EXTENSIONS AND TRACES

1. Extensions.

In this lecture we first consider the problem of extending $u \in W^{k,p}(U)$ to $u \in W^{k,p}(\mathbb{R}^n)$. The motivation for doing so is that in many cases the boundary ∂U is rather annoying.

Note that such extension is far from trivial.

Example 1. Let $U = (0,1)$ and $u \equiv 1$ on U. Then clearly $u \in W^{k,p}(U)$ for any k, p. Now we try to extend this function to a function on $\mathbb{R}^n$:

1. Define $u \equiv 0$ outside U. But one can check that the extended function $\tilde{u}$ is not in $W^{k,p}(\mathbb{R})$ for any $k \geqslant 1$.
2. Seeing that the problem in the previous extension is that discontinuities are created, we extend u to $\tilde{u} \equiv 1$ over $\mathbb{R}$. But clearly this extension is not even in L^p.

We see from the example that, to obtain a good extension, we need to

1. Keep some "continuity" across ∂U,
2. "Cut-off" somewhere outside U.

Theorem 2. *Assume U is bounded and ∂U is C^1*[1]*. Select a bounded open set V such that $U \Subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \mapsto W^{1,p}(\mathbb{R}^n) \tag{1}$$

such that for each $u \in W^{1,p}(U)$:

i. $Eu = u$ a.e. in U;

ii. Eu has support within V;

iii.

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \, \|u\|_{W^{1,p}(U)} \tag{2}$$

the constant C depending only on p, U, and V.

We call Eu the extension of u to $\mathbb{R}^n$.

Proof. We sketch the proof. For details see Evans pp.254 – 257.

1. First notice that, due to the approximation results, we only need to consider the case where $u \in C^\infty(\bar{U})$.
2. Next, using a partition of unity, we only need to deal with the extension problem in $U \cap B$ for some ball B which is divided into two parts by ∂U.
3. As ∂U is C^1, we can do a C^1-change of variable so that $U \cap B = B(r) \cap \{x_n \geqslant 0\}$.
4. Now we try to extend u to $x_n < 0$. The idea is to use a linear combination

   $$\tilde{u}(x) := \begin{cases} u(x) & x_n > 0 \\ a\, u(x_1, \ldots, x_{n-1}, -x_n) + b\, u(x_1, \ldots, x_{n-1}, -x_n/2) & x_n < 0 \end{cases}. \tag{3}$$

   We require $\tilde{u}$ and $\frac{\partial \tilde{u}}{\partial x_n}$ to be continuous across $x_n = 0$. This leads to two linear equations for a, b and thus determine them.
5. Summing all the extension together we get our desired extension. Note that the constant C in

   $$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leqslant C \, \|u\|_{W^{1,p}(U)} \tag{4}$$

1. *This time it cannot be relaxed.*

is determined by the number of partitions, the size of the ball B, the power p, and the constants a, b. Thus C depends only on U, V, p. $\square$

Remark 3. It is clear that the same proof works for $W^{k,p}$, if we assume ∂U is C^k.

The only difference is that, instead of

$$\tilde{u}(x) := \begin{cases} u(x) & x_n > 0 \\ a\,u(x_1, \ldots, x_{n-1}, -x_n) + b\,u(x_1, \ldots, x_{n-1}, -x_n/2) & x_n < 0 \end{cases}. \tag{5}$$

we need the more complicated

$$\tilde{u}(x) := \begin{cases} u(x) & x_n > 0 \\ \displaystyle\sum_{j=1}^{k+1} a_1\; u(x_1, \ldots, x_{n-1}, -x_n) + a_2\,u\Big(x_1, \ldots, x_{n-1}, -\frac{x_n}{2}\Big) + \cdots + a_{k+1}\,u\Big(\ldots, -\frac{x_n}{k+1}\Big) & x_n < 0 \end{cases}. \tag{6}$$

2. Traces.

When dealing with PDEs, we often need to perform integration by parts, or equivalently Gauss' theorem for C^1 functions

$$\int_U \nabla \cdot F = \int_{\partial U} \boldsymbol{n} \cdot F. \tag{7}$$

If we would like to use Sobolev spaces in the study of PDEs, we need to extend this formula to $W^{1,p}$ functions. Thus we need to find a good way to define the boundary values of arbitrary $W^{1,p}$ functions.

Theorem 4. (Trace Theorem) *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T\colon\; W^{1,p}(U) \mapsto L^p(\partial U) \tag{8}$$

such that

i. $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$, and

ii.

$$\|Tu\|_{L^p(\partial U)} \leqslant C\, \|u\|_{W^{1,p}(U)}, \tag{9}$$

for each $u \in W^{1,p}(U)$, with the constant C depending only on p and U.

Definition 5. *We call Tu the trace of u on ∂U.*

Proof. We sketch the main steps. for details see Evans pp. 258–259.

1. It is clear that for $u \in C^\infty(\bar{U})$, we should define $Tu = u|_{\partial U}$. Now since $C^\infty(\bar{U})$ is dense in $W^{1,p}(U)$, the only reasonable way to define Tu is through the limiting process:

$$Tu = \lim_{m \nearrow \infty} Tu_m \tag{10}$$

where $u_m \in C^\infty(\bar{U})$ converges to u in $W^{1,p}(U)$. We need to settle several issues.

   i. For any $\{u_m\}$, this limit exists;

   ii. This limit does not depend on the choice of the sequence $\{u_m\}$;

   iii. The limit satisfies

$$\|Tu\|_{L^p(\partial U)} \leqslant C\, \|u\|_{W^{1,p}(U)}, \tag{11}$$

From the last point, we see that the convergence of Tu_m should take place in L^p, and we need to establish

$$\|Tu_m\|_{L^p(\partial U)} \leqslant C\, \|u_m\|_{W^{1,p}(U)}, \tag{12}$$

for every u_m, with a constant C independent of m.

2. Existence of limit. Take any $\{u_m\} \subset C^\infty(\bar{U})$ approximating u in $W^{1,p}(U)$. All we need to show is

$$\|u_m - u_n\|_{L^p(\partial U)} \leqslant C \, \|u_m - u_n\|_{W^{1,p}(U)} \tag{13}$$

for a uniform constant C. It suffices to show that

$$\|v\|_{L^p(\partial U)} \leqslant C \, \|v\|_{W^{1,p}(U)} \tag{14}$$

for any $v \in C^\infty(\bar{U})$. We argue through as follows.

Through a partition of unity and change of variables (straighting the boundary), we only need to consider the case $U = B \cap \{x_n > 0\}$ and $u = 0$ on ∂B. In this case, we have

$$\begin{aligned} \int_{B \cap \{x_n = 0\}} |v|^p \, \mathrm{d}y &= \int_{B \cap \{x_n > 0\}} (|v|^p)_{x_n} \, \mathrm{d}x \\ &= \int_U p \, |v|^{p-1} \operatorname{sgn}(v) \, v_{x_n} \, \mathrm{d}x \\ &\leqslant C \int_U |v|^p + |v_{x_n}|^p \, \mathrm{d}x. \end{aligned} \tag{15}$$

Here we have used the Young's inequality

$$a\, b \leqslant \frac{a^p}{p} + \frac{b^q}{q} \tag{16}$$

where $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. The Young's inequality can be proved using the concavity of $\ln x$.

3. The limit is unique. If we take another approximating sequence $\{u_m'\}$, the above argument shows that

$$\|u_m - u_m'\|_{L^p(\partial U)} \leqslant C \, \|u_m - u_m'\|_{W^{1,p}(U)} \tag{17}$$

Therefore the limits are the same.

4. The bound holds. This is clear from the above. □

Now it is easy to see that the Gauss theorem

$$\int_U \nabla \cdot F = \int_{\partial U} \boldsymbol{n} \cdot F \tag{18}$$

still holds for $F \in W^{1,p}(U)$.

Another application of the trace operator is an alternative characterization of $W_0^{1,p}(U)$. Recall that, by definition $W_0^{1,p}(U)$ is the closure of $C_0^\infty(U)$ in $W^{1,p}(U)$. But this characterization is almost useless in practice. We have the following more user-friendly one.

Theorem. *Assume U is bounded and ∂U is C^1. Suppose furthermore that $u \in W^{1,p}(U)$. Then*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U. \tag{19}$$

Proof. "$\Longrightarrow$" is trivial. Now we prove the other direction, that is $Tu = 0$ implies $u \in W_0^{1,p}(U)$. The difficulty of this direction lies in the fact that $Tu = 0$ does not imply that we can find approximating sequences with zero boundary values.

However, due to

$$\|Tu\|_{L^p(\partial U)} \leqslant C \, \|u\|_{W^{1,p}(U)}, \tag{20}$$

We know that for any approximating sequence $u_m \in C^\infty(\bar{U})$,

$$\|u_m\|_{L^p(\partial U)} \longrightarrow 0 \tag{21}$$

as $m \nearrow \infty$. The idea now is to modify u_m and obtain an approximating sequence in $C_0^\infty(U)$.

The most natural way to do this is to "cut-off". Take $V_m \Subset U$ and let $\zeta \in C_0^\infty(U)$ be such that $\zeta_m \equiv 1$ on V_m. Now let $v_m := \zeta \, u_m \in C_0^\infty$. The question is, do we have

$$\|v_m - u_m\|_{W^{1,p}(U)} \longrightarrow 0 \tag{22}$$

as $m\nearrow\infty$?

To make things simple, we notice that the $u\in W^{1,p}(U)$ and $Tu=0$ does not change under change of variables. Thus we can apply a partition of unity and then a "straightening of boundary", to reduce our problem to the case

$$u\in W^{1,p}(\mathbb{R}^n_+)\cap C_0^\infty(\mathbb{R}^n),\qquad Tu=0. \tag{23}$$

Now take $V_m=\{x_n>1/m\}$ and take ζ_m accordingly. We estimate

$$\begin{aligned}\|\zeta_m u-u\|_{W^{1,p}(\mathbb{R}^n_+)} &\leqslant \|\zeta_m u-u\|_{L^p(\mathbb{R}^n_+)}+\sum_{i=1}^n \|\partial_{x_i}(\zeta_m u)-\partial_{x_i}u\|_{L^p(\mathbb{R}^n_+)}\\ &\leqslant \|\zeta_m u-u\|_{L^p(\mathbb{R}^n_+)}+\sum_{i=1}^n \|\zeta_m(\partial_{x_i}u)-\partial_{x_i}u\|_{L^p(\mathbb{R}^n_+)}\\ &\quad+\|(\partial_{x_n}\zeta_m)\,u\|_{L^p(0<x_n<1/m)}.\end{aligned} \tag{24}$$

where we have used the fact that $\partial_{x_i}\zeta_m=0$ for all $i\neq n$, and also $\partial_{x_n}\zeta_m=0$ for $x_n>1/m$.

The first two terms clearly converges to 0 as $m\nearrow\infty$. For the 3rd term, we have $\partial_{x_n}\zeta_m\sim 1/m$, thus we need to show that

$$m\,\|u\|_{L^p(0<x_n<1/m)} \tag{25}$$

can be as small as we like, given that $\|u\|_{L^p(x_n=0)}$ is as small as we like. To see this, we use

$$u(y,x_n)=u(y,0)+\int_0^{x_n}(\partial_{x_n}u)\,\mathrm{d}x_n. \tag{26}$$

This leads to

$$\begin{aligned}\|u\|^p_{L^p(0<x_n<1/m)} &= \int_0^{1/m}\left[\int_{\mathbb{R}^{n-1}} u(y,x_n)^p\,\mathrm{d}y\right]\mathrm{d}x_n\\ &\leqslant \int_0^{1/m}\left[\int\left(|u(y,0)|+\int_0^{x_n}(\partial_{x_n}u)\,\mathrm{d}s\right)^p\mathrm{d}y\right]\mathrm{d}x_n\\ &\leqslant C\int_0^{1/m}\int u(y,0)^p\,\mathrm{d}y\,\mathrm{d}x_n+\int_0^{1/m}\left[\int\left(\int_0^{x_n}|Du|\,\mathrm{d}s\right)^p\mathrm{d}y\right]\mathrm{d}x_n\end{aligned} \tag{27}$$

Clearly for any $\varepsilon>0$, the first term can be taken smaller than ε/m^p and thus is no problem. We need to show that the 2nd term is also of order $o(1/m^p)$. To show this, we use Hölder inequality

$$\int_0^{x_n}|Du|\,\mathrm{d}x_n\leqslant\left(\int_0^{x_n}1\right)^{\frac{p-1}{p}}\left(\int_0^{x_n}|Du|^p\,\mathrm{d}x_n\right)^{1/p}. \tag{28}$$

This leads to

$$\begin{aligned}\int_0^{1/m}\left[\int\left(\int_0^{x_n}|Du|\,\mathrm{d}s\right)^p\mathrm{d}y\right]\mathrm{d}x_n &\leqslant \int\left[\int_0^{1/m}\left(x_n^{p-1}\int_0^{x_n}|Du|^p\,\mathrm{d}s\right)\mathrm{d}x_n\right]\mathrm{d}y\\ &\leqslant \int\left[\int_0^{1/m}\left(x_n^{p-1}\int_0^{1/m}|Du|^p\,\mathrm{d}s\right)\mathrm{d}x_n\right]\mathrm{d}y\\ &= \left(\int_0^{1/m}x_n^{p-1}\,\mathrm{d}x_n\right)\left(\int_{0<x_n<1/m}|Du|^p\,\mathrm{d}x\right)\\ &= Cm^{-p}\left(\int_{0<x_n<1/m}|Du|^p\,\mathrm{d}x\right).\end{aligned} \tag{29}$$

The desired result follows as

$$\int_{0<x_n<1/m}|Du|^p\,\mathrm{d}x\longrightarrow 0 \tag{30}$$

as $m\nearrow\infty$. □