Second-Order Elliptic Equations: Weak Solutions

1. Definitions.

In this and the following two lectures we will study the boundary value problem

$$L u = f \quad \text{in } U; \qquad u = 0 \quad \text{on } \partial U. \tag{1}$$

Here

$$L u = -\sum_{i,j}^{n} \left(a^{ij}(x)\, u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)\, u \equiv -\nabla \cdot (A(x)\, D u) + b(x) \cdot D u + c(x)\, u, \tag{2}$$

or

$$L u = -\sum_{i,j=1}^{n} a^{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)\, u \equiv A(x) : D^2 u + b(x) \cdot D u + c(x)\, u. \tag{3}$$

In the first case it's said the equation is in *divergence form*, in the second *nondivergence form*.

Remark 1. If $a^{ij} \in C^1$ then the two forms are basically equivalent.

Remark 2. For the equation to be elliptic, we need to assume $A(x)$ symmetric, and positive definite.

Remark 3. Non-zero boundary values can easily be incorporated. In the following we will see that the right setting for the weak solution is $u \in H^1(U)$. In this case, if $u = g \neq 0$ on the boundary, we can extend g to a H^1 function G in U[1], and then the equation for $v = u - G$ satisfies $v = 0$ on ∂U.

In the following, we require $a^{ij}, b^i, c \in L^\infty$, that is uniformly bounded.

Remark 4. Note that under such assumption, the divergence and nondivergence forms are not equivalent anymore.

Under such assumption, the following definition is meaningful. Denote by $B[\,\cdot\,,\cdot\,]$ the bilinear form

$$B[u, v] := \int_U \left[\sum_{i,j=1}^{n} a^{ij}\, u_{x_i}\, u_{x_j} + \sum_{i=1}^{n} b^i\, u_{x_i}\, v + c\, u\, v \right] \mathrm{d}x \tag{4}$$

for $u, v \in H_0^1(U)$.

Definition 5. *We say that $u \in H_0^1(U)$ is a weak solution of the problem*

$$L u = f \quad \textit{in } U; \qquad u = 0 \quad \textit{on } \partial U. \tag{5}$$

if

$$B[u, v] = (f, v) \equiv \int_U f\, v\, \mathrm{d}x \tag{6}$$

for all $v \in H_0^1(U)$.

Remark 6. Note that the boundary condition $u = 0$ does not appear in the equation $B[u, v] = (f, v)$. It's instead guaranteed by the requirement $u \in H_0^1(U)$.[2]

We try to establish a complete theory of such elliptic equations, and hope that we can obtain results similar to that of the Poisson equation, which is the special case $a^{ij} = \delta_{ij}$, $b^i = 0$, $c = 0$. In particular, we would like to be able to show that $f \in H^k \Longrightarrow u \in H^{k+2}$. It turns out that to fulfill this, we need to make one further assumption, that is the operator L is uniformly elliptic.

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1. Some regularity is required for g. More precisely we need $g \in H^{1/2}(\partial U)$. Of course, since g should be the trace of u, it must satisfy such requirement.

2. Suppose we try to form a weak solution formulation for the Neumann problem $L u = f$, $\frac{\partial u}{\partial n} = g$, what function space should we take?

Definition 7. *We say the partial differential operator L is uniformly elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\,\xi_i\,\xi_j \geqslant \theta\,|\xi|^2 \tag{7}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

2. Existence and uniqueness of weak solution.

In our general case here, it is not possible to find an explicit formula as we did for the Poisson equation. Therefore we need to show existence implicitly. One way is through the Lax-Milgram theorem. We assume $f \in L^2(U)$.

Theorem 8. (Lax-Milgram Theorem) *Let H be a real Hilbert space, with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let $\langle\cdot,\cdot\rangle$ denote the pairing of H with its dual. Let*

$$B[\cdot,\cdot]\colon H \times H \mapsto \mathbb{R} \tag{8}$$

be a bilinear mapping. Let $f\colon H \mapsto \mathbb{R}$ be a bounded linear functional on H.

With the above setting, if ther are constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \leqslant \alpha\,\|u\|\,\|v\| \qquad \textit{(boundedness)} \tag{9}$$

and

$$\beta\,\|u\|^2 \leqslant B[u,u] \qquad \textit{(Coercivity)} \tag{10}$$

then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v\rangle \tag{11}$$

for all $v \in H$.

Remark 9. In our problem, H is the space $H_0^1(U)$, whose inner product is

$$(u,v) := \int u\,v + Du \cdot Dv\,\mathrm{d}x \tag{12}$$

Its dual is the space H^{-1} which contains L^2, and for $f \in L^2 \subset H^{-1}$, the pairing with any $v \in H_0^1$ is given by

$$\langle f, v\rangle := \int_U f\,v\,\mathrm{d}x. \tag{13}$$

For the characterization of H^{-1}, see Evans 5.9.1.

Proof. First we define an operator $\tilde{A}$ through

$$\left\langle \tilde{A}\,u, v\right\rangle = B[u,v] \tag{14}$$

for all $v \in H$. As $B[u,v] \leqslant \alpha\,\|u\|\,\|v\|$, we conclude that $\tilde{A}\,u \in H^*$. Now apply the Riesz representation theorem, we can define another operator $A\colon H \mapsto H$ such that

$$(A u, v) = \left\langle \tilde{A}u, v\right\rangle \tag{15}$$

for all $v \in H$. It is easy to check that A is a bounded linear operator. Apply the Riesz representation theorem again, we can find $\tilde{f} \in H$ such that

$$\left(\tilde{f}, v\right) = \langle f, v\rangle \tag{16}$$

for all v.

Now what we need to show becomes

- Existence: For all $\tilde{f} \in H$, we can find u such that $(A u, v) = \left(\tilde{f}, v\right)$ or equivalently $A u = \tilde{f}$. In other words existence is equivalent to that $A\colon H \mapsto H$ is onto.

- Uniqueness: For any $\tilde{f} \in H$ there is at most one u such that $Au = \tilde{f}$. In other words, the mapping A is one-to-one.

It is easy to check that uniqueness follows immediately from coercivity of B. Now we show that A is onto, that is $R(A) = H$. Assume the contrary, that is $R(A)$ is a genuine closed subspace of H. Then since H is a Hilbert space, we can find a nonzero $v \in H$ such that $v \perp R(A)$. Now compute

$$0 = (Av, v) = B(v, v) \geqslant \beta \|v\|^2 \Longrightarrow v = 0 \tag{17}$$

Contradiction! □

Now we are ready to show the existence and uniqueness of our problem. Note that the boundedness of the coefficients a^{ij}, b^i, c guarantees the boundedness of B. However coercivity is not always satisfied.

Theorem 10. *There is a number $\gamma \geqslant 0$ such that for each $\mu \geqslant \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary-value problem*

$$Lu + \mu u = f \quad \text{in } U; \qquad u = 0 \quad \text{on } \partial U. \tag{18}$$

Proof. See Evans pp.300–301. □

Remark 11. From the proof we can see that for the Poisson equation, γ can be taken to be 0.[3]

3. Regularity.

Theorem 12. (Interior H^2-regularity) *Assume $a^{ij} \in C^1(U)$, b^i, $c \in L^\infty(U)$, and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE*

$$Lu = f \qquad \text{in } U. \tag{19}$$

Then

$$u \in H^2_{\text{loc}}(U) \tag{20}$$

and for each open subset $V \Subset U$, we have the estimate

$$\|u\|_{H^2(V)} \leqslant C\,(\|f\|_{L^2} + \|u\|_{L^2}). \tag{21}$$

The constant C depending only on V, U and the coefficients of L.

Remark 13. There are several points worth noticing.

1. As a^{ij} is assumed to be in C^1, it doesn't matter whether L is in divergence or nondivergence form. Note that this assumption is indeed necessary in the proof.
2. We does not require $u \in H_0^1(U)$. Thus the important point here is that, even if u only satisfies the weak form of the equation locally, then it is H^2_{loc} there.
3. "Interior" refers to the fact that our estimate

$$\|u\|_{H^2(V)} \leqslant C\,(\|f\|_{L^2} + \|u\|_{L^2}). \tag{22}$$

   cannot reach the boundary. As we will see in the proof, the constant C tends to infinity as V gets larger.

Proof. First we illustrate the idea through Poisson equation, whose weak formulation is

$$\int Du \cdot Dv = \int f v. \tag{23}$$

Now if we can set $v = -\triangle u$, then after integration by parts we have

$$\int (\triangle u)^2 = \int f \triangle u \leqslant 2 \int f^2 + \frac{1}{2} \int (\triangle u)^2 \Longrightarrow \int (\triangle u)^2 \leqslant 4 \int f^2. \tag{24}$$

3. In fact we can take γ to be negative by taking advantage of the Poincare inequalities.

Furthermore, if we can set $v = u$, the weak formulation gives

$$\int |Du|^2 = \int f u \leqslant \int f^2 + \int u^2. \tag{25}$$

The above gives

$$\int (\triangle u)^2 + \int |Du|^2 + \int u^2 \leqslant C\left(\int f^2 + \int u^2 \right) \tag{26}$$

which leads to our desired result.

Of course the major problem in the above "proof" is that we cannot take $v = -\triangle u$ and $v = u$. The way to fix this is to "cut-off".

Fix any $V \Subset U$, and choose an open set W such that $V \Subset W \Subset U$. Then we can construct a smooth function ζ such that

$$\zeta \equiv 1 \text{ on } V; \qquad \zeta \equiv 0 \text{ on } \mathbb{R}^n - W; \qquad 0 \leqslant \zeta \leqslant 1. \tag{27}$$

Now the idea is, instead of using $\triangle u$, we use $\triangle(\zeta u)$. This test function vanishes at the boundary, which is necessary for it to be in H_0^1. However, since $u \in H^1$ only, we cannot take two derivatives. So finally we have to take the following more tricky version:

$$v := -D_k^{-h}\left(\zeta^2 D_k^h u\right) \tag{28}$$

where the difference quotient

$$D_k^h u(x) := \frac{u(x + h\, e_k) - u(x)}{h}. \tag{29}$$

Now we have

$$\begin{aligned}
\int \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} \,\mathrm{d}x &= -\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} \left[D_k^{-h}\left(\zeta^2 D_k^h u\right) \right]_{x_j} \\
&= \sum_{i,j=1}^n \int_U D_k^h\left(a^{ij} u_{x_i}\right) \left(\zeta^2 D_k^h u\right)_{x_j} \\
&= \sum_{i,j=1}^n \int_U a^{ij}(x + h\, e_k) \left(D_k^h u_{x_i}\right) \left(\zeta^2 D_k^h u\right)_{x_j} \\
&\quad + \left(D_k^h a^{ij}\right) u_{x_i} \left(\zeta^2 D_k^h u\right)_{x_j} \\
&= \sum_{i,j=1}^n \int_U \zeta^2 a^{ij}(x + h\, e_k)\left(D_k^h u_{x_i}\right) \left(D_k^h u\right)_{x_j} \\
&\quad + \sum_{i,j=1}^n \int_U 2\,\zeta\,\zeta_{x_j} a^{ij}(x + h\, e_k) \left(D_k^h u_{x_i}\right) \left(D_k^h u\right) \\
&\quad + \sum_{i,j=1}^n \int_U 2\,\zeta\,\zeta_{x_j} \left(D_k^h a^{ij}\right) u_{x_i} \left(D_k^h u\right) \\
&\quad + \sum_{i,j=1}^n \int_U \zeta^2 \left(D_k^h a^{ij}\right) u_{x_i} \left(D_k^h u\right)_{x_j}.
\end{aligned} \tag{30}$$

Now the first term can be estimated as

$$\sum_{i,j=1}^n \int_U \zeta^2 a^{ij}(x + h\, e_k)\left(D_k^h u_{x_i}\right) \left(D_k^h u\right)_{x_j} \geqslant \theta \int_U \zeta^2 \left|D_k^h D u\right|^2 \geqslant \theta \int_V \left|D_k^h D u\right|^2. \tag{31}$$

If we can find a uniform (in h) upper bound for this term, we are done.

To do this, we need to give upper bounds for the other three terms. The key observation is that, each of them is bounded either by

$$C \int_U \left|D^2 u\right| |Du| \tag{32}$$

or by

$$C \int_U |Du|^2. \tag{33}$$

Same is true for all the terms coming from $b \cdot Du, c\,u$ and f. Thus we obtain

$$\|u\|_{H^2(V)} \leqslant C\,(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{34}$$

To obtain the desired result, we need to estimate $\|u\|_{H^1(U)}$ by $\|f\|_{L^2}$ and $\|u\|_{L^2}$.[4]

Unfortunately this is not possible. On the other hand fortunately the above argument works not only for U but also for any $\tilde{U} \Subset U$ satisfying $V \Subset \tilde{U}$. Thus we have

$$\|u\|_{H^2(V)} \leqslant C \left(\|f\|_{L^2(\tilde{U})} + \|u\|_{H^1(\tilde{U})} \right). \tag{35}$$

To estiamte $\|u\|_{H^1(\tilde{U})}$, we take $v = \zeta^2\,u$ with $\zeta \equiv 1$ on $\tilde{U}$ instead of V, and go through estimates similar (but simpler) to what we have done above. □

Remark 14. There is no difficulty extending this result to higher order, basically getting $f \in H^m \Longrightarrow u \in H^{m+2}_{\text{loc}}(U)$, as long as we are willing to assume $a^{ij}, b^i, c \in C^{m+1}(U)$. See Evans pp. 314–316.

Now what happens at the boundary?

Theorem 15. (Boundary H^2 regularity) *Assume $a^{ij} \in C^1(\bar{U})$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose that $u \in H^1_0(U)$ is a weak solution of the elliptic boundary-value problem*

$$Lu = f \text{ in } U; \qquad u = 0 \text{ on } \partial U. \tag{36}$$

Assume finally ∂U is C^2. Then $u \in H^2(U)$ and we have the estimate

$$\|u\|_{H^2(U)} \leqslant C\,(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{37}$$

The constant C depending only on U and the coefficients of L.

Remark 16. Note the differences!

1. $a^{ij} \in C^1(\bar{U})$ instead of $C^1(U)$.
2. Assumption on the regularity of ∂U is necessary.
3. u needs to solve the boundary value problem, not just satisfy the equation.

Proof. We only sketch the main ideas. For details see Evans pp.317– 322.

- First consider the case where the boundary is $x_n = 0$. Now we can take the cuf-off function ζ such that $\zeta \equiv 1$ on $B(0, 1/2)$ but $\equiv 0$ outside $B(0, 1)$. Now it is easy to check that

$$v := -D_k^{-h}(\zeta^2\,D_k^h u) \tag{38}$$

  can serve as a test function (that is, in H^1_0) as long as $k \neq n$. This way we can obtain bounds for all entries in $D^2 u$ except $u_{x_n x_n}$.

- To estimate $u_{x_n x_n}$, write the equation in nondivergence form, and move all terms to the RHS except for $a^{nn}\,u_{x_n x_n}$.

- Now we have

$$\|u\|_{H^2(U)} \leqslant C\,(\|f\|_{L^2} + \|u\|_{H^1}). \tag{39}$$

  This time no trick is needed to estimate $\|u\|_{H^1}$, since $u \in H^1_0$ allows us to use Poincare inequality.

- Finally, in the general case, we first do a partition of unity and then apply a change of variables to "straighten" the boundary locally. It turns out that, under the assumption $\partial U \in C^2$, the new equation still satisfies the conditions in the theorem. □

4. If we only want a bound for $\|u\|_{H^1(U)}$, no work needs to be done as u is assumed to be in H^1.

Remark 17. One can also obtain higher order versions of boundary regularity under the assumption that

$$a^{ij}, b^i, c \in C^{m+1}(\bar{U}), \quad \partial U \in C^{m+2}. \tag{40}$$

See Evans pp.323–326.