SOBOLEV INEQUALITIES AND COMPACT EMBEDDING

In the lecture we discuss the relation between different Sobolev spaces, as well as between Sobolev spaces and Hölder spaces.

1. Scaling.

It is pretty hard to remember all the Sobolev inequalities. Therefore it is important to have a way to quickly tell what kind of inequalities are possible. This can be done through scaling. Or more precisely, it is done through understanding how "bump" function behave in different function spaces.[1]

Consider a "bump" function u supported in a ball of radius l (the length scale) and with height h. Then we have

$$\|u\|_{L^q} \sim (h^q l^n)^{1/q}; \qquad \|\nabla u\|_{L^p} \sim \left(\left(\frac{h}{l}\right)^p l^n\right)^{1/p}; \qquad \sup|u| \sim h; \quad [u]_{C^{0,\alpha}} \sim \frac{h}{l^\alpha} \tag{1}$$

Then an inequality like

$$\|u\|_{L^q} \leqslant C\, \|\nabla u\|_{L^p}^a \tag{2}$$

translates to

$$h\, l^{n/q} \leqslant C\, h^a\, l^{\frac{n-p}{p}a}. \tag{3}$$

As the constant C does not depend on the specific function u (otherwise such an inequality would be useless), (3) has to hold for all possible l and h.

Varying h, we see that (3) can hold for a universal C only if $a = 1$.

Now suppose our domain is the whole space $\mathbb{R}^n$. In this case the radius of the support, l, can approach both 0 and ∞. As a consequence, we must have

$$\frac{n}{q} = \frac{n-p}{p} \Longleftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \tag{4}$$

In other words, an inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C\, \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{5}$$

can hold for all $u \in W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ only if

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \tag{6}$$

It is clear from here that $p \leqslant n$. We will see soon that for $p < n$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C\, \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{7}$$

indeed holds.

Remark 1. When $p = n$, the above gives $q = \infty$ which corresponds to

$$\|u\|_{L^\infty} \leqslant \|\nabla u\|_{L^n}. \tag{8}$$

which is incorrect. One should always be cautious when L^∞ gets involved.

For $p > n$, we have

$$\|\nabla u\|_{L^p} \sim h\, l^{\frac{n-p}{p}} = \frac{h}{l^{\frac{p-n}{p}}} \sim [u]_{C^{0,\alpha}}, \qquad \alpha = 1 - \frac{n}{p}. \tag{9}$$

This suggests

$$[u]_{C^{0,\alpha}} \leqslant C\, \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{10}$$

which also turns out to be true (See Evans 5.6.2).

1. Somehow to me, the following is easier to do than the "re-scaling" argument in Evans (p.262). However different people may feel differently.

Such inequalities, once true, tells us inclusion relations between spaces. For example, Consider $1 \leqslant p < \infty$, and $q = \frac{n\,p}{n-p}$ (that is $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$), then for any $u \in W^{1,p}(\mathbb{R}^n)$, the inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C\, \|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{11}$$

tells us that $u \in L^q$. In other words, we have

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n). \tag{12}$$

Remark 2. Such analysis applies to inequalities involving higher derivatives too. It is easy to show that for

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C\, \|D^k u\|_{L^p(\mathbb{R}^n)} \tag{13}$$

to hold, necessarily

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}. \tag{14}$$

This of course leads to the corresponding embedding

$$W^{k,p} \subset L^q. \tag{15}$$

The same analysis can also be applied to the situation involving more than two norms. For example, one can figure out the correct parameters through scaling for the following Gagliardo-Nirenberg type inequality

$$\|D u\|_{L^p} \leqslant \|u\|_{L^q}^a\, \|D^2 u\|_{L^r}^{1-a}. \tag{16}$$

Remark 3. Note that the scaling

$$h\, l^{n/q} \leqslant C\, h\, l^{\frac{n-p}{p}} \tag{17}$$

corresponds to both

$$\|u\|_{L^q} \leqslant C\, \|\nabla u\|_{L^p} \tag{18}$$

and

$$\|\nabla u\|_{L^p} \leqslant C\, \|u\|_{L^q}. \tag{19}$$

However, it is clear that it is not possible for the latter to be true.

Remark 4. If instead of $\mathbb{R}^n$, we consider a bounded domain U, then l can only approach 0, but not infinity. In other words,

$$l^a \leqslant C\, l^b \tag{20}$$

is true for $a \geqslant b$, instead of $a = b$ in the whole space case. Thus for example we have

$$\|u\|_{L^q} \leqslant C\, \|\nabla u\|_{L^p} \tag{21}$$

for all $q \leqslant \frac{n\,p}{n-p}$. But the tradeoff is that, except for $q = \frac{n\,p}{n-p}$, the constant C would depend on not only p, n, but also U.

2. Gagliardo-Nirenberg-Sobolev inequality.

We prove the follwoing. We denote

$$p^* := \frac{n\,p}{n-p} \tag{22}$$

which is usually called the Sobolev conjugate of p.

Theorem 5. (Gagliardo-Nirenberg-Sobolev inequality) *Assume $1 \leqslant p < n$. There exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\, \|D u\|_{L^p(\mathbb{R}^n)} \tag{23}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Remark 6. Note that the compact support is necessary, as the example $u \equiv 1$ shows.

Proof.

1. The case $p = 1$. In this case $p^* = \frac{n}{n-1}$.

   Since u has compact support, we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, y_i, \ldots, x_n) \, \mathrm{d}y_i, \tag{24}$$

therefore

$$|u(x)|^{\frac{n}{n-1}} \leqslant \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}. \tag{25}$$

Integrating with respect to x_1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} \mathrm{d}x_1 &\leqslant \int_{-\infty}^{\infty} \prod \left(\int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\ &= \left(\int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1. \end{aligned} \tag{26}$$

Now using the general Hölder inequality

$$\int \prod f_i \leqslant \prod \left(\int f_i^{p_i} \right)^{1/p_i} \tag{27}$$

as long as

$$p_i > 0, \qquad \sum \frac{1}{p_i} = 1, \tag{28}$$

we have

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \leqslant \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, \mathrm{d}x_1 \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}. \tag{29}$$

This gives

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} \mathrm{d}x_1 \leqslant \left(\int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, \mathrm{d}x_1 \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}. \tag{30}$$

Now integrate with respect to x_2. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} \mathrm{d}x_1 \, \mathrm{d}x_2 \leqslant \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, \mathrm{d}x_1 \, \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i \neq 2}^{n} I_i^{\frac{1}{n-1}} \mathrm{d}x_2 \tag{31}$$

with

$$I_1 = \int_{-\infty}^{\infty} |Du| \, \mathrm{d}y_1, I_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| \, \mathrm{d}x_1 \, \mathrm{d}y_i. \tag{32}$$

Using Hölder's inequality again, we obtain

$$\int \int |u|^{\frac{n}{n-1}} \mathrm{d}x_1 \, \mathrm{d}x_2 \leqslant \left(\int \int |Du| \, \mathrm{d}x_1 \, \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \left(\int \int |Du| \, \mathrm{d}y_1 \, \mathrm{d}x_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int \int \int |Du| \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}y_i \right)^{\frac{1}{n-1}}. \tag{33}$$

Integrating with respect to $x_3, \ldots, x_n$ successively, we finally obtain

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \mathrm{d}x \leqslant \left(\int_{\mathbb{R}^n} |Du| \, \mathrm{d}x \right)^{\frac{n}{n-1}}. \tag{34}$$

Thus proves the $p = 1$ case.

2. Now for the general $1<p<n$, we set $v:=|u|^\gamma$ with γ to be fixed, and apply the $p=1$ estimate to v:

$$\begin{aligned}\left(\int |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} &\leqslant \int \left|D|u|^\gamma\right| \mathrm{d}x \\ &= \gamma\int |u|^{\gamma-1}|Du|\,\mathrm{d}x \\ &\leqslant \gamma\left(\int |u|^{(\gamma-1)q}\right)^{1/q}\left(\int |Du|^p\right)^{1/p}. \end{aligned} \tag{35}$$

Here

$$\frac{1}{q}+\frac{1}{p}=1 \iff q=\frac{p}{p-1}. \tag{36}$$

Now we choose γ such that

$$\frac{\gamma n}{n-1}=(\gamma-1)\,q=\frac{(\gamma-1)\,p}{p-1} \Longrightarrow \frac{\gamma n}{n-1}=p^*. \tag{37}$$

Thus ends the proof of the general case. □

Remark 7. Similar results hold for bounded domains. See Theorems 2 and 3 on page 265 of Evans.

Now we mention the main result for the $p>n$ case.

Theorem 8. *Assume $n<p\leqslant\infty$. Then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)}\leqslant C\,\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{38}$$

for all $u\in C^1(\mathbb{R}^n)$, where

$$\gamma:=1-\frac{n}{p}. \tag{39}$$

Proof. See Evans pp. 266 - 268. □

3. Compact embedding.

We have shown that

$$W^{1,p}(U)\subset L^{p^*}(U), \tag{40}$$

and furthermore when U is bounded,

$$W^{1,p}(U)\subset L^q(U) \tag{41}$$

for all $q<p^*$ too. Now we show that in the latter case, this inclusion is in fact compact. That is, any bounded set in $W^{1,p}(U)$, when viewed in $L^q(U)$, is in fact compact. In particular, if we have a sequence u_m uniformly bounded in $W^{1,p}(U)$, we can subtract a subsequence which is converging in $L^q(U)$.

Theorem 9. (Rellich-Kondrachov Compactness Theorem) *Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1\leqslant p<n$. Then*

$$W^{1,p}(U)\Subset L^q(U) \tag{42}$$

for each $1\leqslant q<p^*$.

Proof. We sketch the proof. For details see Evans pp.272 – 274.

1. Take $\{u_m\}$ uniformly bounded in $W^{1,p}$. We need to find a subsequence which is Cauchy in L^q.
2. Use the extension theorem to extend u_m to a larger set V and such that u_m vanishes outside V.
3. Now let $u_m^\varepsilon:=\eta_\varepsilon * u_m$. It turns out that

$$u_m^\varepsilon\longrightarrow u_m \tag{43}$$

uniformly in L^q.[2]

4. Show that for each fixed $\varepsilon > 0$, $\{u_m^\varepsilon\}$ is uniformly bounded and equicontinuous. Thus by the Arzela-Ascoli theorem, for each fixed $\varepsilon > 0$, there is a subsequence of $\{u_m^\varepsilon\}$ converges uniformly, and thus converges in L^q.

5. Now for any $\delta > 0$, we can take $\varepsilon > 0$ such that
$$\|u_m^\varepsilon - u_m\|_{L^q} < \delta/3. \tag{44}$$
Now for this particular ε, we can find a subsequence, still denote as u_m^ε, and $M > 0$ such that
$$\|u_m^\varepsilon - u_{m'}^\varepsilon\| < \delta/3 \tag{45}$$
when $m, m' > M$. This means
$$\|u_m - u_{m'}\|_{L^q} < \delta/3 \tag{46}$$
as long as $m, m' > M$.

6. Now taking $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and repeatly subtract subsequences, we obtain a Cauchy sequence via the standard diagonal argument. □

2. We know that $u^\varepsilon \longrightarrow u$ for any $u \in L^q$, thus the significant thing here is that the convergence if uniform in m. This is due to the uniform boundedness of u_m in $W^{1,p}$.