WAVE EQUATIONS: UNIQUENESS AND ASYMPTOTICS

In this lecture we prove the uniqueness for the wave equations. We also prove some asymptotic decay results.

Recall that to prove uniqueness for the Laplace/Poisson and the heat equations, we have two approaches. The first one is via maximum principles, the second one via energy estimate. However, for the wave equation, no maximum principle holds, as can be seen by setting $g = u\,|_{t=0} = 0$, and $h = u_t\,|_{t=0}$ to be 1 for $x \in (-R, R)$ and 0 for $x \notin (-R-1, R+1)$ and positive everywhere else in the 1D wave equation. Therefore, the only choice left is energy estiamte.

1. Uniqueness via energy method.

Consider the wave equation in a bounded domain $\Omega \subset \mathbb{R}^n$.

$$\begin{aligned} \Box u \equiv u_{tt} - \triangle u &= f && \Omega \times (0,T) && (1)\\ u &= g && \Omega \times \{0\} \text{ and } \partial\Omega \times [0,T] && (2)\\ u_t &= h && \Omega \times \{0\}. && (3)\end{aligned}$$

It is clear that the uniqueness of this problem is equivalent to that the following equation

$$\begin{aligned} \Box u \equiv u_{tt} - \triangle u &= 0 && \Omega \times (0,T) && (4)\\ u &= 0 && \Omega \times \{0\} \text{ and } \partial\Omega \times [0,T] && (5)\\ u_t &= 0 && \Omega \times \{0\}. && (6)\end{aligned}$$

having only 0 solution.

Now we prove this. Multiply the equation by u_t and integrate over $\Omega \times (0,T)$, we have

$$\begin{aligned} 0 &= \int_{\Omega\times(0,T)} (u_{tt} - \triangle u)\, u_t \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega\times(0,T)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} u_t^2\right) \mathrm{d}x\,\mathrm{d}t + \int_{\Omega\times(0,T)} -\triangle u\, u_t \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega\times(0,T)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} u_t^2\right) \mathrm{d}x\,\mathrm{d}t + \int_{\Omega\times(0,T)} \nabla u \cdot \nabla u_t \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega\times(0,T)} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)\right] \mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \left[\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)\right](x,T)\,\mathrm{d}x - \int_\Omega \left[\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)\right](x,0)\,\mathrm{d}x \\ &= \int_\Omega \left[\frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)\right](x,T)\,\mathrm{d}x. \end{aligned} \tag{7}$$

This implies u is a constant at time T. But this constant must be 0 according to the boundary value.

Remark 1. If we know the solutions decays at infinity, we can use the same method when Ω is unbounded and obtain the same result.

2. Domain of dependence.

We have seen from the formulas that the value of $u(x,t)$ only depends on the initial values in the ball $B_t(x)$. In other words, if $g = h = 0$ in $B_r(x)$, then u must vanish in the cone

$$|x| + t \leqslant r. \tag{8}$$

We prove this fact now.

Denote by C_r the above mentioned cone. and for $T < r$ denote by U_T the following domain

$$U_T \equiv \{(x,t) \in C_r,\ 0 \leqslant t \leqslant T\}. \tag{9}$$

Then naturally the boundary of U_T consists of three parts

$$\partial U_T = S_T + S_0 + S_{\text{side}} \tag{10}$$

where

$$S_u = \{(x,t) \in C_r,\ \ t = u\}, \quad S_{\text{side}} = \partial C_r \cap \bar{U}_T. \tag{11}$$

Now we compute

$$\begin{aligned}
0 &= \int_{U_T} (u_{tt} - \triangle u)\, u_t \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{U_T} u_{tt}\, u_t - \triangle u\, u_t \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{U_T} \partial_t \left(\frac{1}{2} u_t^2\right) - \nabla \cdot (\nabla u\, u_t) + \nabla u \cdot \nabla u_t \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{U_T} \partial_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2\right) - \nabla \cdot (\nabla u\, u_t) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{U_T} \nabla_{t,x} \cdot \begin{pmatrix} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \\ -\nabla u\, u_t \end{pmatrix} \mathrm{d}x\,\mathrm{d}t \\
&= \int_{\partial U_T} n_{t,x} \cdot \begin{pmatrix} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \\ -\nabla u\, u_t \end{pmatrix} \mathrm{d}S \\
&= \int_{S_T} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \int_{S_0} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \\
&\quad + \int_{S_{\text{side}}} \begin{pmatrix} n_t \\ n_x \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \\ -\nabla u\, u_t \end{pmatrix} \mathrm{d}S \\
&= \int_{S_T} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \int_{S_0} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \\
&\quad + \int_{S_{\text{side}}} \frac{n_t}{2} u_t^2 + \frac{n_t}{2} |\nabla u|^2 - n_x \cdot \nabla u\, u_t \,\mathrm{d}S.
\end{aligned} \tag{12}$$

For the last term, we notice that the equation for S_{side} is $|x| + t = r$ which means $n_t = |n_x|$ and consequently

$$\frac{n_t}{2} u_t^2 + \frac{n_t}{2} |\nabla u|^2 - n_x \cdot \nabla u\, u_t \geqslant 0. \tag{13}$$

Thus we have shown that

$$\int_{S_T} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \leqslant \int_{S_0} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 \tag{14}$$

for all $T < r$ and the conclusion follows.

3. Decay of the solution.

We prove the following.

Proposition 2. *Let u solve*

$$u_{tt} - \triangle u = 0 \qquad \text{in } \mathbb{R}^3 \times (0, \infty) \tag{15}$$

$$u = g \quad u_t = h \qquad \text{on } \mathbb{R}^3 \times \{t = 0\} \tag{16}$$

where g, h are smooth and have compact support. Then there is a constant C such that

$$|u(x,t)| \leqslant C/t \tag{17}$$

for all (x,t).

Proof. Recall the Kirchhoff formula:

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} t\, h(w) + g(w) + \nabla g(w) \cdot (y - x) \,\mathrm{d}S_w. \tag{18}$$

Since $h, g, \nabla g$ vanishes outside their respective supports, we can write

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B_t(x) \cap A} t\, h(w) + g(w) + \nabla g(w) \cdot (y - x) \,\mathrm{d}S_w \tag{19}$$

where A is the union of the three supports. Now the conclusion easily follows after we notice that the area of $\partial B_t(x) \cap A$ is bounded by a constant independent of t. $\square$

Remark 3. The above estimate behaves badly when t is small. But this is easily remedied by noticing that when t is small, the area of $\partial B_t(x) \cap A$ scales as t^2 and therefore u is uniformly bounded. Integrating this observation into the estimate gives

$$|u(x,t)| \leqslant C\,(1+t)^{-1}. \tag{20}$$

Proposition 4. *Let u solve*

$$\begin{aligned} u_{tt} - \triangle u \;&=\; 0 \qquad \text{in } \mathbb{R}^2 \times (0,\infty) &(21)\\ u \;=\; g \quad u_t \;&=\; h \qquad \text{on } \mathbb{R}^2 \times \{t=0\} &(22)\end{aligned}$$

where g, h are smooth and have compact support. Then there is a constant C such that

$$|u(x,t)| \leqslant C\,(1+t)^{-1/2}\,(1+|t-|x||)^{-1/2}. \tag{23}$$

for all (x,t).

Proof. Assume that the supports of g, h are contained in the ball B_R. Recall the Poisson's formula:

$$u(x,t) = \frac{1}{2\,\pi\,t^2} \int_{D_t(x)} \frac{t\,g(y) + t^2\,h(y) + t\,\nabla g \cdot (y-x)}{\left(t^2 - |y-x|^2\right)^{1/2}}\,\mathrm{d}y \tag{24}$$

By taking the supreme of $g, h, \nabla g$ and noticing that $|y-x| \leqslant t$ we have

$$|u(x,t)| \leqslant C \int_{D_t(x)} \frac{\mathrm{d}y}{(t-|y-x|)^{1/2}\,(t+|y-x|)^{1/2}} \leqslant C\,t^{-1/2} \int_{D_t(x)} \frac{\mathrm{d}y}{(t-|y-x|)^{1/2}}. \tag{25}$$

Now let $z = y - x$ we have

$$|u(x,t)| \leqslant C\,t^{-1/2} \int_{D_t} \frac{\mathrm{d}z}{(t-|z|)^{1/2}}. \tag{26}$$

Here note that the integral is in fact over $D_t \cap \{|z+x| \leqslant R\}$. We have

- $|x| > t + R$: $u(x,t) \equiv 0$.
- $t - 2\,R < |x| < t + R$: We use polar coordinates, note that the angle is of order R/t (we only consider the case $t \gg R$ here), thus we have

$$\begin{aligned} \int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t-|z|)^{1/2}} \;&\lesssim\; \frac{R}{t} \int_{|x|-R}^{\min(t,|x|+R)} \frac{r\,\mathrm{d}r}{(t-r)^{1/2}} \\ &\leqslant\; R \int_{|x|-R}^{\min(t,|x|+R)} (t-r)^{-1/2}\,\mathrm{d}r \\ &\leqslant\; C\,(t-(|x|-R))^{1/2} \\ &\leqslant\; C\,(3\,R)^{1/2}. \end{aligned} \tag{27}$$

since $t - 2\,R < |x| < t + R$ implies $t - 3\,R < |x| - R < t$. As

$$-R < t - |x| < 2\,R, \tag{28}$$

we have

$$1 + |t - |x|| < 2\,R \implies (1 + |t-|x||)^{-1/2} > (2\,R)^{-1/2}. \tag{29}$$

Thus

$$\int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t-|z|)^{1/2}} \leqslant C(R)\,(1+|t-|x||)^{-1/2} \tag{30}$$

in this case.

– $|x| < t - 2R$: We have $(t - |z|) \geqslant t - (R + |x|) = (t - |x|) - R \geqslant \frac{1}{2}(t - |x|) \geqslant \frac{R}{2R+1}(1 + t - |x|)$[1], and therefore

$$\int_{D_t \cap \{|z+x| \leqslant R\}} \frac{\mathrm{d}z}{(t - |z|)^{1/2}} \leqslant C(R)\,(1 + t - |x|)^{-1/2}. \tag{32}$$

Combining the above, we see that when t is large (for example $t > 100 \max(R, 1)$), we have

$$|u(x,t)| \leqslant C t^{-1/2}\,(1 + |t - |x||)^{-1/2}. \tag{33}$$

On the other hand, when $t \leqslant 100 \max(R, 1)$ we have

$$\begin{aligned} |u(x,t)| &\leqslant C \int_{D_t} \frac{\mathrm{d}z}{(t^2 - |z|^2)^{1/2}} \\ &= C \int_0^t \frac{r\,\mathrm{d}r}{(t^2 - r^2)^{1/2}} \\ &= C t \leqslant C(R). \end{aligned} \tag{34}$$

Thus u is bounded by a constant when t is small and by $C\,t^{-1/2}\,(1 + |t - |x||)^{-1/2}$ when t is large, as a consequence, we can write

$$|u(x,t)| \leqslant C\,(1 + t)^{-1/2}\,(1 + |t - |x||)^{-1/2}. \tag{35}$$

as desired. Note that the constant C is heavily dependent on R. □

Remark 5. In general, we have

– $n > 1$ odd:

$$|u(t,x)| \leqslant C\,(1 + t)^{-\frac{n-1}{2}}; \tag{36}$$

– $n > 1$ even:

$$|u(t,x)| \leqslant C\,(1 + t)^{-\frac{n-1}{2}}\,(1 + |t - |x||)^{-\frac{n-1}{2}}. \tag{37}$$

Remark 6. Such algebraic decays are also characteristic in other dispersive equations, for example the Schrödinger equation.

Remark 7. It is clear that no decay can be expected for the solutions to the 1D wave equation:

$$u(x,t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{38}$$

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1. Since $|x| < t - 2R$, we have

$$t - |x| > 2R \Longrightarrow \frac{1}{t - |x|} < \frac{1}{2R} \Longrightarrow 1 + \frac{1}{t - |x|} < \frac{2R+1}{2R}. \tag{31}$$