Similarity Solutions

In previous lectures, we have derived solution formulas for quite a few linear and nonlinear PDEs. We have also seen that this can only be done for simple equations. Furthermore, even when we can obtain explicit or semi-explicit formulas, such formulas may be too complicated to be useful. For example, for the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1}$$

the Lax-Oleinik formula

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right) = \frac{x - y(x,t)}{t} \tag{2}$$

does not tell us much about the solutions due to the fact that we do not know where $y(x, t)$ is. It turns out that, except for the simplest equations, certain special solutions help much more in our understanding of the equations.

For example, for the Burgers equation, we consider the special case when the initial value is either $u_0 = \left\{ \begin{array}{ll} 0 & x<0 \\ 1 & x>0 \end{array} \right.$ or $u_0 = \left\{ \begin{array}{ll} 1 & x<0 \\ 0 & x>0 \end{array} \right.$. We have seen that in the former case the solution can be written as

$$u(x,t) = U(\xi), \qquad \xi = \frac{x}{t}, \tag{3}$$

and in the latter case

$$u(x,t) = U(\xi), \qquad \xi = x - \sigma t. \tag{4}$$

By assembling such special solutions, we can get immediate understanding of the problem with more complicated initial values.

The two special solutions are of the two most popular types of similarity solutions, the former is called a "self-similar solution", where the solution at different times can be obtained through re-scaling; the latter is called "traveling wave solution", where the solution at different times can be obtained through translation.

Remark 1. The similarity assumption $u(x,t) = U(\xi)$ reduces the original PDE to an ODE, which is usually easier to solve. Before the age of computers, this is basically the only way to obtain some concrete understanding of the equations. Nowadays, such understanding can be obtained better through numerical computation. However, similarity solutions are still important due to

1. Similarity solutions often describe certain asymptotical behavior of the general solutions;
2. Similarity solutions can help in the theoretical study via properties of equations, like the maximum principles, which allows us to compare general solutions with similarity solutions.

Remark 2. If we let $x = \ln y, t = \ln \tau$, then

$$x - \sigma t = \ln\left(\frac{x}{t^\sigma}\right). \tag{5}$$

This turns a traveling wave to a self-similar solution.

1. Self-similar solutions.

Example 3. (Rarefaction wave) We consider the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{6}$$

and try to find self-similar solutions of the form

$$u(x,t) = U(\xi), \qquad \xi = \frac{x}{t}. \tag{7}$$

Substituting this ansatz into the equation we have

$$-\xi U' + U U' = 0 \implies U = \xi. \tag{8}$$

In other words

$$u(x,t)=\frac{x}{t}. \tag{9}$$

Remark 4. One can "guess" the form of the self-similar ansatz $\xi=\frac{x}{t}$ through dimensional analysis. Consider the change of variables $x=\lambda\,x', t=\mu\,t'$. Let $u'(x',t')=u(x,t)$. Then

$$u_t=\mu^{-1}\,u'_{t'}, \qquad \left(\frac{u^2}{2}\right)_x=\lambda^{-1}\left(\frac{u'^2}{2}\right)_{x'}. \tag{10}$$

We see that the equation does not change as long as $\lambda=\mu$. In this case the ratio x/t does not change. Therefore we should try using x/t as the new variable.

Example 5. (Porous medium) The porous medium equation reads

$$u_t-\triangle(u^\gamma)=0 \tag{11}$$

with $u\geqslant 0$ and $\gamma>1$.

We look for a solution having the form

$$u(x,t)=\frac{1}{t^\alpha}\,v\!\left(\frac{x}{t^\beta}\right). \tag{12}$$

Substituting into the equation we obtain

$$\alpha\,t^{-(\alpha+1)}\,v(y)+\beta t^{-(\alpha+1)}\,y\cdot Dv(y)+t^{-(\alpha\gamma+2\beta)}\,\triangle(v^\gamma)(y)=0. \tag{13}$$

Since the idea of introducing the self-similar ansatz is to simplify the equation (by reducing the number of variables), we try to cancel the t terms, by setting

$$\alpha+1=\alpha\,\gamma+2\,\beta. \tag{14}$$

Then the equation reduces to

$$\alpha\,v+\beta\,y\cdot Dv+\triangle(v^\gamma)=0. \tag{15}$$

This is still a PDE with n variables. To be able to solve it, we consider only the radial symmetric solution

$$v(y)=w(|y|). \tag{16}$$

Then the equation for w is

$$\alpha\,w+\beta\,r\,w'+(w^\gamma)''+\frac{n-1}{r}\,(w^\gamma)'=0. \tag{17}$$

Now if we further set

$$\alpha=n\,\beta, \tag{18}$$

we have (after multiplying the equation by r^{n-1})

$$\left(r^{n-1}\,(w^\gamma)'\right)'+\beta\,(r^n\,w)'=0 \implies r^{n-1}(w^\gamma)'+\beta\,r^n\,w=a. \tag{19}$$

To settle a, we assume that w decays at infinity, that is w, $w'\longrightarrow 0$ as $r\longrightarrow\infty$. This leads to $a=0$. Therefore

$$(w^\gamma)'=-\,\beta\,r\,w. \tag{20}$$

This gives

$$-\,\beta\,r\,w=\gamma\,w^{\gamma-1}\,w' \implies -\,\beta\,r=\gamma\,w^{\gamma-2}\,w' \implies \left(w^{\gamma-1}\right)'=-\frac{\gamma-1}{\gamma}\,\beta\,r. \tag{21}$$

consequently

$$w^{\gamma-1}=b-\frac{\gamma-1}{2\,\gamma}\,\beta\,r^2 \tag{22}$$

which is

$$w=\max\left\{\left(b-\frac{\gamma-1}{\gamma}\,\beta\,r^2\right)^{\frac{1}{\gamma-1}},0\right\}. \tag{23}$$

Finally, returning to the original variables, we have the *Barenblatt solution*

$$u(x,t) = \left\{ \frac{1}{t^{\alpha}} \left(b - \frac{\gamma - 1}{2\,\gamma}\,\beta\,\frac{|x|^2}{t^{2\beta}} \right)^{\frac{1}{\gamma-1}}, 0 \right\}. \tag{24}$$

Remark 6. As the porous medium equation has maximum principles, the Barenblatt solution has been used to show properties of general solutions.

2. Traveling waves.

Example 7. (Solitons) Consider the KdV equation

$$u_t + 6\,u\,u_x + u_{xxx} = 0. \tag{25}$$

We look for solutions of the form

$$u(x,t) = v(x - \sigma\,t). \tag{26}$$

Substituting into the equation we have

$$-\sigma\,v' + 6\,v\,v' + v''' = 0. \tag{27}$$

Integrating once

$$-\sigma\,v + 3\,v^2 + v'' = a. \tag{28}$$

To integrate further, we multiply the equation by v'

$$-\sigma\,v\,v' + 3\,v^2\,v' + v'\,v'' = 0 \implies \frac{(v')^2}{2} = -\,v^3 + \frac{\sigma}{2}\,v^2 + a\,v + b. \tag{29}$$

We try to find solutions which decay to 0, this leads to $a = b = 0$. And the equation becomes

$$v' = \pm\,v\,(\sigma - 2\,v)^{1/2}. \tag{30}$$

Note that, if we let $s \rightarrow -\,s$, then v' changes sign but the RHS does not. Therefore once we solve the equation for one sign, setting $s \rightarrow -\,s$ would give us the solution to the equation with the opposite sign.

We take the minus sign. Let $v = \frac{\sigma}{2}\,w^2$. Then the equation for w is

$$w' = -\,\frac{\sqrt{\sigma}}{2}\,w\,\left(1 - w^2\right)^{1/2}. \tag{31}$$

Now inspired by $\left(1 - w^2\right)^{1/2}$ and keeping in mind that we need w decay at infinity, we set

$$w = \operatorname{sech}\theta = \frac{2}{e^{\theta} + e^{-\theta}}, \qquad \theta > 0. \tag{32}$$

We have

$$w' = -\,\frac{2\left(e^{\theta} - e^{-\theta}\right)}{\left(e^{\theta} + e^{-\theta}\right)^2}\,\theta',\ \left(1 - w^2\right)^{1/2} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}. \tag{33}$$

Substituting into the equation we obtain

$$\theta' = \frac{\sqrt{\sigma}}{2} \implies \theta(s) = \frac{\sqrt{\sigma}}{2}\,(s - c) \tag{34}$$

where c is a constant.

Putting everything together we have

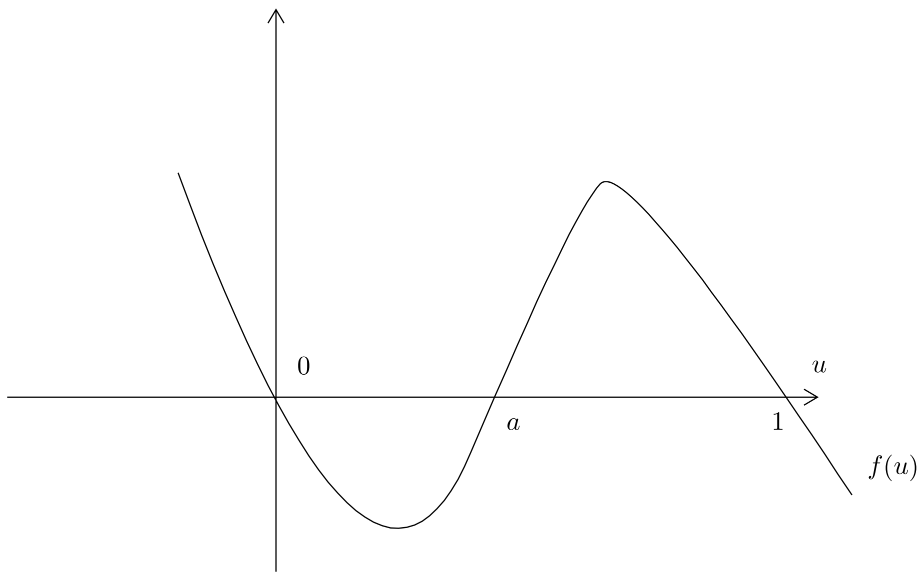
$$u(x,t) = \frac{\sigma}{2}\operatorname{sech}^2\!\left(\frac{\sqrt{\sigma}}{2}(x - \sigma\,t - c) \right). \tag{35}$$

Note that there is no restriction on σ. In other words, there exists traveling wave solutions for each speed σ.

Example 8. (A bistable equation) Consider the reaction-diffusion equation

$$u_t - u_{xx} = f(u) \tag{36}$$

with f having a "cubic-like" shape



If we omit the u_{xx} term, then we see that 0 and 1 are stable fixed points, while a is unstable. Thus we try to find a traveling wave solution satisfying $u(-\infty)=0, u(+\infty)=1$.

Setting $u(x,t)=v(x-\sigma t)$, the equation becomes

$$-\sigma v' - v'' = f(v) \tag{37}$$

Our task is to show that one can find σ such that the desired solution exists. Thus this is a non-linear "eigenvalue" problem.

Now let $w=v'$, we have the following ODE system

$$v' = w, \tag{38}$$
$$w' = -\sigma w - f(v). \tag{39}$$

This system has three fixed points: $(0,0),(a,0),(1,0)$. Linearize at any fixed point we have

$$v' = w \tag{40}$$
$$w' = -f' v - \sigma w \tag{41}$$

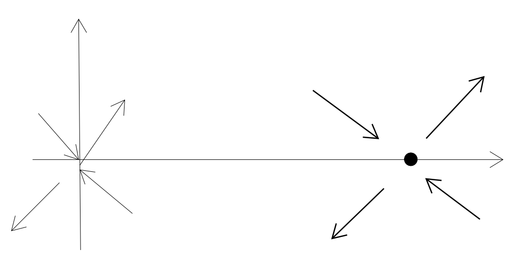
whose eigenvalue equation is

$$\lambda^2 + \sigma \lambda + f' = 0. \tag{42}$$

with eigenvectors

$$w = \lambda v. \tag{43}$$

Now at 0 and 1, we have $f' < 0$ which means we have one positive eigenvalue and one negative eigenvalue with corresponding eigenvectors as follows:

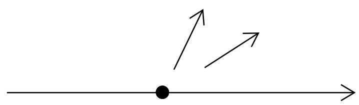


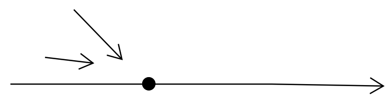
On the other hand, at $(a, 0)$ we have two cases:

i. $\sigma > 0$. In this case both eigenvalues are positive.

ii. $\sigma < 0$. In this case both eigenvalues are negative.

The two cases are illustrated as follows:





From this analysis we know that there is one trajectory starting from (0, 0) into the first quadrant. Now the task is to show the existence of σ such that this trajectory ends at $(1, 0)$.

We first consider the case $\sigma < 0$, as in the book. The first task is to show that this trajectory cannot end at $(a, 0)$. To see this, consider the functional

$$E(v, w) := \frac{w^2}{2} + \int_0^v f(z)\,\mathrm{d}z. \tag{44}$$

As $f < 0$ on $(0, a)$ and > 0 on $(a, 1)$, we can draw the level curves of E as on page 178, if we assume $\int_0^1 f(z)\,\mathrm{d}z > 0$. Now differentiating E we find out

$$\frac{\mathrm{d}}{\mathrm{d}t} E \geqslant 0 \tag{45}$$

with "=" only when $w = 0$ or $\sigma = 0$.

From this analysis we see that the trajectory leaving $(0, 0)$ stays out side of the region $E(v, w) \leqslant E(0, 0)$, and therefore has to cross the vertical line $L: v = a + \varepsilon$. Same holds for the (backward) trajectory from $(1, 0)$. Now we need to show that there is $\sigma < 0$ such that the intersections of these two trajectories with L are the same.

Denote these two points by $(a + \varepsilon, w_0(\sigma))$ and $(a + \varepsilon, w_1(\sigma))$. First we see that $w_0(0) < w_1(0)$. On the other hand, one can show that when $\sigma < 0$ is negative enough, $w_0(\sigma) > w_1(\sigma)$ (for details see p. 179-180 of Evans). Thus there must be $\sigma < 0$ such that the two trajectories are "connected into one".

Remark 9. For $\sigma > 0$ naturally one should assume $\int_0^1 f(z)\,\mathrm{d}z < 0$. Does the proof still work?

Further Reading. For a thorough discussion of similarity solutions, see G. I. Barenblatt "Scaling, self-similarity, and intermediate asymptotics", Cambridge University Press, 1996.