SECOND-ORDER ELLIPTIC EQUATIONS: EIGENVALUES AND EIGENFUNCTIONS

In this lecture we study the boundary-value problem

$$L w = \lambda w \quad \text{in } U; \qquad w = 0 \quad \text{on } \partial U. \tag{1}$$

Such problem is called "eigenvalue problem". When it admits a non-zero solution (note that 0 is automatically a solution for all λ), the corresponding λ is called an "eigenvalue", and any one of the non-zero solution is called an "eigenfunction". The importance of understanding the properties of the eigenvalues/eigenfunction is that one can expand other functions in them.[1]

1. Eigenvalues of symmetric elliptic operators.

We consider

$$L u = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} \tag{2}$$

with $a^{ij} \in C^\infty(\bar{U})$. As usual, we assume that $a^{ij} = a^{ji}$ and the corresponding bilinear form is bounded an coercive. Then it turns out that the eigenfunctions have properties similar to that of the Fourier modes.

Theorem 1. (Eigenvalues of symmetric elliptic operators)

i. *Each eigenvalue of L is real;*

ii. *If we repeat each eigenvalue according to its (finite) multiplicity, we ahve*

$$\sum = \{\lambda_k\}_{k=1}^\infty, \tag{3}$$

where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \tag{4}$$

and

$$\lambda_k \longrightarrow \infty \qquad \text{as } k \longrightarrow \infty. \tag{5}$$

iii. *There exists an orthonormal basis $\{w_k\}_{k=1}^\infty$ of $L^2(U)$, where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k:*

$$L w_k = \lambda_k w_k \text{ in } U; \qquad w_k = 0 \text{ on } \partial U \tag{6}$$

for $k = 1, 2, \ldots$

Remark 2. Note that each w_k is in fact $C^\infty(U)$. Furthermore if $\partial U \in C^\infty$, so is w_k.

Proof. For i, assume λ is an eigenvalue, let w be a corresponding eigenfunction. We have

$$L w = \lambda w \implies L w^* = \lambda^* w^* \tag{7}$$

where * denotes the complex conjugate. Now we have (taking w such that $\int |w|^2 = 1$)

$$\lambda = \int (L w) w^* = \int w (L w^*) = \lambda^*. \tag{8}$$

Thus eigenvalues must be real.

To see that the eigenfunctions corresponding to different eigenvalues are orthogonal, we check

$$\int \lambda w \tilde{w} = \int (L w) \tilde{w} = \int w (L \tilde{w}) = \int \tilde{\lambda} w \tilde{w}. \tag{9}$$

As $\lambda \neq \tilde{\lambda}$, we must have $\int w \tilde{w} = 0$.

It is also easy to see that the eigenvalues are positive.

1. For example, when $U = (0,1)$, the eigenfunctions are just the Fourier (sine) modes $\sin(\pi n x)$. For a general domain U, one cannot use Fourier analysis as the Fourier modes do not respect the boundary conditions. In these cases the eigenfunctions are used.

For the remaining properties, the idea is the consider the inverse operator

$$S := L^{-1}:\ L^2 \mapsto L^2 \tag{10}$$

which exists due to the existence theory.

It is clear that S is linear. Now we show that S is bounded. Recall that $Sf = u$ if and only if $Lu = f$. Now using the coercivity of L, we have

$$\theta \, \|u\|^2 \leqslant \langle Lu, u\rangle = \langle f, u\rangle \leqslant \|f\|_{L^2} \|u\| \implies \|Sf\|_{H^1} = \|u\| \leqslant \theta^{-1} \|f\|_{L^2}. \tag{11}$$

Next we show that S is compact. Taking f_i bounded in L^2. From the above we see that $\|Sf_i\|_{H^1}$ are uniformly bounded. The compact embedding $H^1 \Subset L^2$ tells us that we can subtract a convergent (in L^2) subsequence from $\{Sf_i\}$.

In summary, $S = L^{-1}$ is a linear bounded compact operator from L^2 to itself. The properties follow from genearl operator theory. $\square$

Next we study the first eigenvalue and its corresponding eigenfunction. Usually $\lambda_1 > 0$ is called the *principal eigenvalue*.

Theorem 3. (Variational principle for the principal eigenvalue)

i. We have

$$\lambda_1 = \min \left\{ B[u, u] \mid u \in H_0^1(U), \|u\|_{L^2} = 1 \right\}. \tag{12}$$

ii. Furthermore, the minimum is attained. The minimizer is an eigenfunction.

iii. The eigenspace corresponding to λ_1 is one-dimensional. And its eigenfunctions are either positive or negative.

Proof.

i. Since $\{w_k\}$ form an orthonormal basis of L^2, we can write any function $u \in H_0^1$ as

$$u = \sum_1^\infty u_k \, w_k. \tag{13}$$

$\|u\|_{L^2} = 1 \iff \sum u_k^2 = 1.$

Now compute[2]

$$B[u, u] = \sum \lambda_k \, u_k^2. \tag{17}$$

It is clear that

$$B[u, u] \geqslant \lambda_1 \sum u_k^2 = \lambda_1 \tag{18}$$

and the minimum is λ_1.

ii. It is clear that the minimum is attained at a function which is a combination of eigenfunctions corresponding to λ_1, and thus is itself an eigenfunction corresponding to λ_1.

2. A technical issue here. As $\{w_k\}$ is an orthonormal basis for L^2, the relation $u = \sum_1^\infty u_k \, w_k$ only holds in L^2. In other words, the infinite sum in the RHS converges in L^2. As $B[u, u]$ involves Du, we need the infinite sum to converge in H^1 too to be able to write

$$B[u, u] = \sum_{l, m} B[u_l \, w_l, u_m \, w_m]. \tag{14}$$

We overcome this difficulty as follows. First note that, thanks to the coercivity of the operator and the Poincare inequality, $(B[u, u])^{1/2}$ is an equivalent norm on H_0^1. Thus we have

$$\|w_k\|_{H^1} \sim (B[w_k, w_k])^{1/2} = \lambda_k^{1/2} \, \|w_k\|_{L^2} = \lambda_k^{1/2}. \tag{15}$$

To show that $\sum_1^N u_k \, w_k$ is a Cauchy sequence on H_0^1, it suffices to show that $\sum_1^N u_k \, \lambda_k^{1/2}$ is a Cauchy sequence on $\mathbb{R}$. But this follows from the fact that $u \in H_0^1$, which translates to

$$\sum_1^\infty u_k^2 \, \lambda_k < \infty. \tag{16}$$

iii. First we show that any eigenfunction corresponding to λ_1 is either positive or negative. Let u be such that

$$L u = \lambda_1 u, \qquad u = 0 \text{ on } \partial U. \tag{19}$$

We need to show that either u^+ or u^- must vanish.

First we show that $u^\pm$ must also be eigenfunctions corresponding to λ_1. Note that $u^\pm$ cannot be nonzero at the same time, therefore

$$\lambda_1 = B[u, u] = B\left[u^+, u^+\right] + B\left[u^-, u^-\right] \geqslant \lambda_1 \int_U \left(u^+\right)^2 + \lambda_1 \int_U \left(u^-\right)^2 = \lambda_1. \tag{20}$$

This implies that

$$B\left[u^\pm, u^\pm\right] = \lambda_1 \int \left(u^\pm\right)^2 \tag{21}$$

which in turn gives

$$L u^\pm = \lambda_1 u^\pm, \qquad u^\pm = 0 \text{ on } \partial U. \tag{22}$$

But then the strong maximum principle tells us that if $u^\pm = 0$ at one point, it has to vanish everywhere. As a consequence, either $u^+ \equiv 0$ or $u^- \equiv 0$.

Now let $u, \tilde{u}$ be two different eigenfunctions corresponding to λ_1. We can always find a number $\mu \in \mathbb{R}$ such that $\int u - \mu \tilde{u} = 0$. Thus $u - \mu \tilde{u}$ is either identically zero or takes both positive and negative values. But $u - \mu \tilde{u}$ is also an eigenfunction for λ_1. Therefore we must have

$$u - \mu \tilde{u} = 0. \tag{23}$$

Thus ends the proof. □

2. Eigenvalues of nonsymmetric elliptic operators.

Now we consider the nondivergence form.

$$L u = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + c u. \tag{24}$$

Assume the coefficients are all smooth up to the boundary, which is also smooth. Further assume that a^{ij} is symmetric, and $c \geqslant 0$. Note that in this case L is not symmetric anymore. More specifically, we do not have

$$\int (L w) v = \int w (L v) \tag{25}$$

and as a consequence, the eigenvalues are not real anymore. Nevertheless, we still have the following.

Theorem 4. (Principal eigenvalue for nonsymmetric elliptic operators)

i. There exists a real eigenvalue λ_1 for the operator L, taken with zero boundary conditions, such that if $\lambda \in \mathbb{C}$ is any other eigenvalue, we have

$$\Re \lambda \geqslant \lambda_1. \tag{26}$$

ii. There exists a corresponding eigenfunction $w_1 > 0$.

iii. Any other eigenfunction corresponding to λ_1 is a multiple of w_1.

Proof. The proof is highly technical. It seems hard to even sketch "main ideas" here. Those interested should read Evans pp. 341–344. □