HAMILTON-JACOBI EQUATION: EXPLICIT FORMULAS

In this lecture we try to apply the method of characteristics to the Hamilton-Jacobi equation:

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{2}$$

To avoid confusion, we use the following notation:

$$\begin{pmatrix} x \\ t \end{pmatrix} \longleftrightarrow \tilde{x}, \quad u \longleftrightarrow z, \quad \begin{pmatrix} Du \\ u_t \end{pmatrix} \longleftrightarrow \tilde{p} = \begin{pmatrix} p \\ p_{n+1} \end{pmatrix}. \tag{3}$$

Then we can re-write the equation to

$$F(Du, u_t, u, x, t) = 0 \tag{4}$$

where

$$F(\tilde{p}, z, \tilde{x}) := p_{n+1} + H(p, x). \tag{5}$$

The characteristics ODEs then are

$$\begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix} = \dot{\tilde{x}} = D_{\tilde{p}} F = \begin{pmatrix} D_p H \\ 1 \end{pmatrix}, \tag{6}$$

$$\dot{z} = D_{\tilde{p}} F \cdot \tilde{p} = \begin{pmatrix} D_p H \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p \\ p_{n+1} \end{pmatrix} = p_{n+1} + D_p H \cdot p = D_p H \cdot p - H(p, x), \tag{7}$$

$$\begin{pmatrix} \dot{p} \\ \dot{p}_{n+1} \end{pmatrix} = \dot{\tilde{p}} = -(D_z F)\, p - D_x F = -\begin{pmatrix} D_x H \\ 0 \end{pmatrix}. \tag{8}$$

1. Method of characteristics.

We try to solve the characteristic ODEs. First notice that, since $\dot{t} = 1$, we can simply use t as the parameter instead of s. Thus the equations become

$$\dot{x} = D_p H, \tag{9}$$

$$\dot{z} = D_p H \cdot p - H(p, x), \tag{10}$$

$$\dot{p} = -D_x H, \tag{11}$$

$$p_{n+1} = p_{n+1}|_{t=0}. \tag{12}$$

It is clear the all we need to do is to solve the first 3 equations.

Losing a bit rigor, we assume (for now only) H is differentiable and strictly convex. We also assume H grows super-linearly at infinity:

$$\lim_{|p| \nearrow \infty} \frac{H(p)}{|p|} = +\infty, \tag{13}$$

Noticing that

$$p_0 = \operatorname{argmax}_{p \in \mathbb{R}^n} \{D_p H(p_0, x) \cdot p - H(p, x)\}. \tag{14}$$

We can define

$$q := D_p H(p, x) \tag{15}$$

which also give p as a function of q as $D_p^2 H$ is non-singular due to the convexity of H. We write

$$L(q, x) := \sup_{p \in \mathbb{R}^n} \{q \cdot p - H(p, x)\}. \tag{16}$$

As a consequence the z equation becomes

$$\dot{z} = L(q, x) \tag{17}$$

where q satisfies

$$q = D_p H(p, x). \tag{18}$$

Therefore the solution u is given by

$$u(x) = u(x_0) + \int_0^t L(q(\tau), x(\tau))\, \mathrm{d}\tau. \tag{19}$$

where x and x_0 are related by

$$x = X(s) \tag{20}$$

where X solves

$$\frac{\mathrm{d}}{\mathrm{d}t} X = D_p H = q, \qquad X(0) = x_0. \tag{21}$$

To further simplify the system, we notice that

$$\dot{x} = D_p H, \qquad \dot{p} = -D_x H \tag{22}$$

implies

$$-\frac{\mathrm{d}}{\mathrm{d}s}(D_q L) + D_x L = 0 \tag{23}$$

which implies that q, x minimizes

$$\int_0^t L(q(\tau), x(\tau))\, \mathrm{d}\tau \tag{24}$$

with $x(0), x(t)$ fixed. (See Evans 3.3.1 for details).

To see this, write $L(q, x) = q \cdot p(q, x) - H(p(q, x), x))$, and compute

$$D_q L = p + q \cdot D_q p - D_p H \cdot D_q p = p, \tag{25}$$

$$D_x L = q \cdot D_x p - D_x H - D_p H \cdot D_x p = -D_x H, \tag{26}$$

where we have used $q = D_p H$. Now the equation $\dot{p} = -D_x H$ gives what we want.

Thus we see that the Hamilton-Jacobi equation can be solved as soon as we find out the trajectories $x(t)$ and $q(t)$. Below we will see that in a special case, this can indeed be done (in some sense).

2. The Hopf-Lax formula.

This special case is when H is independent of x, that is $H = H(Du)$. The characteristic equations can then be further simplified to

$$\begin{aligned} \dot{x} &= D_p H, & (27)\\ \dot{z} &= D_p H \cdot p - H(p) = L(q), & (28)\\ \dot{p} &= D_x H = 0, & (29)\\ p_{n+1} &= p_{n+1}|_{t=0}. & (30) \end{aligned}$$

We see that p is a constant vector along the characteristic curve, and as a consequence $\dot{x} = D_p H$ is a constant vector, and therefore the characteristics $x(t)$ are straight lines. Furthermore we know that the velocity $q = \dot{x}$ is constant.

Thus if $x(0) = y$ and $x(t) = x$, we must have

$$q = \frac{x - y}{t}. \tag{31}$$

As a consequence

$$\frac{\mathrm{d}}{\mathrm{d}t} z = L(q) = L\left(\frac{x - y}{t}\right) \Longrightarrow z(t) = z(0) + t\, L\left(\frac{x - y}{t}\right) = g(y) + t\, L\left(\frac{x - y}{t}\right). \tag{32}$$

Now the only problem is that y is not known.

Now think of $g(y)$ as not merely an "initial function", but as an intermediate record. In other words, instead of starting at $t = 0$, imagine our system starts from $t =$, say, -1. We consider all possible trajectories emanating from some point at $t = -1$, passing y at $t = 0$, and finally reach time t at x. Think of $g(y)$ as the record of work done from $t = -1$ to $t = 0$. Obviously the correct trajectory should be the one that is the minimizer among them all.

Following this idea, we reach the following Hopf-Lax formula:

$$u(x,t) = z(t) = \inf_{y \in \mathbb{R}^n} \left\{ t\, L\left(\frac{x-y}{t}\right) + g(y) \right\}. \tag{33}$$

Remark 1. It can be shown that L grows superlinearly at infinity. As a consequence, if we assume g to be Lipschitz continuous, then the infimum is actually a minimum.

Remark 2. The relation

$$L(q) = H^*(q) := \sup_{p \in \mathbb{R}^n} \{q \cdot p - H(p)\} \tag{34}$$

is called "Legendre transform" and is very useful. It can be shown that the following theorem holds (Evans p. 122)

Theorem 3. *Let $H = H(p)$ be convex, and satisfies*

$$\lim_{|p| \nearrow \infty} \frac{H(p)}{|p|} = +\infty, \tag{35}$$

then

i. $H^*(q)$ is also convex,

$$\lim_{|q| \nearrow \infty} \frac{H(q)}{|q|} = +\infty, \tag{36}$$

ii. $H = (H^*)^*$.

Inspecting the proof, one sees that i still holds even if H is not convex, but convexity is necessary for ii (If H is not convex, then it cannot be the same as $(H^*)^*$, which is convex by i).

Remark 4. Note that convex functions are continuous. The proof can go roughly as follows. First one can show that f (the convex function) is bounded, let the bound be denoted M. Then using the definition of convexity we have, for any fixed x, y,

$$u(y + \alpha(x-y)) \leqslant u(y) + \alpha\,(u(x) - u(y)) \leqslant u(y) + 2\,\alpha\, M. \tag{37}$$

Letting $\alpha \to 0$ we see that

$$\limsup_{x_n \to x} u(x_n) \leqslant u(x). \tag{38}$$

On the other hand, for any $x_n \to x$ we have, by convexity

$$u(x) \leqslant \frac{1}{2}\,[u(x_n) + u(2\,x - x_n)]. \tag{39}$$

This gives

$$u(x) \leqslant \frac{1}{2} \liminf_{x_n \to x}\, [u(x_n) + u(2\,x - x_n)]. \tag{40}$$

Continuity then follows.

One can in fact prove that any convex function is Lipschitz continuous, see e.g. B. Dacorogna **Direct Methods in the Calculus of Variations**, 2nd ed., Springer, 2008, §2.3.

3. Solution of the Hamilton-Jacobi equation.

Now we show that the Hopf-Lax formula

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ t\, L\left(\frac{x-y}{t}\right) + g(y) \right\}. \tag{41}$$

indeed solves the Hamilton-Jacobi equation, albeit only "almost everywhere".

Remark 5. It is easy to see that in general one cannot expect the existence of classical solutions due to possible intersections of characteristics.

There are three things to show.

1. $u = g$ on $\mathbb{R}^n \times \{t = 0\}$,
2. u_t, Du exist almost everywhere,
3. $u_t + H(Du) = 0$ a.e.

We show them one by one.

1. $u = g$ on $\mathbb{R}^n \times \{t = 0\}$.

   Recall the formula:
$$u(x,t) = \min_y \left\{ t\, L\left(\frac{x-y}{t} \right) + g(y) \right\}. \tag{42}$$
   Taking $y = x$ we have
$$u(x,t) \leqslant g(x) + t\, L(0) \implies \limsup_{t \searrow 0} u(x,t) \leqslant g(x). \tag{43}$$
   On the other hand, we compute
$$\begin{aligned} u(x,t) &= \min_y \left\{ t\, L\left(\frac{x-y}{t} \right) + g(y) \right\} \\ &= g(x) + \min_y \left\{ t\, L\left(\frac{x-y}{t} \right) + g(y) - g(x) \right\} \\ &\geqslant g(x) - \max_y \left\{ \mathrm{Lip}(g)\, |y-x| - t\, L\left(\frac{x-y}{t} \right) \right\} \\ &= g(x) - t \max_z \{ \mathrm{Lip}(g)\, |z| - L(z) \} \\ &= g(x) - t \max_{w \in B_{\mathrm{Lip}(g)}} \left\{ \max_z \{ w \cdot z - L(z) \} \right\} \\ &= g(x) - t \max_{w \in B_{\mathrm{Lip}(g)}} H(w). \end{aligned} \tag{44}$$
   As H is continuous, we have
$$\liminf_{t \searrow 0} u(x,t) \geqslant g(x). \tag{45}$$
   Thus ends the proof.

2. u_t, Du exist almost everywhere.

   It suffices to show that u is Lipschitz with respect to x and to t.

   – u is Lipschitz w.r.t. x. We estimate $u(\hat{x}, t) - u(x,t)$.

     Choose y such that
$$u(x,t) = t\, L\left(\frac{x-y}{t} \right) + g(y). \tag{46}$$
     Then
$$u(\hat{x}, t) - u(x,t) = \min \left\{ t\, L\left(\frac{\hat{x}-z}{t} \right) + g(z) - t\, L\left(\frac{x-y}{t} \right) - g(y) \right\}. \tag{47}$$
     Taking $z = \hat{x} - x + y$ (such that $\hat{x} - z = x - y$) we have
$$u(\hat{x}, t) - u(x,t) \leqslant g(\hat{x} - x + y) - g(y) \leqslant \mathrm{Lip}(g)\, |\hat{x} - x|. \tag{48}$$
     Similarly we can show
$$u(x,t) - u(\hat{x}, t) \leqslant \mathrm{Lip}(g)\, |\hat{x} - x|. \tag{49}$$
     The Lipschitz continuity of u then follows.

   – u is Lipschitz w.r.t. t. This follows from the following property of the Hopf-Lax formula:
$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s)\, L\left(\frac{x-y}{t-s} \right) + u(y,s) \right\}. \tag{50}$$
     That this should hold is intuitively very clear following our derivation of the formula. For a proof see Evans p. 126.

Using this formula, we see that estimating $u(x,t)-u(x,s)$ is no different than estimating $u(x,t)-g(x)$. Thus a similar argument as in Step 1. gives

$$|u(x,t)-u(x,s)| \leqslant C\,|t-s|. \tag{51}$$

3. $u_t + H(Du) = 0$ a.e.

Fix any $q \in \mathbb{R}^n$, we compute

$$\begin{aligned} u(x+h\,q, t+h) &= \min\left\{h\,L\left(\frac{x+h\,q-y}{h}\right)+u(y,t)\right\} \\ &\leqslant h\,L(q)+u(x,t). \end{aligned} \tag{52}$$

This implies

$$u_t + q\cdot Du \leqslant L(q) \iff -u_t \geqslant Du\cdot q - L(q) \tag{53}$$

for all $q\in\mathbb{R}^n$. Therefore

$$-u_t \geqslant \max_q \{Du\cdot q - L(q)\} = H(Du) \tag{54}$$

and

$$u_t + H(Du) \leqslant 0. \tag{55}$$

For the other direction (that is $u_t + H(Du) \geqslant 0$), we only need to find one q such that

$$u_t + q\cdot Du \leqslant L(q) \tag{56}$$

or more specifically

$$\frac{u(x,t)-u(y,s)}{t-s} \leqslant L(q) \tag{57}$$

where $x-y$ is in the direction of q.

As u is a minimum, to get the $u(x,t)-u(y,s)\leqslant$ something, we get rid of the minimum in $u(x,t)$. Take z such that

$$u(x,t) = t\,L\left(\frac{x-z}{t}\right)+g(z). \tag{58}$$

Now that $q=\frac{x-z}{t}$ is already chosen, y has to be on the line segment connecting x and z. Thus we take

$$s = t-h, \quad y = \frac{s}{t}\,x + \left(1-\frac{s}{t}\right)z. \tag{59}$$

Then we have $\frac{x-z}{t} = \frac{y-z}{s} = q$. We compute

$$\begin{aligned} u(x,t)-u(y,s) &\geqslant t\,L\left(\frac{x-z}{t}\right)+g(z)-\left[s\,L\left(\frac{y-z}{s}\right)+g(z)\right] \\ &= (t-s)\,L\left(\frac{x-z}{t}\right). \end{aligned} \tag{60}$$

As

$$\frac{u(x,t)-u(y,s)}{t-s} \to u_t + \frac{x-z}{t}\cdot Du, \tag{61}$$

we get

$$u_t + \frac{x-z}{t}\cdot Du \geqslant L\left(\frac{x-z}{t}\right) \tag{62}$$

and finish the proof.