The Single Conservation Law: Existence, Uniqueness, Asymptotics

In the last lecture we see that an appropriate notion of solutions to the single conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0 \tag{1}$$

where $f'' > 0$ is the *entropy solution*, which is a weak solution satisfying the entropy condition:

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t} \tag{2}$$

for all $a > 0, t > 0$ and a fixed constant E. In this lecture we will first establish the existence and uniqueness of such solutions, and then study their asymptotic behaviors.

1. Existence of entropy solution.

We will not go into the details of the existence proofs, but just mention several approaches.

1. Explicit formula.

   Consider the conservation law

   $$u_t + f(u)_x = 0 \tag{3}$$

   with f uniformly convex.[1] Now consider the Hamilton-Jacobi equation

   $$v_t + f(v_x) = 0. \tag{4}$$

   If we differentiate it with respect to x and then let $u := v_x$, we see that we have recovered the conservation law.

   Furthermore, notice that when studying conservation law, we usually require u_0 to be bounded, which translates to the boundedness of v_0', that is the Lipschitz continuity of v_0.

   We know that the Hopf-Lax formula

   $$v(x,t) = \min\left\{ t\,L\left(\frac{x-y}{t}\right) + v_0(y) \right\} \tag{5}$$

   gives the correct weak solution for the Hamilton-Jacobi equation, where L is the Legendre transform of f. This leads to the conjecture that

   $$u(x,t) := \frac{\partial}{\partial x}\left[\min\left\{ t\,L\left(\frac{x-y}{t}\right) + v_0(y) \right\} \right] \tag{6}$$

   gives the entropy solution for the single conservation law.

   Let $y(x,t)$ be the minimizer, we have,

   $$u(x,t) = \frac{\partial}{\partial x}\left[t\,L\left(\frac{x - y(x,t)}{t}\right) + v_0(y(x,t)) \right]. \tag{7}$$

   The a.e. differentiability of the function in the parenthesis is established in the proof of Theorem 1 on page 146 of Evans. Taking the derivative we have

   $$u(x,t) = L'\left(\frac{x - y(x,t)}{t}\right)(1 - y_x(x,t)) + \frac{\partial}{\partial x} v_0(y(x,t)). \tag{8}$$

   Now notice that as $y(x,t)$ is the minimizer of

   $$t\,L\left(\frac{x-y}{t}\right) + v_0(y), \tag{9}$$

   x is the minimizer of

   $$t\,L\left(\frac{x - y(z,t)}{t}\right) + v_0(y(z,t)). \tag{10}$$

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1. One can show that for solutions of single conservation laws, $\sup_x |u(x,t)| \leqslant \sup_x |u_0(x)|$, therefore if u_0 is initial bounded and $f \in C^2$, $f'' > 0 \Longrightarrow f'' > \theta > 0$. So the uniform convexity is not really some extra requirement.

As a consequence

$$L'\left(\frac{x-y(x,t)}{t}\right)(-y_x(x,t))+\frac{\partial}{\partial x}v_0(y(x,t))=0. \tag{11}$$

This leads to

$$u(x,t)=L'\left(\frac{x-y(x,t)}{t}\right). \tag{12}$$

Now what is L'? Notice that as $L=f^*$, we have

$$L(x)=\sup_y \{x\,y-f(y)\}. \tag{13}$$

For almost every x, we can find a maximizer $y=y(x)$. Now simple calculation gives

$$L'(x)=y(x), \qquad f'(y(x))=x. \tag{14}$$

Thus

$$f'(L'(x))=x. \tag{15}$$

In other words, L' is the inverse function of f'.

One can check that the solution given by

$$u(x,t)=(f')^{-1}\left(\frac{x-y(x,t)}{t}\right) \tag{16}$$

is indeed a weak solution and satisfy the entropy condition. See pp. 148 - 150 of Evans.

2. Approximating using discrete solution.

   This approach is taken by J. Smoller in "Shock Waves and Reaction-Diffusion Equations" (Chapter 16, also see Lecture 20 of Fall 2008 Math 527). The basic idea is as follows.

   Consider the following finite difference discretization of the equation

$$\frac{u_n^{k+1}-\left(u_{n+1}^k+u_{n-1}^k\right)/2}{\Delta t}+\frac{f\left(u_{n+1}^k\right)-f\left(u_{n-1}^k\right)}{2\,\Delta x}=0 \tag{17}$$

   with initial values

$$u_n^0\equiv u_0(n\,\Delta x) \tag{18}$$

   It can be shown that the discrete solution exists for all (discrete) time, and satisfies the discrete versions of the conditions for the entropy solution. Then one can show that as the grid size tend to 0, a subsequence of the discrete solutions converges. Finally one shows that the limit function is an entropy solution.

3. Approximating using vanishing viscosity.

   This is a very popular approach in studying conservation laws. Consider the following reaction-diffusion equation

$$u_t+f(u)_x=\varepsilon\,u_{xx}. \tag{19}$$

   One can show that for any $\varepsilon>0$, the classical solution to this equation exists for all time. Then one can try to show that as $\varepsilon\searrow 0$, the solutions converge to an entropy solution of the conservation law.

4. Semigroup method.

   This approach applies the abstract theory of semigroups. See 2.6 of Denis Serre "Systems of Conservation Laws 1: Hyperbolicity, Entropies, Shock Waves" if you are interested.

2. Uniqueness of entropy solution.

To show uniqueness, we need to show that if u,v are two entropy solutions, then necessarily $u-v=0$ for almost all (x,t). Since $u,v\in L^1$, we finally see that the only thing need to be shown is

$$\int (u-v)\,\phi=0 \tag{20}$$

for all ϕ in, say, C_0^1.

- Main idea of the proof.

  What we have is the weak formulation which is satisfied by both u and v:

$$\int\int u\,\psi_t + f(u)\,\psi_x + \int_{t=0} u_0\,\psi = 0, \qquad \int\int v\,\psi_t + f(v)\,\psi_x + \int_{t=0} v_0\,\psi = 0 \tag{21}$$

  where ψ is any C_0^1 function.

  Subtracting the two equations, and remembering $u_0 = v_0$, we have

$$\int\int\ (u-v)\left[\psi_t + \frac{f(u)-f(v)}{u-v}\,\psi_x\right] = 0. \tag{22}$$

  Now setting $F(x,t) \equiv \frac{f(u)-f(v)}{u-v}$, all we need to do is to show that for any $\phi \in C_0^1$, we can find $\psi \in C_0^1$ such that

$$\psi_t + F(x,t)\,\psi_x = \phi. \tag{23}$$

  For initial conditions, we assume $\phi = 0$ for $t > T$ and take $\psi = 0$ along $t = T$.

- Where's the catch.

  The above transport equation can be solved (formally) by the method of characteristics. Let $x(t)$ solves

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x(t),t), \tag{24}$$

  we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(x(t),t) = \phi(x(t),t), \qquad \psi\,|_{t=T} = 0 \tag{25}$$

  which leads to

$$\psi(x(t),t) = \int_T^t \phi(x(s),s)\,\mathrm{d}s. \tag{26}$$

  Remember that we want $\psi \in C^1$ and at the same time has compact support.

  - Does ψ have compact support?

    Recall first that we solve ψ by setting $\psi = 0$ for $t \geqslant T$. Next notice that ψ can be non-zero only along those characteristics which passes the support of ϕ. Now since F is uniformly bounded, the slope of the characteristics are uniformly bounded and the boundedness of ψ's support follows.

  - Is $\psi \in C^1$?

    This is where the catch is. As u, v are only in L^1, $F(x,t)$ is in general not Lipschitz and therefore the characteristics may collide with one another. When that happens, ψ is not in C^1 anymore. In fact, as $F(x,t)$ can only be expected to be in $L^1 \cap L^\infty$, even the existence of the solution is questionable!

- Fixing the problem.

  We have seen that the obstacle is that F is not Lipschitz. To overcome this, we replace F by a smooth approximation F^ε such that $F^\varepsilon \to F$ locally in L^1, and call the corresponding solution ψ^ε. Then we have

$$\int\int\ (u-v)\,\phi = \int\int\ (u-v)\,[\psi_t^\varepsilon + F^\varepsilon(x,t)\,\psi_x^\varepsilon]\,\mathrm{d}x\,\mathrm{d}t. \tag{27}$$

  Comparing with the definition of weak solutions, we obtain

$$\int\int\ (u-v)\,\phi = \int\int\ (u-v)\,[F(x,t) - F^\varepsilon(x,t)]\,\psi_x^\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{28}$$

  As soon as we have shown the uniform boundedness of ψ_x^ε, we can take $\varepsilon \searrow 0$ and obtain

$$\int\int\ (u-v)\,\phi = 0 \tag{29}$$

and finish the proof.

- Uniform boundedness of ψ_x^ε.

  If we naïvely mollify F, there is no way we can obtain this bound as in general ψ_x^ε grows as the Lipschitz constant of F^ε grows, and the latter grows like ε^{-1}. On the other hand, we can make F smooth in a more sophisticated manner, which allows us to take advantage of the entropy condition (which hasn't been used so far!).

  Instead of mollifying F directly, we mollify u, v and define (recall the trick we used in our proof of uniqueness for the Hamilton-Jacobi equation!)

$$F^\varepsilon(x,t) = \frac{f(u^\varepsilon) - f(v^\varepsilon)}{u^\varepsilon - v^\varepsilon} = \int_0^1 f'(\theta\, u^\varepsilon + (1-\theta)\, v^\varepsilon)\, \mathrm{d}\theta. \tag{30}$$

  This gives

$$\frac{\partial F^\varepsilon}{\partial x} = \int_0^1 f''(\theta\, u^\varepsilon + (1-\theta)\, v^\varepsilon) \left[\theta \frac{\partial u^\varepsilon}{\partial x} + (1-\theta) \frac{\partial v^\varepsilon}{\partial x}\right] \mathrm{d}\theta \tag{31}$$

  which is uniformly bounded from above if $\frac{\partial u^\varepsilon}{\partial x}$ and $\frac{\partial v^\varepsilon}{\partial x}$ are so – and this is indeed the case due to the entropy condition.

  Thus we obtain

$$\frac{\partial F^\varepsilon}{\partial x} \leqslant \frac{C}{t} \tag{32}$$

  for some positive constant C.[2]

  From this bound one can show that[3]

$$|\psi_x^\varepsilon(x,t)| \leqslant C \log t^{-1} \tag{33}$$

  using the argument presented on p. 287 of J. Smoller's book.

  Finally, recall that we want to prove

$$\iint_{t \geqslant 0} (u-v)\, [F - F^\varepsilon]\, |\psi_x^\varepsilon|\, \mathrm{d}x\, \mathrm{d}t \longrightarrow 0 \tag{34}$$

  as $\varepsilon \to 0$. It is easy see that $F - F^\varepsilon \to 0$ in L^1. Combine this with the fact tat $F - F^\varepsilon$ is uniformly bounded, we can show that

$$F^\varepsilon \longrightarrow F \qquad \text{in } L^p \text{ for any } 1 \leqslant p < \infty. \tag{35}$$

  Now the desired limit holds as $|\psi_x^\varepsilon|$ is uniformly bounded for any $1 \leqslant q < \infty$.[4]

3. Asymptotics.

Our approaches here follows J. Smoller. In Evans, everything is derived from the explicit formula.

3.1. Uniform decay for initial values with compact support.

First note that we can always replace f by $f - c$ and also do the change of variables $x' = x - f'(0)\, t$ such that the equation reduces to

$$u_t + f(u)_x = 0, \qquad u\,|_{t=0} = u_0 \tag{36}$$

with $f(0) = f'(0) = 0$. Recall that we always assume $f'' > 0$.

We will prove the following theorem.

Theorem 1. (Uniform decay) *Assume that $f'' > 0$, $f(0) = f'(0) = 0$, and that $u_0(x)$ is a bounded measurable function having compact support. Then the unique entropy solution decays to 0 uniformly at a rate $t^{-1/2}$ as $t \nearrow \infty$.*

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2. Note that if we solve a transport equation forward, $u_t + a(x,t)\, u_x = \phi$, then when a is increasing, that is $a_x \geqslant 0$, the characteristics are moving away from one another, which means u_x remain bounded; In the current situation, we are solving backwards from $t = T$ to $t = 0$, thus $a_x \leqslant 0$ is "good" and $a_x > 0$ is "bad". This is why we do not need a lower bound of $\frac{\partial F^\varepsilon}{\partial x}$.

3. Seems a stronger (in time) bound is obtained on p. 153 of Evans.

4. For an elementary – and therefore much more tricky – argument, see pp. 287 – 290 of J. Smoller's book.

Remark 2. We can consider an initial value like

$$u_0(x) = \begin{cases} 0 & x<0, x>1 \\ 1 & x \in (0,1) \end{cases} \tag{37}$$

to see why the solution should decay. We can see that the variation in the solution is "eaten up" by the shocks.

- Main idea.

  Since $f''>0$, there is $\mu>0$ such that $f''>\mu$ for all $u \in \left[-\|u_0\|_{L^\infty}, \|u_0\|_{L^\infty}\right]$ which leads to

  $$f'(u) = f'(u) - f'(0) = f''(\xi)\,u \qquad \Longrightarrow \qquad |u| \leqslant |f'(u)|/\mu. \tag{38}$$

  Now recall the equation for (backward) characteristics,

  $$x = x_0 + f'(u_0(x_0))\,t \quad \Longrightarrow \quad |f'(u_0(x_0))| \leqslant \frac{|x-x_0|}{t}. \tag{39}$$

  Combine the above, we obtain

  $$|u| \leqslant c\,\frac{|x-x_0|}{t}. \tag{40}$$

  The desired result follows if we can show $|x-x_0| < C\,t^{1/2}$ for all (x,t) in the support of u. It suffices to show that the support of $u(\cdot,t)$ grows no faster than $C\,t^{1/2}$.

- A technical remark.

  The equation for characteristics only holds in the smooth part of the solution. According to a theorem by R. DiPerna,[5] when $f''>0$ the solution is always piecewise smooth. Furthermore note that for any entropy solution, the backward characteristic can always reach $t=0$.

- Now we try to obtain the bound on the growth of the support of $u(\cdot,t)$. Denote by $s_+(t)$ the infimum of x such that $u(y,t)=0$ for all $y>x$. $s_-(t)$ is defined similarly. Thus the goal is to show

  $$s_+(t) - s_-(t) \leqslant C\,t^{1/2}. \tag{41}$$

  Fix $t=T$. Note that $u(s_+(T)+,T)=0$. Now if $u(s_+(T)-,T)=0$, then we have $\frac{\mathrm{d}}{\mathrm{d}t} s^+(t) = 0$ at $t=T$;

  If $u(s_+(T)-,T)>0$ (<0 is prohibited by the entropy condition), consider the region enclosed by the backward characteristic from $(s_+(T),T)$, $t=0$, and $x=s_+(T)$. Call the three curves $\Gamma_1, \Gamma_2, \Gamma_3$.

  Integrating the equation

  $$u_t + f(u)_x = 0 \tag{42}$$

  over this region and using the jump condition, we obtain

  $$\int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} u\,\mathrm{d}x - f(u)\,\mathrm{d}t = 0. \tag{43}$$

  Now along Γ_1, $u = u(s_+(T)+,T)$ (denote by $\bar{u}$) which gives

  $$\int_{\Gamma_1} = \int_0^T [u\,f'(u) - f]\,\mathrm{d}t = T\,[\bar{u}\,f'(\bar{u}) - f(\bar{u})]. \tag{44}$$

  Along Γ_2 we have

  $$\int_{\Gamma_2} = -\int_y^\infty u_0(x)\,\mathrm{d}x \tag{45}$$

  where y is the intersection of Γ_1 and $t=0$.

5. R. J. DiPerna, Singularities of solutions of nonlinear hyperbolic systems of conservation laws, Arch. Rat. Mech. Anal., 60, 75–100, 1975.

Along Γ_3 we have

$$\int_{\Gamma_3} = 0. \tag{46}$$

Therefore we have

$$T\,[\bar{u}\,f'(\bar{u}\,) - f(\bar{u}\,)] \leqslant \max_y \int_y^\infty u_0(x)\,\mathrm{d}x \leqslant C. \tag{47}$$

Expanding f at 0, we have

$$u(s_+(T)+,T) \leqslant C\,t^{-1/2} \tag{48}$$

which leads to

$$s^+ \leqslant C\,t^{1/2}. \tag{49}$$

The bound on s^- is estimated similarly.

3.2. Asymptotic profile for initial values with compact support.

We use the following notation:

$$q \equiv \max_y \int_y^\infty u_0(x)\,\mathrm{d}x, \quad -p \equiv \min_y \int_{-\infty}^y u_0(x)\,\mathrm{d}x, \quad k = f''(0). \tag{50}$$

Define

$$w(x,t) = \begin{cases} \dfrac{x}{k\,t} & s_- - \sqrt{2\,k\,p}\,t^{1/2} < x < s_+ + \sqrt{2\,k\,q}\,t^{1/2} \\ 0 & \text{otherwise} \end{cases}. \tag{51}$$

This function is called an "N-wave" due to its shape at every fixed t. Then we have

$$\|u(\,\cdot\,,t) - w(\,\cdot\,,t)\|_{L^1(\mathbb{R})} = O\Big(t^{-1/2}\Big) \qquad \text{as} \qquad t \nearrow \infty. \tag{52}$$

In other words, as $t \nearrow \infty$, all the details in the initial data are lost.

To see why such convergence makes sense, we take any (x,t) and let $y(x,t)$ be the intersection of the backward characteristic and the x-axis. Thus we have

$$u(x,t) = u_0(y,t) \tag{53}$$

and

$$x = y(x,t) + f'(u_0(y(x,t)))\,t = y(x,t) + f'(u(x,t))\,t = y(x,t) + f''(0)\,u\,t + O\big(u^2\big)\,t. \tag{54}$$

Since $u = O\Big(t^{-1/2}\Big)$ as $t \nearrow \infty$, and therefore

$$x = y(x,t) + f''(0)\,u\,t + O\big(u^2\big)\,t = y(x,t) + k\,u\,t + O(1). \tag{55}$$

This leads to (note that as u_0 has compact support, $y(x,t)$ is bounded)

$$u(x,t) = \frac{x}{k\,t} + O\big(t^{-1}\big). \tag{56}$$

Combine with

$$s_+(t) \leqslant s_+ + \Big[\sqrt{2\,k\,q} + O\Big(t^{-1/2}\ln t\Big)\Big]\,t^{1/2}, \qquad s_-(t) \geqslant s_- - \Big[\sqrt{2\,k\,p} + O\Big(t^{-1/2}\ln t\Big)\Big]\,t^{1/2} \tag{57}$$

leads to the result.

For details see J. Smoller book pp. 295 – 297.

3.3. Convergence for periodic solutions.

When the initial value is periodic, we have better decay rate.

Theorem 3. *Let $f'' > 0$ and let $u_0 \in L^\infty(\mathbb{R})$ be piecewise monotonic periodic function of period p. Then we have*

$$|u(x,t) - \bar{u}_0| \leqslant \frac{2\,p}{h\,t} \tag{58}$$

where $\bar{u}_0 = \frac{1}{p}\int_0^p u_0(x)\,\mathrm{d}x$, and $h \equiv \min\big\{f''(u)\colon\ |u| \leqslant \|u_0\|_{L^\infty}\big\}$.

The idea again is the consider backward characteristics. Since u_0 is piecewise monotonic and periodic, so is u at any time t. Let y_1, ..., y_n be the points where u changes from increasing to decreasing or vice versa, at time t. Then due to periodicity,

$$\text{Total variation of } u = 2\sum_{i \text{ odd}} u(y_i, t) - u(y_{i+1}, t) = 2\sum_{i \text{ even}} u(y_i, t) - u(y_{i-1}, t). \tag{59}$$

Here we have assumed that $[y_1, y_2], [y_3, y_4], \ldots$ are the decreasing intervals.

Now if we consider backward characteristics (which are straight lines!) starting from y_i's, we would have

$$p \geqslant \sum_{i \text{ odd}} (y_{i+1} - y_i) + \left[\sum_{i \text{ odd}} u(y_i, t) - u(y_{i+1}, t)\right] t \geqslant \left[\sum_{i \text{ odd}} u(y_i, t) - u(y_{i+1}, t)\right] t. \tag{60}$$

This shows the total variation of u decays like t^{-1}, and the desired result follows.