METHOD OF CHARACTERISTICS

In this lecture we try to solve the first order equation

$$F(Du, u, x) = 0 \qquad \text{in } U; \qquad u = g \quad \text{on } \Gamma \subseteq \partial U. \tag{1}$$

Recall that, when F is quasi-linear,

$$F(Du, u, x) = b(u, x) \cdot Du - f(u, x), \tag{2}$$

we can try to solve the ODE system:

$$\dot{x}(s) = b(z, x) \tag{3}$$
$$\dot{z}(s) = f(z, x) \tag{4}$$

and then try to represent s using x and finally obtain the solution from

$$u(x) = z(s). \tag{5}$$

This approach does not work anymore when F is fully nonlinear. For example, when

$$F(Du, u, x) = |Du|^2 - f(u, x), \tag{6}$$

the above approach would give

$$\dot{x}(s) = Du(x) \tag{7}$$
$$\dot{z}(s) = f(u, x). \tag{8}$$

As Du is not known, the ODE system cannot be solved.

Nevertheless, the main idea: reduce the PDE to a system of ODEs along particular curves $x(s)$, still works.

1. Simplifying the equation.

Consider a curve $x(s)$ to be determined. We try to find out whether it is possible to simplify the PDE along this particular curve.

Clearly $x(s)$ and $z(s) := u(x(s))$ are unknown functions that need to be solved. How about Du? If we only consider the function along $x(s)$, then Du and u has no relation except

$$\frac{\mathrm{d}}{\mathrm{d}s} u = \dot{x}(s) \cdot Du. \tag{9}$$

Therefore $p(s) := Du(x(s))$ should also be treated as an independent unknown function.

Now we derive equations for x, z, p.

First, from the above discussion,

$$\dot{z}(s) = \dot{x}(s) \cdot p(s). \tag{10}$$

To figure out the equation for x, we need to first look at the equation for p, as x will be chosen to make other equations as simple as possible.

Compute

$$\dot{p}(s) = \frac{\mathrm{d}}{\mathrm{d}s} Du(x(s)) = D^2 u(x(s)) \cdot \dot{x}(s). \tag{11}$$

The quantity $D^2 u$ is not known and has to be cancelled.

To cancel it, we turn to the equation. Differentiating $F(Du, u, x) = 0$ we obtain

$$D^2 u \cdot D_p F + \left(D_z F\right) Du + D_x F = 0 \iff D^2 u \cdot D_p F = -\left(D_z F\right) Du - D_x F. \tag{12}$$

We see that the $D^2 u$ in the p equation can be cancelled provided we require

$$\dot{x}(s) = D_p F. \tag{13}$$

Thus the ODE system we look for is

$$\dot{x} = D_p F, \tag{14}$$
$$\dot{z} = \dot{x} \cdot p = D_p F \cdot p, \tag{15}$$
$$\dot{p} = -(D_z F)\, p - D_x F. \tag{16}$$

We see that the system is closed, that is, does not involve any unknown quantities except x, z, p.

Example 1. When F is quasi-linear, that is

$$F(p, z, x) = b(z, x) \cdot p - f(z, x), \tag{17}$$

we have

$$\dot{x} = b(z, x), \tag{18}$$
$$\dot{z} = b(z, x) \cdot p = f(z, x), \tag{19}$$
$$\dot{p} = -(D_z b \cdot p - D_z f)\, p - (D_x b \cdot p - D_x f). \tag{20}$$

We see that the first two equations and the third equation are decoupled. Thus we can solve the first two equations and obtain the solution. There is no need to solve the 3rd equation.

Summarizing, we have shown that if u solves $F(Du, u, x) = 0$, and $\dot{x}(s) = D_p F(Du, u, x)$, then $p(s) := Du(x(s))$, $z(s) := u(x(s))$ solves

$$\dot{z} = D_p F \cdot p, \qquad \dot{p} = -(D_z F)\, p - D_x F. \tag{21}$$

Obviously the opposite direction is more important: Suppose we solve the characteristic ODE system, does the solution gives us the solution to the original equation?

2. Local existence of solutions to $F(Du, u, x) = 0$.

Consider the first order PDE

$$F(Du, u, x) = 0 \qquad \text{in } U; \qquad u = g \quad \text{on } \Gamma \subseteq \partial U. \tag{22}$$

We will show that as long as Γ is non-characteristic (meaning will be clear later), the method of characteristics will give us a solution of the equation in a neighborhood of Γ (which is usually called a "local solution"). We break down the whole argument into several steps.

- The equation. Let p, z, x solve the characteristics system.

  We compute

  $$\frac{\mathrm{d}}{\mathrm{d}s} F(p, z, x) = D_p F \cdot \dot{p} + D_z F \dot{z} + D_x F \cdot \dot{x} = 0. \tag{23}$$

  Therefore we have

  $$F(p, z, x) = 0 \tag{24}$$

  as long as there is s_0 such that

  $$F(p(s_0), z(s_0), x(s_0)) = 0. \tag{25}$$

- Cauchy data. How do we pick s_0? It is clear that we should use the Cauchy data

  $$u = g \qquad \text{on } \Gamma. \tag{26}$$

  We set $x(s_0) \in \Gamma$ and $z(s_0) = g(x(s_0))$. But how do we find out $p(s_0)$?

  First consider the case $\Gamma \subseteq \mathbb{R}^{n-1} \cap \{x_n = 0\}$. Then the knowledge of $u = g$ on Γ also gives us the first $n-1$ component of $p(s_0)$:

  $$p_i(s_0) = D_{x_i} u(x(s_0)) = D_{x_i} g(x(s_0)), \qquad i = 1, 2, \ldots, n-1. \tag{27}$$

  To determine $p_n(s_0)$, we have to use the equation. We must have

  $$F(p_1(s_0), \ldots, p_n(s_0), z(s_0), x_1(s_0), \ldots, x_n(s_0)) = 0. \tag{28}$$

The implicit function theorem tells us that $p_n(s_0)$ can be represented as a function of others (which are all known to us) if

$$F_{p_n} \neq 0. \tag{29}$$

For general Γ, one can use a change of variable to transform Γ to the above "flat" case (F is transformed to G). If after the transformation we have

$$G_{p_n} \neq 0, \tag{30}$$

then we say Γ is non-characteristic.

In the original variables, the non-characteristic condition can then be shown as

$$D_p F \cdot \nu(x_0) \neq 0 \tag{31}$$

where $\nu(x_0)$ is the unit normal vector of Γ at x_0.

– Defining u. Naturally we try to define

$$u(x) = z(x(s)). \tag{32}$$

But then there is the question of whether this can be done for all x in a neighborhood of Γ. In other words, whether the characteristic curves starting from Γ "fill" the neighborhood.

Mathematically, the question can be put as: Let $y_1, \ldots, y_{n-1}$ be the coordinate in Γ. Then does the solution to

$$x = x(y_1, \ldots, y_{n-1}, s) \tag{33}$$

exists for any given x in a neighborhood of Γ?

We have the following lemma guaranteeing that the answer is yes given that Γ is non-characteristic (Without loss of generality, we consider the "flat" case only):

Lemma 2. (Evans p. 106) *Assume we have the noncharacteristic condition $F_{p_n} \neq 0$. Then there exists an open interval $I \subset \mathbb{R}$ containing 0, a neighborhood W of $x^0 \in \Gamma$, and a neighborhood V of x^0 in $\mathbb{R}^n$, such that for each $x \in V$ there exist unique $s \in I$, $y \in W$ such that*

$$x = x(y, s). \tag{34}$$

The mappings $x \mapsto s, y$ are C^2.

Proof. Using the inverse function theorem, all we need to do is to show that

$$\det D_{y,s} x \neq 0 \tag{35}$$

as $y = \begin{pmatrix} x_1^0 \\ \vdots \\ x_{n-1}^0 \end{pmatrix}$, $s = 0$. As

$$D_{y,s} x = \begin{pmatrix} I_{n-1} & \nabla_{p_1, \ldots, p_{n-1}} F \\ 0 & F_{p_n} \end{pmatrix}, \tag{36}$$

The condition is satisfied when Γ is noncharacteristic. □

Thus we see that u is well-defined.

– We have shown that

$$F(p(y_1, \ldots, y_{n-1}, s), u(x), x) = 0 \tag{37}$$

in a neighborhood of Γ. The last step is to show that $p = Du$. This is done as follows.

First from the characteristics equation, we have

$$Du \cdot \frac{\partial x}{\partial s} = \dot{z} = D_p F \cdot p = p \cdot \frac{\partial x}{\partial s}. \tag{38}$$

Next we can show that (Evans pp. 108-109)

$$Du \cdot \frac{\partial x}{\partial y_i} = p \cdot \frac{\partial x}{\partial y_i}, \qquad i = 1, 2, \ldots, n-1. \tag{39}$$

Combining these two, we have

$$D_{y,s}x \cdot (Du - p) = 0. \tag{40}$$

As the matrix $D_{y,s}x$ is nonsingular, we must have

$$p = Du \tag{41}$$

as desired.

Thus we have established the local existence of the solutions.

3. Examples.

Example 3. (Conservation laws) Consider the conservation law

$$u_t + \operatorname{div} F(u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty); \qquad u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{42}$$

We let

$$G(p, z, x) = p_{n+1} + F'(z) \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}. \tag{43}$$

Now the characteristic equations are (with $x_{n+1} = t$)

$$\dot{x} = D_p G = \begin{pmatrix} F'(z) \\ 1 \end{pmatrix}, \tag{44}$$

$$\dot{z} = D_p G \cdot p = 0, \tag{45}$$

$$\dot{p} = \cdots. \tag{46}$$

We didn't write the equation for p explicitly because it is decoupled from the equations for x and z.

Solving the first two equations we have

$$t = s \tag{47}$$

$$x = F'(z_0)\, t + x_0 \tag{48}$$

$$z = z_0 = g(x_0). \tag{49}$$

We see that the solution is given implicitly by

$$u(x, t) = g(x - F'(u)\, t). \tag{50}$$

Example 4. (Hamilton-Jacobi equation) Consider

$$u_t + H(Du, x) = 0. \tag{51}$$

Let

$$F(p, z, x) = p_{n+1} + H\left(\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right). \tag{52}$$

Then we have

$$\dot{x} = D_p F = \begin{pmatrix} D_p H \\ 1 \end{pmatrix} \Longleftrightarrow \dot{x} = D_p H, \quad \dot{t} = 1. \tag{53}$$

$$\dot{z} = D_p F \cdot p = p_{n+1} + D_p H \cdot p = D_p H \cdot p - H, \tag{54}$$

$$\dot{p} = -(D_z F)\, p - D_x F = \begin{pmatrix} -D_x H \\ 0 \end{pmatrix} \Longleftrightarrow \dot{p} = -D_x H, \quad \dot{p}_{n+1} = 0. \tag{55}$$

Note that in the above the notation is a bit messy, as x is used for both the spatial variable x and $\begin{pmatrix} x \\ t \end{pmatrix}$, and p is used for both Du and $\begin{pmatrix} Du \\ u_t \end{pmatrix}$.

As $\dot{t} = 1$, we can replace s by t. Thus the characteristics equations become

$$\dot{x} = D_p H, \quad \dot{z} = \cdots, \quad \dot{p} = -D_x H, \quad \dot{p}_{n+1} = 0. \tag{56}$$

Now we recognize that the x, p equations are

$$\dot{x} = D_p H, \quad \dot{p} = -D_x H \tag{57}$$

which are the so-called Hamiltonian equations which governs the evolution of particles (x is the location and p is the momentum of the particle). This understanding is important in deriving the solution formula for the H-J equation.