Second-Order Elliptic Equations: Maximum Principles

Recall that, for Poisson equation, we have

- Weak maximum principle.
$$-\triangle u \leqslant 0 \text{ in } U \implies u \leqslant \max_{\partial U} U \text{ in } U \tag{1}$$
and
- Strong maximum principle (assuming U is connected)
$$-\triangle u \leqslant 0 \text{ in } U, \ u(x_0) = \max_U u \text{ for some } x_0 \in U \implies u \equiv u(x_0). \tag{2}$$

In this lecture we will try to check whether such properties still hold for the general second order elliptic equations. It turns out, the nondivergence form

$$L u = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} + c\, u \tag{3}$$

is more appropriate for this.

Before listing and proving theorems, we mention a critical difference between the Poisson equation and the general case. It turns out that the sign of c plays an important role. To see this, consider the 1D case

$$-u'' + c\, u = 0. \tag{4}$$

When $c < 0$, we easily check that $u = \sin\left(\sqrt{c}\, x\right)$ solves the equation. Therefore no maximum principle can hold.

1. Weak maximum principle.

Theorem 1. (Weak maximum principle) *Assume $u \in C^2(U) \cap C\left(\bar{U}\right)$ and $c \equiv 0$. Then*

$$L u \leqslant 0 \implies \max_{\bar{U}} u = \max_{\partial U} u, \tag{5}$$

and

$$L u \geqslant 0 \implies \min_{\bar{U}} u = \min_{\partial U} u. \tag{6}$$

Proof. It is clear that we only need to prove either one of the two. Without loss of generality we prove

$$L u \leqslant 0 \implies \max_{\bar{U}} u = \max_{\partial U} u, \tag{7}$$

First consider the case $L u < 0$. Assume that there is $x_0 \in U$ such that $u(x_0) = \max_U u(x_0)$. Then we have

$$D u(x_0) = 0, \qquad D^2 u(x_0) \text{ negative semi-definite.} \tag{8}$$

Substitute into the equation, we see that we can have a contradiction if we can show the following:

$$A, B \text{ positive semi-definite} \implies \operatorname{tr}(A\, B) = \sum_{i,j} A_{ij} B_{ij} \geqslant 0. \tag{9}$$

Take any $\varepsilon > 0$. We know that $B_\varepsilon := B + \varepsilon\, I$ is positive definite and thus we can find C positive definite such that $C^2 = B_\varepsilon$. Now we have

$$\operatorname{tr}(A\, B_\varepsilon) = \operatorname{tr}\left(C\, A\, B_\varepsilon\, C^{-1}\right) = \operatorname{tr}(C\, A\, C) \geqslant 0. \tag{10}$$

Letting $\varepsilon \searrow 0$ we get the desired result.

Now consider the general case $L u \leqslant 0$. Write $u^\varepsilon(x) := u(x) + \varepsilon\, e^{\lambda x_1}$ for some $\lambda > 0$ to be determined later. We compute

$$L u^\varepsilon = L u + \varepsilon\, L\left(e^{\lambda x_1}\right) \leqslant \varepsilon\, e^{\lambda x_1} \left[-\lambda^2 a^{11} + \lambda\, b^1\right]. \tag{11}$$

Now choose λ large enough so that the RHS is negative. We see that

$$\max_{\bar{U}} u^{\varepsilon} = \max_{\partial U} u^{\varepsilon}. \tag{12}$$

Taking $\varepsilon \searrow 0$ we have proved the theorem. □

Theorem 2. (Weak maximum principle for $c \geqslant 0$) *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \geqslant 0$. Then*

$$L u \leqslant 0 \implies \max_{\bar{U}} u \leqslant \max_{\partial U} u^{+}, \tag{13}$$

and

$$L u \geqslant 0 \implies \min_{\bar{U}} u \leqslant \min_{\partial U} u^{-}. \tag{14}$$

Remark 3. As we have seen in the example at the beginning of the lecture, when $c < 0$ we should not expect maximum principles to hold.

Remark 4. Note that the $\leqslant$ here cannot be replaced by $''=''$. A counterexample is $u \equiv -1$. However, if $L u = 0$, we can indeed conclude the equality

$$\max_{\bar{U}} |u| = \max_{\partial U} |u|. \tag{15}$$

Proof. Clearly we only need to prove the first one. Let $V := \{x \in U \mid u(x) > 0\}$. Let $K u := L u - c u$. Then we have

$$K u \leqslant -c u \leqslant 0 \tag{16}$$

in V, and therefore we can apply the case $c = 0$ here. The proof is finished after noticing that

$$\max_{\partial V} u = \max_{\partial U} u^{+}. \tag{17}$$

□

2. Strong maximum principle.

Again we have two cases.

Theorem 5. (Strong maximum principle) *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$. Suppose also U is connected, open and bounded. Then u has to be a constant if either*

- *$L u \leqslant 0$ and u attains its maximum at an interior point, or*
- *$L u \geqslant 0$ and u attains its minimum at an interior point.*

Theorem 6. (Strong maximum principle with $c \leqslant 0$) *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \geqslant 0$. Suppose also U is connected, open and bounded. Then u has to be a constant if either*

- *$L u \leqslant 0$ and u attains a nonnegative maximum at an interior point, or*
- *$L u \geqslant 0$ and u attains a nonnegative minimum at an interior point.*

Recall that we establish the strong maximum principle for the Poisson equation using the mean value formula. Unfortunately for the general equation, the mean value formula does not hold anymore. Instead, we need the following more technical Hopf lemma.

Lemma 7. (Hopf's Lemma) *Assume $u \in C^2(U) \cap C(\bar{U})$, $L u \leqslant 0$ in U, and there exists a point $x^0 \in \partial U$ such that*

$$u(x^0) > u(x) \tag{18}$$

for all $x \in U$. Assume finally that U satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset U$ such that $x^0 \in \partial B$. Then

- *If $c \equiv 0$, then $\frac{\partial u}{\partial \nu}(x_0) > 0$ where ν is the outer normal to B at x^0;*

– *If $c \geqslant 0$, then the same holds if $u(x_0) \geqslant 0$.*

Remark 8. The lemma is nontrivial because instead of $\frac{\partial u}{\partial \nu} \geqslant 0$, we can conclude the strict inequality.

Remark 9. With the help of the Hopf's lemma, the proof of strong maximum principle is rather easy. Let $V \subset U$ be the points where $u < \max u$. As $u \in C^2$, $\partial V \cap U$ satisfies the interior ball condition. Take any $x^0 \in \partial V \cap U$, we have, by Hopf's Lemma, $\frac{\partial u}{\partial \nu} > 0$. But this x^0 is at the same time an interior maximizer of u which means $D u = 0 \Longrightarrow \frac{\partial u}{\partial \nu} = 0$. Contradiction.

Proof. It is clear that we can simply take $U = B$. The idea is to construct an auxliary function v with $\frac{\partial v}{\partial \nu} > 0$ on ∂B and furthermore $u + \varepsilon v$ still reaches maximum at x^0.

Without loss of generality, assume B is in fact the ball $B_r(0)$. As we cannot specify where x^0 is, necessarily v should be radially symmetric. Furthermore since we would like $u + \varepsilon v$ to reach maximum at x^0, we should take $v \equiv 0$ on ∂B.

Guided by this, we set

$$v(x) := e^{-\lambda |x|^2} - e^{-\lambda r^2} \tag{19}$$

with λ to be specified later.

Thus $u + \varepsilon v = u$ on ∂B and therefore $u\left(x^0\right) = \max_{\partial B} (u + \varepsilon v)$. To conclude that $u\left(x^0\right)$ is also larger than $u + \varepsilon v$ inside B, we need to show that

$$L(u + \varepsilon v) \leqslant 0 \tag{20}$$

inside. Or equivalently we need $L v \leqslant 0$.

We compute

$$\begin{aligned} L v &= -\sum_{i,j}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} + c v \\ &= e^{-\lambda |x|^2} \left[\sum_{i,j}^{n} a^{ij} \left(-4 \lambda^2 x_i x_j + 2 \lambda \delta_{ij} \right) - \sum_{i=1}^{n} b^i \, 2 \lambda x_i \right] + c \left(e^{-\lambda |x|^2} - e^{-\lambda r^2} \right) \\ &\leqslant e^{-\lambda |x|^2} \left(-4 \theta \lambda^2 |x|^2 + 2 \lambda \operatorname{tr} A + 2 \lambda |b| \, |x| + c \right). \end{aligned} \tag{21}$$

Now it is clear that, no matter what λ we choose, $L v \leqslant 0$ cannot hold in the whole ball B.

However, if we consider the annular region $R := B \backslash B_{r/2}(0)$, then we can take λ large enough that $L v \leqslant 0$ in R. Thus we have

$$L(u + \varepsilon v) \leqslant 0 \qquad \text{in } R; \qquad \max_{\partial B} (u + \varepsilon v) = u\left(x^0\right). \tag{22}$$

To apply the weak maximum principle, we need

$$\max_{\partial B_{r/2}} (u + \varepsilon v) \leqslant u\left(x^0\right). \tag{23}$$

This is done as follows. Since $u\left(x^0\right) > u(x)$ for all $x \in \partial B_{r/2}$, $u\left(x^0\right) > \max_{\partial B_{r/2}} u$. Thus the above is true as long as ε is small enough. ∎