## HAMILTON-JACOBI EQUATION: WEAK SOLUTION

We continue the study of the Hamilton-Jacobi equation:

$$u_t + H(Du) = 0 \quad \mathbb{R}^n \times (0, \infty); \qquad u = g \quad \mathbb{R}^n \times \{t = 0\}. \tag{1}$$

We have shown that

1. In general we cannot expect the existence of classical solutions (that is $u \in C^1(\mathbb{R}^n \times (0, \infty))$ satisfying the equation everywhere);
2. The Hopf-Lax formula
$$u(x, t) = \min_y \left\{ t\, L\left( \frac{x - y}{t} \right) + g(y) \right\} \tag{2}$$
satisfies the equation almost everywhere.

We see that existence is guaranteed as soon as we only require the equation to be satisfied almost everywhere. Thus it is natural to propose the following as the definition for "weak solutions":

> $u$ Lipschitz is a weak solution if $u$ satisfies the equation a.e. and takes $g$ as its initial value.

However such definition leads to non-uniqueness, as shown in the following example.

**Example 1.** Consider the 1D H-J equation

$$u_t + |u_x|^2 = 0 \quad \mathbb{R} \times (0, \infty); \qquad u = 0 \quad \mathbb{R} \times \{t = 0\}. \tag{3}$$

It is clear that $u \equiv 0$ is a classical solution. However, we can try to find other solutions using separation of variables. Set $u = T(t) + X(x)$. Then the equation gives

$$T'(t) + (X'(x))^2 = 0 \tag{4}$$

which gives

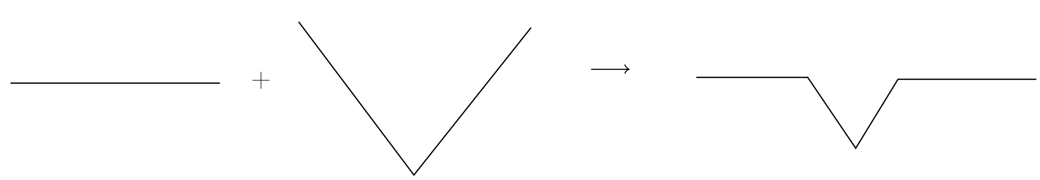
$$T'(t) = -\lambda^2, \qquad (X'(x))^2 = \lambda^2. \tag{5}$$

Obviously if we require the above to hold everywhere, then necessarily $\lambda = 0$ (Recall Exercise 5 in Homework 1). However, if we only require the equation to be satisfied almost everywhere, then there are infinitely many solutions. More specifically, we can pick any $X$ such that $X' = \pm\lambda$ piecewise, and then solve $T' = -\lambda^2$. We can even take different $\lambda$ for different $x, t$. One example of "almost everywhere" solutions obtained this way is

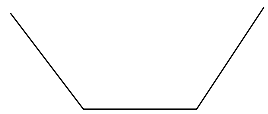
$$u(x, t) = \begin{cases} 0 & |x| \geqslant t \\ x - t & 0 \leqslant x \leqslant t \\ -x - t & -t \leqslant x \leqslant 0 \end{cases} \tag{6}$$

Note that $u(x, t)$ is in fact "assembled" using two "solutions": 0 and $\begin{cases} x - t & x \geqslant 0 \\ -x - t & x \leqslant 0 \end{cases}$. It is easy to see that one can assemble any solutions in anyway to get a function satisfying the equation almost everywhere, the only thing to be careful is to make the result be consistent with the initial value.

For example:



gives a "almost everywhere" solution (note that $T' \leqslant 0$ so the "wedge" is moving downward, thus satisfying the initial value). In contrast,



is not a good construction, as it does not satisfy the initial value.

It is clear that the solution is not unique. How do we regain uniqueness? Or more specifically, how to do we get rid of all non-zero solutions (as $u \equiv 0$ is obviously a reasonable solution for this problem).

One way is to add "viscosity". Consider

$$u_t^\varepsilon + |u_x^\varepsilon|^2 = \varepsilon\, u_{x x}^\varepsilon \tag{7}$$

and then let $\varepsilon \searrow 0$. In many cases $u^\varepsilon$ will then converge to one of the many almost everywhere solutions of the "inviscid" problem. In other words, $u^\varepsilon$ is a "smoothed" version of the correct solution.

Smoothing the "wedge" solution, we see that $x = 0$ is a minimizer of $u^\varepsilon$. At this point, we have

$$u_t < 0, \qquad u_x = 0, \qquad \varepsilon\, u_{x x} > 0 \tag{8}$$

a contradiction! Therefore the limit of $u^\varepsilon$ cannot be the "wedge" solution![1]

From the above example we see that "wedges" should not appear in the "correct" weak solution. In other words, a "correct" solution should not have $u'' = +\infty$ anywhere. Motivated by this, we introduce the following notion.

**Definition 2. (Semi-concavity)** *$f$ is called semi-concave if there is $C > 0$ such that*

$$f(x + z) - 2\, f(x) + f(x - z) \leqslant C\, |z|^2 \tag{9}$$

*for any $x, z$.*

It is easy to see that the condition is equivalent to

$$f(x) - \frac{C}{2}\, |x|^2 \tag{10}$$

is concave. Hence the name "semi-concave".

Now we can try the following definition of "weak solutions" for the H-J equation:

**Definition 3.** *A Lipschitz function $u$ is said to be a weak solution if*

1. *$u$ takes $g$ as initial value;*
2. *$u$ satisfies the equation a.e.;*
3. *$u$ is semi-concave:*

$$u(x + z, t) - 2\, u(x, t) + u(x - z, t) \leqslant C\left(1 + \frac{1}{t}\right) |z|^2. \tag{11}$$

Now we try to establish well-posedness. More specifically, we will show that under certain assumptions, the solution given by the Hopf-Lax formula is the unique weak solution.

**Theorem 4. (Existence)** *The function $u(x, t)$ given by the Hopf-Lax formula is a weak solution if either one of the following is true.*

a) *$g$ is semi-concave;*

b) *$H$ is uniformly convex, that is*

$$\xi^T D^2 H\, \xi \geqslant \theta\, |\xi|^2 \tag{12}$$

*for all $p, \xi$.* [2]

1. Of course, for this problem, one can directly show that $u^\varepsilon \equiv 0$. However the argument used here is more appropriate for motivating the general theory.

**Proof.**

a) Since $g$ is semi-concave, there is $C>0$ such that

$$g(x+z)-2\,g(x)+g(x-z)\leqslant C\,|z|^2. \tag{13}$$

Now for any $t$, let $y$ be such that

$$u(x,t)=t\,L\left(\frac{x-y}{t}\right)+g(y). \tag{14}$$

Then we have[3]

$$u(x\pm z,t)\leqslant t\,L\left(\frac{x-y}{t}\right)+g(y\pm z) \tag{16}$$

and the conclusion follows.

b) Again we choose $y$ such that

$$u(x,t)=t\,L\left(\frac{x-y}{t}\right)+g(y). \tag{17}$$

Now

$$u(x\pm z,t)\leqslant t\,L\left(\frac{x\pm z-y}{t}\right)+g(y). \tag{18}$$

Thus

$$u(x+z,t)-2\,u(x,t)+u(x-z,t)\leqslant t\left[L\left(\frac{x+z-y}{t}\right)-2\,L\left(\frac{x-y}{t}\right)+L\left(\frac{x-z-y}{t}\right)\right]. \tag{19}$$

The conclusion follows from the fact that

$$\begin{aligned} H \text{ uniformly convex} &\Longrightarrow H\left(\frac{p_1+p_2}{2}\right)-\frac{1}{2}\,H(p_1)-\frac{1}{2}\,H(p_2)\leqslant-\frac{\theta}{8}\,|p_1-p_2|^2\\ &\Longrightarrow L\left(\frac{q_1+q_2}{2}\right)-\frac{1}{2}\,L(q_1)-\frac{1}{2}\,L(q_2)\geqslant-\frac{1}{8\,\theta}\,|q_1-q_2|^2. \end{aligned} \tag{20}$$

Which is Problem 3.5.9.[4] □

Now we establish uniqueness.

**Theorem 5. (Uniqueness)** *Assume $H$ is $C^2$, convex, with super-linear growth at infinity and $g$ Lipschitz. Then there exists at most one weak solution of the H-J equation.*

**Proof.** Suppose $u,\tilde{u}$ are two weak solutions. then letting $w=u-\tilde{u}$, we have $w$ is differentiable a.e. and

$$\begin{aligned} w_t(x,t) &= u_t(x,t)-\tilde{u}_t(x,t)\\ &= H(D\,\tilde{u}(x,t))-H(D u(x,t))\\ &= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}H(r\,Du+(1-r)\,D\,\tilde{u})\,\mathrm{d}r\\ &= -\int_0^1 DH(r\,Du+(1-r)\,D\tilde{u})\cdot Dw(x,t)\,\mathrm{d}r\\ &= -\left[\int_0^1 DH(r\,Du+(1-r)\,D\tilde{u})\,\mathrm{d}r\right]\cdot Dw(x,t). \end{aligned} \tag{21}$$

Letting

$$b(x,t)\equiv\int_0^1 DH(r\,Du+(1-r)\,D\tilde{u})\,\mathrm{d}r \tag{22}$$

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2. *Equivalently, $H-\frac{\theta}{2}\,|p|^2$ is still convex.*

3. By the Hopf-Lax formula

$$u(x\pm z,t)=\min_{y'}\left\{t\,L\left(\frac{x\pm z-y'}{t}\right)+g(y')\right\} \tag{15}$$

Take $y'=y\pm z$.

4. Hint: For any $q_1,q_2$, take $p_1,p_2$ such that $H(p_i)+L(q_i)=p_i\cdot q_i$.

we conclude that $w$ satisfies

$$w_t + b \cdot D w = 0 \qquad a.e.; \qquad w\,|_{t=0} = 0. \tag{23}$$

If $w$ is a classical solution to the above equation, then obviously $w \equiv 0$ and we are done. But this is not the case. Nevertheless, if $b$ is Lipschitz, we still can show $w \equiv 0$ as follows.

Let $x(t)$ be such that $\dot{x} = b$. Then $w$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} w(x(t), t) = 0 \qquad a.e. \tag{24}$$

As $w$ is Lipschitz in $x$, $t$ and $x(t)$ Lipschitz in $t$, $W(t) := w(x(t), t)$ is also Lipschitz and thus absolutely continuous. As a consequence we have

$$W(t) - W(0) = \int_0^t W'(s)\,\mathrm{d}s = 0 \tag{25}$$

as $W'(s) = 0$ almost everywhere. Now since $W(0) = 0$ we conclude that $W \equiv 0$. Then it's clear that $w = 0$ almost everywhere.

However the problem is that

$$b(x,t) \equiv \int_0^1 DH(r\,Du + (1-r)\,D\tilde{u})\,\mathrm{d}r \tag{26}$$

is definitely not Lipschitz. The idea now is to approximate $b$ by a Lipschitz function $b^{\varepsilon}$, get some estimate and then let $\varepsilon \searrow 0$. Then we can write

$$w_t + b^{\varepsilon} \cdot D w = (b^{\varepsilon} - b) \cdot D w \qquad a.e. \tag{27}$$

If we try to do "energy-type" estimate of the above, we quickly realize that a uniform upper bound of $\nabla \cdot b^{\varepsilon}$ is needed. This means that the "naïve" way

$$b^{\varepsilon} = \eta^{\varepsilon} * b \tag{28}$$

will not work as this way $\nabla \cdot b^{\varepsilon} \sim 1/\varepsilon$.

The correct smoothing here is (we use subscript to emphasize that this is not the usual mollification)

$$b_{\varepsilon} := \int_0^1 DH(r\,Du^{\varepsilon} + (1-r)\,D\tilde{u}^{\varepsilon})\,\mathrm{d}r \tag{29}$$

where $u^{\varepsilon}, \tilde{u}^{\varepsilon}$ are usual mollification $f^{\varepsilon} = \eta^{\varepsilon} * f$.

This way we have

$$\nabla \cdot b_{\varepsilon} = \int_0^1 D^2 H \colon \left[ r\,D^2 u^{\varepsilon} + (1-r)\,D^2 \tilde{u}^{\varepsilon} \right] \tag{30}$$

where $A\colon B \equiv \sum_{i,j} A_{ij} B_{ij}$. Now using the fact that both $u$, $\tilde{u}$ are semi-concave, we have $D^2 u^{\varepsilon}, D^2 \tilde{u}^{\varepsilon} \leqslant C\left(1 + \frac{1}{t}\right) I$. Note that as the mollification is taken over space-time, such bound only holds for $t > C\,\varepsilon$. Using this we have

$$\nabla \cdot b_{\varepsilon} \leqslant C\left(1 + \frac{1}{t}\right). \tag{31}$$

Now let $v := \phi(w) \geqslant 0$ to be fixed later. Then $v$ satisfies the same equation as $w$. Set

$$R := \max\{|DH(p)|\,|\,|p| \leqslant \max\{\mathrm{Lip}(u), \mathrm{Lip}(\tilde{u})\}\}. \tag{32}$$

We try to show that $v \equiv 0$ in the cone[5]

$$C := \{(x,t)\,|\,0 \leqslant t \leqslant t_0, |x - x_0| \leqslant R\,(t_0 - t)\}. \tag{33}$$

Set

$$e(t) := \int_{B(x_0, R(t_0 - t))} v(x,t)\,\mathrm{d}x. \tag{34}$$

5. Recall our proof of the uniqueness for the wave equation! Here the "slope" $R$ is an upper bound of the speed of characteristics.

We compute

$$\begin{aligned}\dot{e}(t) &= \int v_t - R\int_{\partial B(x_0,R(t_0-t))} v\,\mathrm{d}S\\ &= \int_B -\nabla\cdot(b_\varepsilon\, v) + (\nabla\cdot b_\varepsilon)\, v + (b_\varepsilon - b)\cdot Dv\,\mathrm{d}x - - R\int_{\partial B(x_0,R(t_0-t))} v\,\mathrm{d}S\\ &= -\int_{\partial B} v\,(b_\varepsilon\cdot \boldsymbol{n} + R)\,\mathrm{d}S + \int_B (\nabla\cdot b_\varepsilon)\, v + \int (b_\varepsilon - b)\cdot Dv. \qquad (35)\end{aligned}$$

The first term is $\leqslant 0$ due to the definitions of $b_\varepsilon$ and $R$. The second term is bounded by $C\left(1+\frac{1}{t}\right)e(t)$, the third term vanishes as $\varepsilon \searrow 0$ due to dominated convergence. Thus we have

$$\dot{e}(t) \leqslant C\left(1+\frac{1}{t}\right)e(t) \qquad (36)$$

for a.e. $0 < t < t_0$.

Finally, taking $\phi$ such that $\phi(z) = 0$ for $|z| \leqslant \varepsilon\,[\mathrm{Lip}(u) + \mathrm{Lip}(\tilde{u})]$ and positive otherwise, we have

$$v = \phi(w) = 0 \qquad (37)$$

for $t \leqslant \varepsilon$. Now for $t > \varepsilon$, $1 + \frac{1}{t} \leqslant 1 + \frac{1}{\varepsilon}$ and we have

$$\dot{e}(t) \leqslant C\left(1+\frac{1}{\varepsilon}\right)e(t) \qquad a.e., t > \varepsilon; \qquad e(t) = 0 \qquad t \leqslant \varepsilon. \qquad (38)$$

which gives $e(t) = 0$. This means

$$|u - \tilde{u}| = |w| \leqslant \varepsilon\,[\mathrm{Lip}(u) + \mathrm{Lip}(\tilde{u})] \qquad (39)$$

in the cone $C$. By the arbitrariness of $\varepsilon$ we conclude that $u - \tilde{u} = 0$ almost everywhere and ends the proof. □