SECOND-ORDER PARABOLIC EQUATIONS

In this lecture we study the initial/boundary-value problem

$$\begin{cases} u_t + L u &= f \text{ in } U_T \\ u &= 0 \text{ on } \partial U \times [0, T] \\ u &= g \text{ on } U \times \{t = 0\} \end{cases} \tag{1}$$

Here U is an open bounded subset of $\mathbb{R}^n$, and $U_T := U \times (0, T]$ for some fixed $T > 0$.

The operator L either has the divergence form

$$L u = -\sum_{i,j}^n \left(a^{ij}(x,t)\, u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i(x,t)\, u_{x_i} + c(x,t)\, u \equiv -\nabla \cdot (A(x,t)\, D u) + b(x,t) \cdot D u + c(x,t)\, u, \tag{2}$$

or the nondivergence form

$$L u = -\sum_{i,j=1}^n a^{ij}(x,t)\, u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)\, u_{x_i} + c(x,t)\, u \equiv A(x,t) \colon D^2 u + b(x,t) \cdot D u + c(x,t)\, u. \tag{3}$$

We say the operator $\partial_t + L$ is uniformly parabolic if L is uniformly elliptic, that is, if there exists a constant θ such that

$$\sum_{i,j=1}^n a^{ij}(x)\, \xi_i \xi_j \geqslant \theta\, |\xi|^2 \tag{4}$$

for a.e. $(x,t) \in U_T$ and all $\xi \in \mathbb{R}^n$.

The simplest example of a parabolic equation is the heat equation.

1. Weak solutions.

1.1. Definition.

We assume

$$a^{ij}, b^i, c \in L^\infty(U_T), \qquad f \in L^2(U_T), \qquad g \in L^2(U). \tag{5}$$

We also always assume $a^{ij} = a^{ji}$.

As usual, we try to obtain the correct integral formulation by multiplying the equation with a test function $v \in C_0^\infty(U)$ and then integrate.

$$\int_U f\, v\, \mathrm{d}x = \int_U (u_t + L u)\, v\, \mathrm{d}x \;=\; \int_U \left[u_t\, v + \sum_{i,j=1}^n a^{ij}(\,\cdot\,, t)\, u_{x_i} v_{x_i} + \sum_{i=1}^n b^i(\,\cdot\,, t)\, u_{x_i} v + c(\,\cdot\,, t)\, u\, v \right] \mathrm{d}x. \tag{6}$$

Inspecting the above, we see that we should require $u \in H_0^1(U)$ for every t, and require $u_t \in H^{-1}(U)$ for every t. Thus the solution u should be a mapping[1]

$$u \colon [0, T] \mapsto H_0^1(U) \tag{7}$$

such that

$$u' \colon [0, T] \mapsto H^{-1}(U). \tag{8}$$

Satisfying

$$\langle u', v \rangle + B[u, v; t] = (f, v) \tag{9}$$

where

$$B[u, v; t] := \int_U \left[\sum_{i,j=1}^n a^{ij}(\,\cdot\,, t)\, u_{x_i} v_{x_i} + \sum_{i=1}^n b^i(\,\cdot\,, t)\, u_{x_i} v + c(\,\cdot\,, t)\, u\, v \right] \mathrm{d}x. \tag{10}$$

1. At this stage there are two different approaches. The first one is to treat t and x as equivalent and consider weak solutions through integration by parts in space-time; The second one is the single t out and treat the PDE as an ODE in abstract Banach spaces. Here we take the second approach.

Remark 1. The weak derivative u' is defined aas follows. Let X be a Banach space, and let $u \in L^1(0, T; X)$. A function $v \in L^1(0, T; X)$ is said to be the weak derivative of u, written

$$u' = v \tag{11}$$

provided

$$\int_0^T \phi'(t)\, u(t)\, \mathrm{d}t = -\int_0^T \phi(t)\, v(t)\, \mathrm{d}t \tag{12}$$

for all scalar function $\phi \in C_0^\infty(0, T)$. See Evans E.5 for the definition of Banach-space valued integrals.

More formally, we have

Definition 2. *We say a function*

$$u \in L^2\big(0, T; H_0^1(U)\big) \text{ with } u' \in L^2\big(0, T; H^{-1}(U)\big) \tag{13}$$

is a weak solution of the parabolic initial-boundary-value problem provided $u(0) = g$ *and*

$$\langle u', v\rangle + B[u, v; t] = (f, v) \tag{14}$$

for each $v \in H_0^1(U)$ *and a.e. time* $0 \leqslant t \leqslant T$.

Remark 3. The notation $u \in L^2(0, T; X)$, where X is a function space, means the norm $\|u\|_X$, which is a function of t, belongs to $L^2(0, T)$.

Remark 4. In fact, one can show that

$$u \in L^2\big(0, T; H_0^1(U)\big) \text{ together with } u' \in L^2\big(0, T; H^{-1}(U)\big) \Longrightarrow u \in C\big([0, T]; L^2(U)\big). \tag{15}$$

See Evans p.287 Theorem 3. Thus $u(0)$ (as the limit $\lim_{t \searrow 0} u(t)$) is well-defined.

1.2. Existence.

The dominant approach in the study of existence for parabolic equations is the Galerkin's method, which takes advantage of the eigenvalue/eigenfunction theory of the elliptic equations. A special case is when U is the torus (periodic boundary condition), in that case Galerkin's method reduces to the spectral method.

The idea is as follows. Let $\{w_k\}_{k=1}^\infty$ be the orthonormal basis of $L^2(U)$ consisting of eigenfunctions of L. Note that it is also an orthogonal basis for H_0^1. Then we expect

$$u(t) = \sum_1^\infty d^k(t)\, w_k. \tag{16}$$

Substituting this into the PDE, and equating coefficients for each w_k to 0, we obtain an ODE system with infinite size. Now we "cut-off" this system by disgarding all equations for w_k, $k > m$. The resulting $m \times m$ system usually admits a global solution. Finally we let $m \nearrow \infty$ and try to prove that the solutions converge. The limit is the solution to the original problem.

Fix an integer m. Consider the m equations

$$\big(u_m', w_k\big) + B[u_m, w_k; t] = (f, w_k), \qquad k = 1, \ldots, m. \tag{17}$$

To make this system well-posed, we look for u_m of the form

$$u_m(t) := \sum_1^m d_m^k(t)\, w_k. \tag{18}$$

with

$$d_m^k(0) = (g, w_k). \tag{19}$$

Thus we have m equations and m unknowns $d_m^1, \ldots, d_m^m$. Substituting the formula for $u_m(t)$ into the equations, we see that the ODE system is actually linear, and therefore has a unique solution that exists for all time.

Now the task is to show that we can take limit $\lim_{m \nearrow \infty} u_m$ and the limit is the solution to the original problem. The idea is to obtain uniform bounds on them and then apply certain compactness theorems.

Theorem 5. (Energy estimates) *There exists a constant C, depending only on U, T and the coefficients of L, such that*

$$\max_{0 \leqslant t \leqslant T} \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2(0,T;H_0^1(U))} + \|u_m'\|_{L^2(0,T;H^{-1}(U))} \leqslant C \left(\|f\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)} \right). \tag{20}$$

for $m = 1, 2, \ldots$

Proof. We sketch the main steps. For details see Evans pp. 354–356. The basic idea is the use u_m as the test function in

$$\left(u_m', v\right) + B[u_m, v; t] = (f, v), \qquad v \in \operatorname{span}\{w_1, \ldots, w_m\}. \tag{21}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|_{L^2(U)}^2 \leqslant \frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|_{L^2(U)}^2 + 2\,\beta\,\|u_m\|_{H_0^1(U)}^2 \leqslant C_1 \|u_m\|_{L^2(U)}^2 + C_2 \|f\|_{L^2(U)}^2 \tag{22}$$

which in turn gives us the correct bound for $\max_{0 \leqslant t \leqslant T} \|u_m\|_{L^2(U)}$ through Gronwall's inequality.

Next integrate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|_{L^2(U)}^2 + 2\,\beta\,\|u_m\|_{H_0^1(U)}^2 \leqslant C_1 \|u_m\|_{L^2(U)}^2 + C_2 \|f\|_{L^2(U)}^2 \tag{23}$$

from 0 to T, and use the bound for $\max_{0 \leqslant t \leqslant T} \|u_m\|_{L^2(U)}$, we easily obtain the correct bound for $\|u_m\|_{L^2(0,T;H_0^1(U))}$.

Finally, take arbitrary $v \in H_0^1(U)$. We can decompose $v = v_1 + v_2$ where $v_1 \in \operatorname{span}\{w_1, \ldots, w_m\}$ and $v_2 \perp \operatorname{span}\{w_1, \ldots, w_m\}$ in H_0^1. We have

$$\left(u_m', v\right) = \left(u_m', v_1\right) = (f, v_1) - B[u_m, v_1; t] \tag{24}$$

using the equation. This leads to

$$\left|\left(u_m', v\right)\right| \leqslant C \left(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)} \right) \|v_1\|_{H_0^1(U)} \leqslant C \left(\|f\|_{L^2(U)} + \|u_m\|_{H_0^1(U)} \right) \|v\|_{H_0^1(U)}. \tag{25}$$

As this holds for all $v \in H_0^1(U)$, we obtain the correct bound for $\int_0^T \|u_m'\|_{H^{-1}(U)}$. ☐

Theorem 6. (Existence) *There exists at least one weak solution.*

Proof. From the energy estimates, we know that

- $\{u_m\}$ is uniformly bounded in $L^2\big(0, T; H_0^1(U)\big)$,
- $\{u_m'\}$ is uniformly bounded in $L^2\big(0, T; H^{-1}(U)\big)$.

Thus there exists subsequence converging weakly to some function $u \in L^2\big(0, T; H_0^1(U)\big)$ with $u' \in L^2\big(0, T; H^{-1}(U)\big)$. One can show that u is a weak solution of the original problem. See Evans p.357 for details. ☐

1.3. Uniqueness.

As the equation is linear, it suffices to show that $u \equiv 0$ if $f = g = 0$. Take u as the test function. Then we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2(U)}^2 \leqslant C\,\|u\|_{L^2(U)}^2 \tag{26}$$

with $\|u\|_{L^2(U)}^2 = 0$ at $t = 0$. The conclusion follows from Gronwall's inequality.

1.4. Regularity.

If we assume more on g and f, we can conclude that u has higher regularity than in the definition of the weak solution. We will not go into details here. See Evans 358 – 367. We just mention that if we assume

$$g \in C^\infty\big(\bar{U}\big), \qquad f \in C^\infty \tag{27}$$

then

$$u\in C^{\infty}\big(\bar{U}_T\big). \tag{28}$$

2. Maximum principles.

We assume that L takes the nondivergence form.

$$L u = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + c\, u. \tag{29}$$

We assume the coefficients are continuous. Let Γ_T be the reduced/parabolic boundary. Then we have

Theorem 7. (Weak maximum principle) *Assume $u\in C_1^2(U_T)\cap C\big(\bar{U}_T\big)$ and $c\equiv 0$. Then*

$$u_t + L u \leqslant 0 \Longrightarrow \max_{\bar{U}_T} u = \max_{\Gamma_T} u; \tag{30}$$

$$u_t + L u \geqslant 0 \Longrightarrow \min_{\bar{U}_T} u = \min_{\Gamma_T} u. \tag{31}$$

Proof. The proof is almost identical to the one for the heat equation. □

As before, weak maximum principles continue to hold in the case $c\geqslant 0$, if the modify the conclusions to

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u^+ \quad \text{and} \quad \min_{\bar{U}_T} u = \min_{\Gamma_T} u^-. \tag{32}$$

Using the weak maximum principle we can prove the following Harnack's inequality. See Evans pp.371–374.

Theorem 8. (Parabolic Harnack inequality) *Assume $u\in C_1^2(U_T)$ solves $u_t + L u = 0$ in U_T, and also $u\geqslant 0$ in U_T. Then for any $V\Subset U$, and each $0<t_1<t_2\leqslant T$, there exists a constant C such that*

$$\sup_V u(\,\cdot\,,t_1) \leqslant C \inf_V u(\,\cdot\,,t_2). \tag{33}$$

The constant C depends only on V, t_1, t_2 and the coefficients of L.

Remark 9. Note that $t_1<t_2$. So the bound is only one sided.

Finally we mention the strong maximum principle.

Theorem 10. (Strong maximum principle) *Assume $u\in C_1^2(U_T)\cap C\big(\bar{U}_T\big)$ and $c\equiv 0$ in U_T. Suppose also that U is connected. Then u is a constant on U_{t_0} if either*

$$u_t + L u \leqslant 0, \qquad u \text{ attains maximum at } (x_0,t_0) \tag{34}$$

or

$$u_t + L u \geqslant 0, \qquad u \text{ attains minimum at } (x_0,t_0). \tag{35}$$

Remark 11. For the case $c\geqslant 0$, we modify the conclusions in the standard way.

Proof. Recall that, in the case of the heat equation, we prove strong maximum principle using mean value formula. Such formula is not available in the general case.

Here instead we use the Harnack's inequality. Take a smooth, open set $W\Subset U$, with $x_0\in W$. Let v solve

$$v_t + L v = 0 \tag{36}$$

in W and takes u as its boundary values. As $u_t + L u\leqslant 0$, weak maximum principle yields $u\leqslant v$. Since v takes u as its boundary value, we know $v\leqslant u(x_0,t_0)$. Thus

$$u\leqslant v\leqslant u(x_0,t_0) \tag{37}$$

and in particular $v(x_0, t_0) = u(x_0, t_0)$.

Now let $\tilde{v} := u(x_0, t_0) - v$. We have $\tilde{v}_t + L\tilde{v} = 0$, $\tilde{v} \geqslant 0$. For any $V \Subset W$, with $x_0 \in V$, we now can apply the Harnack inequality to conclude that $\tilde{v} \equiv 0$ in V_{t_0}. Thus $\tilde{v} \equiv 0$ in W_{t_0}, and $v \equiv u(x_0, t_0)$ in W_{t_0}. But this implies $u \equiv u(x_0, t_0)$ along the reduced boundary of W_{t_0}. Using weak maximum principle again, we see that $u \equiv u(x_0, t_0)$ in W_{t_0}.

The strong maximum principle then follows from the arbitrariness of W. $\square$