\|u'\|_{L^2(U)}^2 + A[u,u;t]\right) \tag{7}$$

where

$$A[u,u;t] := \int_U \sum a^{ij} u_{x_i} u_{x_j}. \tag{8}$$

And Gronwall's inequality would lead to $u\equiv 0$ and uniqueness follows.

The problem here is that, as $u'\in L^2(0,T;L^2(U))$ instead of $L^2(0,T;H_0^1(U))$, we cannot use it as a test function.

Proof. (of uniqueness) Fix $0\leqslant s\leqslant T$ and set

$$v(t) := \left\{\begin{array}{ll} \int_t^s u(\tau)\,\mathrm{d}\tau & 0\leqslant t\leqslant s \\ 0 & s\leqslant t\leqslant T \end{array}\right. . \tag{9}$$

Then $v(t) \in H_0^1(U)$ for each $0 \leqslant t \leqslant T$ and can be used as a test function. Plug into the equation, we have

$$\langle u'', v \rangle + B[u, v; t] = 0. \tag{10}$$

Integrate from 0 to s and then integrate the first term, we have

$$\int_0^s -(u', v') + B[u, v; t]\,\mathrm{d}t = 0. \tag{11}$$

For $0 \leqslant t \leqslant s$, by definition $v' = -u$. Thuswe have

$$\int_0^s (u', u) - B[v', v; t]\,\mathrm{d}t = 0. \tag{12}$$

Now note that

$$(u', u) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_{L^2(U)}^2\right) \tag{13}$$

and

$$B[v', v; t] = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}B[v, v; t] + C[u, v; t] - D[v, v; t] \tag{14}$$

with

$$C[u, v; t] := -\int_U \sum b^i\, v_{x_i}\, u + \frac{1}{2} b^i_{x_i}\, u\, v\,\mathrm{d}x \tag{15}$$

$$D[u, v; t] := \frac{1}{2}\int_U \sum a_t^{ij}\, u_{x_i}\, v_{x_j} + \sum b_t^i\, u_{x_i}\, v + c_t\, u\, v\,\mathrm{d}x. \tag{16}$$

Thus we have

$$\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_{L^2(U)}^2 - \frac{1}{2}B[v, v; t]\right)\mathrm{d}t = -\int_0^s C[u, v; t] + D[v, v; t]\,\mathrm{d}t \tag{17}$$

It follows that

$$\frac{1}{2}\|u\|_{L^2(U)}^2(s) - \frac{1}{2}B[v(0), v(0); t] = -\int_0^s C[u, v; t] + D[v, v; t]\,\mathrm{d}t \tag{18}$$

where we have used the fact that $u(0) = 0$ and $v(s) = 0$.

Using the coercivity of B and boundedness of C, D, we have

$$\|u(s)\|_{L^2(U)}^2 + \|v(0)\|_{H_0^1(U)}^2 \leqslant C\left[\int_0^s \left(\|v\|_{H_0^1(U)}^2 + \|u\|_{L^2(U)}^2\right)\mathrm{d}t + \|v(0)\|_{L^2(U)}^2\right]. \tag{19}$$

Now recall the definition of v:

$$v(t) := \begin{cases} \int_t^s u(\tau)\,\mathrm{d}\tau & 0 \leqslant t \leqslant s \\ 0 & s \leqslant t \leqslant T \end{cases}. \tag{20}$$

If we let

$$w(t) := \int_0^t u(\tau)\,\mathrm{d}\tau \tag{21}$$

then for $0 \leqslant t \leqslant s$,

$$v(t) = w(s) - w(t). \tag{22}$$

Substituting into the estimate, we have

$$\begin{aligned} \text{RHS} &= C\left[\int_0^s \left(\|v\|_{H_0^1(U)}^2 + \|u\|_{L^2(U)}^2\right)\mathrm{d}t + \|v(0)\|_{L^2(U)}^2\right] \\ &= C\left[\int_0^s \left(\|w(s) - w(t)\|_{H_0^1}^2 + \|u\|_{L^2(U)}^2\right)\mathrm{d}t\right] + C\,\|w(s)\|_{L^2}^2 \\ &\leqslant C\left[\int_0^s \|w(s)\|_{H_0^1}^2 + \|w(t)\|_{H_0^1}^2 + \|u\|_{L^2}^2\,\mathrm{d}t\right] + C\int_0^s \|u(t)\|_{L^2}^2\,\mathrm{d}t \\ &= C s\,\|w(t)\|_{H_0^1}^2 + C'\int_0^s \|w(t)\|_{H_0^1}^2 + \|u\|_{L^2}^2\,\mathrm{d}t. \end{aligned} \tag{23}$$

On the other hand, we have

$$\text{LHS} = \|w(s)\|_{H_0^1}^2 + \|u\|_{L^2}^2. \tag{24}$$

Now we take s so small that $C s < 1/2$. Then

$$\|w(s)\|_{H_0^1}^2 + \|u(s)\|_{L^2}^2 \leqslant C \int_0^s \|w(t)\|_{H_0^1}^2 + \|u\|_{L^2}^2 \, \mathrm{d}t. \tag{25}$$

Application of Gronwall's inequality gives $u \equiv 0$ on $[0, T_1]$ for $T_1 < \frac{1}{2C}$. We now can apply the same argument to $[T_1, 2T_1], [2T_1, 3T_1]$ and so on. □

1.4. Regularity.

We only mention that

$$g, h \in C^\infty(\bar{U}), \;\; f \in C^\infty(\bar{U}_T) \implies u \in C^\infty(\bar{U}_T) \tag{26}$$

given that the data is compatible.

See Evans pp.389–393 for details.

2. Propagation of disturbance.

Recall that, for the 1D wave equation, we have shown that if u, u_t is initially 0 in a ball, then the solution is 0 in a cone with slope -1. We show here that the same is true in the general case. For simplicity we consider the simple case

$$L u = -\sum a^{ij} u_{x_i x_j} \tag{27}$$

where the coefficients are smooth, independent of time.

Consider the Hamilton-Jacobi equation

$$p_t - \left(\sum a^{ij} p_{x_i} p_{x_j}\right)^{1/2} = 0. \tag{28}$$

If we write

$$p(x, t) = q(x) + t - t_0, \tag{29}$$

then q solves

$$\sum a^{ij} q_{x_i} q_{x_j} = 1 \text{ in } \mathbb{R}^n - \{x_0\}, \qquad q(x_0) = 0. \tag{30}$$

Now we write

$$C := \{(x, t) \mid p(x, t) < 0\} = \{(x, t) \mid q(x) < t_0 - t\}. \tag{31}$$

For each $t > 0$, we further define

$$C_t := \{x \mid q(x) < t_0 - t\} \tag{32}$$

which is the cross section of C at time t.

Theorem 2. (Finite propagation speed) *Assume u is a smooth solution of the hyperbolic equation. If $u \equiv u_t \equiv 0$ on C_0, then $u \equiv 0$ within the cone C.*

Proof. Define the energy

$$e(t) := \frac{1}{2} \int_{C_t} u_t^2 + \sum a^{ij} u_{x_i} u_{x_j} \, \mathrm{d}x. \tag{33}$$

Differentiating, we have

$$\dot{e}(t) = \int_{C_t} u_t u_{tt} + \sum a^{ij} u_{x_i} u_{x_j t} - \frac{1}{2} \int_{\partial C_t} \left(u_t^2 + \sum a^{ij} u_{x_i} u_{x_j}\right) \frac{1}{|Dq|} \, \mathrm{d}S \tag{34}$$

where we have used the Co-area formula.

Integrating by parts, we have

$$\begin{aligned} \int_{C_t} u_t u_{tt} + \sum a^{ij} u_{x_i} u_{x_j t} &= \int_{C_t} u_t \left[u_{tt} - \sum \left(a^{ij} u_{x_i}\right)_{x_j}\right] \mathrm{d}x + \int_{\partial C_t} \sum a^{ij} u_{x_i} \nu^j u_t \, \mathrm{d}S \\ &= -\int_{C_t} u_t \sum a^{ij}_{x_j} u_{x_i} \, \mathrm{d}x + \int_{\partial C_t} \sum a^{ij} u_{x_i} \nu^j u_t \, \mathrm{d}S. \end{aligned} \tag{35}$$

For the first term, we estimate

$$-\int_{C_t} u_t \sum a^{ij}_{x_j} u_{x_i} \,\mathrm{d}x \leqslant C \int_{C_t} u_t^2 + |Du|^2 \leqslant C\, e(t). \tag{36}$$

For the second term, we estimate

$$\left| \sum a^{ij} u_{x_i} \nu^j \right| \leqslant \left(\sum a^{ij} u_{x_i} u_{x_j} \right)^{1/2} \left(\sum a^{ij} \nu^i \nu^j \right)^{1/2}. \tag{37}$$

Note that since $q = t_0 - t$ on C_t, $\nu = \frac{Dq}{|Dq|}$. And the equation for q then gives

$$\sum a^{ij} \nu^i \nu^j = \frac{1}{|Dq|^2}. \tag{38}$$

Therefore

$$\left| \int_{\partial C_t} \sum a^{ij} u_{x_i} \nu^j u_t \,\mathrm{d}S \right| \leqslant \frac{1}{2} \int_{\partial C_t} \left(u_t^2 + \sum a^{ij} u_{x_i} u_{x_j} \right) \frac{1}{|Dq|} \,\mathrm{d}S. \tag{39}$$

As a consequence we have

$$\dot{e}(t) \leqslant C\, e(t). \tag{40}$$

Combining with $e(0) = 0$, we have $e(t) \equiv 1$ and $u \equiv 0$ follows. □