SOBOLEV SPACES: DEFINITIONS AND BASIC PROPERTIES

1. Motivation.

The invention and development of Sobolev spaces are motivated by the study of elliptic PDEs, for example the Poisson equation

$$\triangle u = f \tag{1}$$

and more importantly the more complicated general 2nd order elliptic equations

$$\sum a_{ij}(x)\frac{\partial u}{\partial x_i \partial x_j} + \sum b_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u(x) = f. \tag{2}$$

For PDEs, a natural question is the regularity of solutions. For example, assuming f has certain regularity, how regular can u be?

In calculus, the most natural regularity setting is differentiability. The regularity of a function is measured by how many derivatives one can take of it. Thus naturally one would ask, suppose $f \in C^k$, then $u \in C^?$?

Let's consider the Poisson equation. The situation in 1D is very clear. In this case the equation reduces to

$$u'' = f \tag{3}$$

and naturally $f \in C^k \Longrightarrow u \in C^{k+2}$. It is also clear that this result is optimal.

However the situation changes once we move to higher dimensions. $f \in C^k \Longrightarrow u \in C^{k+2}$ is not true anymore.

Example 1. (f continuous but $u \notin C^2$).

$$\triangle u = f(x) \equiv \frac{x_2^2 - x_1^2}{2\,|x|^2}\left[\frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2\,(-\log|x|)^{3/2}}\right], \qquad x \in B_R \subset \mathbb{R}^n. \tag{4}$$

$f(x)$ is continuous after setting $f(0) = 0$.

However, the solution

$$u(x) = \left(x_1^2 - x_2^2\right)(-\log|x|)^{1/2} \tag{5}$$

has

$$\frac{\partial^2 u}{\partial x_1^2} = 2\,(-\log|x|)^{1/2} + \text{bounded terms} \longrightarrow \infty \qquad x \to 0. \tag{6}$$

Therefore $u \notin C^2$.[1]

However, one can show that the conjecture "The regularity of u is better than that of f by the order of 2" is in fact true, as soon as we use a slightly different way to measure regularity.

Definition 2. (Hölder continuity) *Let $f: \Omega \mapsto \mathbb{R}$, $x_0 \in \Omega$, $0 < \alpha < 1$. The function f is called Hölder continuous at x_0 with exponent α if*

$$\sup_{x \in \Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\alpha}} < \infty. \tag{7}$$

f is called Hölder continuous in Ω if it is Hölder continuous at each $x_0 \in \Omega$ (with the same exponent α), denoted $f \in C^{\alpha}(\Omega)$.

When $\alpha = 1$, f is called Lipschitz continuous at x_0, denoted $f \in \mathrm{Lip}(\Omega)$ or $f \in C^{0,1}(\Omega)$.

1. In fact one can show that there is no classical solution to this problem. Assume otherwise a classical solution v exists, then the difference $u - v$ is a bounded harmonic function in $B_R \backslash \{0\}$. Such functions can be extended as a harmonic function in the whole B_R which means $\nabla^2 u$ must be bounded, a contradiction.

$C^{k,\alpha}(\bar{\Omega})$ contains $f \in C^k(\bar{\Omega})$ whose kth derivatives are uniformly Hölder continuous with exponent α over $\bar{\Omega}$, that is

$$\sup_{x,y\in\bar{\Omega}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} < \infty. \tag{8}$$

$C^{k,\alpha}(\Omega)$ contains $f \in C^k(\Omega)$ whose kth derivatives are uniformly Hölder continuous with exponent α in every compact subset of Ω.

Example 3. The functions $f(x) = |x|^{\alpha}$, $0 < \alpha < 1$, is Hölder continuous with exponent α at $x = 0$. It is Lipschitz continuous when $\alpha = 1$.

We see that $f \in C^{k,\alpha}$ has regularity between C^k and C^{k+1}. It turns out that, not only for the Poisson equation, but also for the general elliptic equation, we have (roughly speaking)

$$f \in C^{k,\alpha} \Longrightarrow u \in C^{k+2,\alpha}. \tag{9}$$

One drawback of the Hölder spaces is that analysis in this framework is usually highly technical. Fortunately there is one other framework which is much more user-friendly and is at least as powerful – the weak solution setting in Sobolev spaces.[2]

The weak solution setting of the Poisson equation is as follows. Recall the Poisson equation

$$\triangle u = f. \tag{10}$$

Now we multiply it with $\phi \in C_0^{\infty}$, integrate, and then integrate by parts, we get

$$-\int \nabla u \cdot \nabla \phi = \int f \phi. \tag{11}$$

One can show that for any $u \in C^2$, if the above holds for all ϕ, then u is a classical solution of the original Poisson equation. Therefore, it makes sense to use this integral relation as a definition for the solution.

Notice that, for the integral relation to make sense, we do not need $u \in C^2$. All we need is $\nabla u \in L^p$ for some p. As a consequence, the natural function spaces for the study of this weak formulation is the space requiring $\{u\colon\ \nabla u \in L^p\}$. It is clear that such a requirement does not follow from $u \in C^{k,\alpha}$ for any k, α.[3] These new spaces are Sobolev spaces.

2. Definitions.

As $\{u\colon\ \nabla u \in L^p\}$ is independent of $u \in C^{k,\alpha}$, we have to deal with the case when $u \notin C^1$, and thus ∇u is not the usual derivative and needs to be defined first.

Definition 4. (Weak partial derivatives) *Suppose $u, v \in L^1_{\mathrm{loc}}(U)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^{\alpha} u = v \tag{12}$$

provided

$$\int_U u\, D^{\alpha}\phi\, \mathrm{d}x = (-1)^{|\alpha|} \int_U v\, \phi\, \mathrm{d}x. \tag{13}$$

One can show that

1. If $u \in C^k$, then for $|\alpha| \leqslant k$, $D^{\alpha} u$ equals the classical α-partial derivative of u.
2. Weak derivatives are uniquely defined. (Evans p.243)

Example 5. Let $u = \begin{cases} x & x > 0 \\ 0 & x < 0 \end{cases}$. Then $u' = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$, $u'' = \delta$.[4]

---

2. There seem to be cases where working in Hölder spaces yield more. An example is the recent development on the global regularity issue of the critical dissipative surface quasi-geostrophic equation. Using Hölder spaces theory, Caffarelli and Vasseur succeeded in proving the global regularity while up to now no one has been successful in establishing this using Sobolev spaces.

3. Unless the domain is bounded.

Now we can define Sobolev spaces. Let $1 \leqslant p \leqslant \infty$ and k be a nonnegative integer.

Definition. (Sobolev Spaces) *The Sobolev space $W^{k,p}(U)$ consists of all $L^1_{\mathrm{loc}}(U)$ functions u such that for each multiindex α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.*

Remark 6. When $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U). \tag{14}$$

It turns out that we can define a norm on $W^{k,p}$ and make it a Banach space. This norm is defined as

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \int_U |D^\alpha u|^p \, \mathrm{d}x \right)^{1/p} & (1 \leqslant p < \infty) \\ \sum_{|\alpha| \leqslant k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty) \end{cases}. \tag{15}$$

Naturally, we say $\{u_m\} \subset W^{k,p}(U)$ converges to $u \in W^{k,p}(U)$, writing

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}(U) \tag{16}$$

if

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0. \tag{17}$$

We also use the notation

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}_{\mathrm{loc}}(U) \tag{18}$$

to mean

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}(V) \tag{19}$$

for every $V \Subset U$.

It is clear that $C_0^\infty(U) \subset W^{k,p}(U)$ for every k, p. As a consequence, we can consider the closure of $C_0^\infty(U)$ in $W^{k,p}(U)$. It turns out that the resulting space is in general smaller than $W^{k,p}(U)$. We denote it by $W_0^{k,p}(U)$. One can easily shown that for any $u \in W_0^{k,p}(U)$, there is $\{u_m\} \subset C_0^\infty(U)$ such that

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}(U). \tag{20}$$

3. Basic Properties.

Theorem 7. (Properties of weak derivatives) *Assume $u, v \in W^{k,p}(U)$, $|\alpha| \leqslant k$. Then*

i. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha\big(D^\beta u\big) = D^{\alpha+\beta} u$ for all multiindices α, β with $|\alpha| + |\beta| \leqslant k$.

ii. For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leqslant k$.

iii. If V is an open subset of U, then $u \in W^{k,p}(V)$.

iv. If $\zeta \in C_0^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u. \tag{21}$$

Proof. See Evans pp.247-248. $\square$

Theorem 8. *$W^{k,p}(U)$ is a Banach space.*

Proof. See Evans p. 249. $\square$

4. Approximation.

4. Of course $\delta \notin L^1_{\mathrm{loc}}$.

One of the main reasons why Sobolev spaces are highly popular is the following approximation property.

Theorem 9. (Global approximation by smooth functions) *Assume U is bounded, and suppose $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$*[5] *such that*

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}(U). \tag{22}$$

Proof. We sketch the major steps. Fix any $\varepsilon > 0$. It suffices to construct $v \in C^\infty(U)$ such that

$$\|v - u\|_{W^{k,p}(U)} < \varepsilon. \tag{23}$$

1. First notice that, for any $U_\delta := \{x \in U \colon \operatorname{dist}(x, \partial U) > \delta\}$, $\rho^\epsilon * u \to u$ in $W^{k,p}(U_\delta)$. In other words, we can find C^∞ function so that the difference in $W^{k,p}(U_\delta)$ is as small as we want.
2. Now we write $U = \cup_{i=1}^\infty U_i$, with
$$U_i := \{x \in U \colon \operatorname{dist}(x, \partial U) > 1/i\}. \tag{24}$$
Let $V_i = U_{i+3} \backslash \bar{U}_{i+1}$. choose V_0 such that $U = \cup_{i=0}^\infty V_i$.
3. Let ζ_i be the partition of unity subordinate to the partition $\{V_i\}$.
4. Now for each i, choose a smooth function v_i such that
$$\|v_i - \zeta_i u\|_{W^{k,p}(U)} < \frac{\varepsilon}{2^{i+1}}. \tag{25}$$
5. Sum up the v_is.

For details see Evans pp.250 – 252. □

Theorem 10. (Global approximation by smooth functions up to the boundary) *Assume U is bounded and ∂U is C^1. Suppose $u \in W^{k,p}(U)$ for some $1 \leqslant p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{U})$*[6] *such that*

$$u_m \longrightarrow u \qquad \text{in } W^{k,p}(U). \tag{26}$$

Proof. See Evans pp.252–254. □

Remark 11. The assumption ∂U is C^1 may be a bit misleading. In fact, checking the proof, we see that no where is the differentiability of the boundary used. All we need is the local representation (after rotation and relabeling of the axes) $U \cap B(x_0, r) = \{x \in B(x_0, r) \colon x_n > \gamma(x_1, \ldots, x_{n-1})\}$ for some function γ.

Remark 12. The benefit of these approximation theorems will be seen soon. Roughly speaking, these theorems allow us to do calculation involving $W^{k,p}$ functions as if they are C^∞.

---

5. *Try to figure out why $C^\infty(U) \notin W^{k,p}(U)$.*
6. *Try to understand the difference between $C^\infty(U)$ and $C^\infty(\bar{U})$.*