WAVE EQUATIONS: EXPLICIT FORMULAS

In this lecture we derive the representation formulas for the wave equation in the whole space:

$$\Box u \equiv u_{tt} - \triangle u = 0, \quad \mathbb{R}^n \times (0, \infty); \qquad u(x,0) = g(x), \qquad u_t(x,0) = h(x). \tag{1}$$

It turns out that the properties of the solutions depend on the dimension. More specifically, there are three cases: $n = 1$, $n > 1$ even; $n > 1$ odd. We will discuss in detail the three representative cases: $n = 1, 2, 3$ (the order is actually $n = 1, 3, 2$, for reasons that will be clear soon).

1. $n = 1$.

We consider the 1D wave equation

$$u_{tt} - u_{xx} = 0, \mathbb{R} \times (0, \infty); \qquad u(x,0) = g(x), \qquad u_t(x,0) = h(x). \tag{2}$$

This equation can be solved via the following change of variables:

$$\xi = x + t; \qquad \eta = x - t, \tag{3}$$

and to make things clearer we set $\tilde{u}(\xi, \eta) = u(x,t)$.

With this change of variable we compute

$$\begin{aligned} u_t &= \tilde{u}_\xi \xi_t + \tilde{u}_\eta \eta_t = \tilde{u}_\xi - \tilde{u}_\eta & (4)\\ u_{tt} &= \tilde{u}_{\xi\xi} - 2\,\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta} & (5)\\ u_x &= \tilde{u}_\xi + \tilde{u}_\eta & (6)\\ u_{xx} &= \tilde{u}_{\xi\xi} + 2\,\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta} & (7) \end{aligned}$$

Therefore

$$u_{tt} - u_{xx} = 0 \Longleftrightarrow \tilde{u}_{\xi\eta} = 0 \Longleftrightarrow \tilde{u}(\xi,\eta) = \phi(\xi) + \psi(\eta) \Longleftrightarrow u(x,t) = \phi(x+t) + \psi(x-t). \tag{8}$$

Now using the initial values we have

$$\phi(x) + \psi(x) = g(x); \qquad \phi'(x) - \psi'(x) = h(x) \tag{9}$$

which yields

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{10}$$

This is d'Alembert's formula.

Theorem 1. *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, define u by*

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y. \tag{11}$$

Then

i. $u \in C^2(\mathbb{R} \times [0, \infty))$;

ii. $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0, \infty)$;

iii. u takes the correct boundary values:

$$\begin{aligned} \lim_{\substack{(x,t) \to (x_0, 0)\\ t>0}} u(x,t) &= g(x_0); & (12)\\ \lim_{\substack{(x,t) \to (x_0, 0)\\ t>0}} u_t(x,t) &= h(x_0). & (13) \end{aligned}$$

Proof. The proof is by direct calculation and is left as an exercise. □

Remark 2. It is easy to generalize the above theorem to the case

$$g \in C^k, \quad h \in C^{k-1} \quad \Longrightarrow \quad u \in C^k. \tag{14}$$

But in general u cannot be smoother (in contrast to the heat equation and the Laplace equation). For example, consider the case $h = g'$, then $u(x, t) = g(x + t)$. It is clear that u cannot have better regularity than g.

Remark 3. One can show that the formula

$$u(x,t) = \phi(x+t) + \psi(x-t) \tag{15}$$

remains true even for distributional solutions of the 1D wave equation.

2. Spherical means and Euler-Poisson-Darboux equation.

The case $n \geqslant 2$ is much more complicated. The idea is to reduce the wave equation to a 1D equation which can be solved explicitly. The reduction is fulfilled through introducing the following auxiliary functions.

Let $u = u(x,t)$. We define at each $x \in \mathbb{R}^n$,

$$U(x;r,t) \equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(w,t)\,\mathrm{d}S_w, \tag{16}$$

$$G(x;r) \equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} g(w)\,\mathrm{d}S_w, \tag{17}$$

$$H(x;r) \equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} h(w)\,\mathrm{d}S_w. \tag{18}$$

Note that when u is continuous, we can recover u from U by taking $r \searrow 0$.

It turns out that $U(x;r,t)$ as a function of r and t satisfies a 1D equation.

Lemma 4. **(Euler-Poisson-Darboux equation)** *Fix $x \in \mathbb{R}^n$. Let $u(x, t) \in C^m$, $m \geqslant 2$ solves the wave equation. Then*

$$U(x;r,t) \equiv \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(w,t)\,\mathrm{d}S_w \tag{19}$$

belongs to $C^m\big(\bar{\mathbb{R}}_+ \times [0,\infty)\big)$, and satisfies

$$U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad \mathbb{R}_+ \times (0,\infty); \qquad U(r,0) = G(r), \quad U_t(r,0) = H(r). \tag{20}$$

Remark 5. Notice that $\partial_{rr} - \frac{n-1}{r}\partial_r$ is just $\triangle$ in $\mathbb{R}^n$ with radial symmetry.

Proof. Recall that

$$U_r(x;r,t) = \frac{r}{n} \frac{1}{|B_r(x)|} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y = \frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y \tag{21}$$

This shows $U \in C^1$, and we can define $U_r(x;0,t) = 0$ since its limit as $r \searrow 0$ is 0.

Differentiating w.r.t r again,

$$\begin{aligned} U_{rr}(x;r,t) &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{B_r(x)} \triangle_y u(y,t)\,\mathrm{d}y\right] \\ &= \frac{1-n}{n} \frac{1}{|B_r|} \int_{B_r(x)} \triangle_y u + \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \triangle_y u. \end{aligned} \tag{22}$$

This shows $U \in C^2$ and also $U_{rr}(x;0,t)$ can be defined.

As

$$\frac{1}{|B_r|} \int_{B_r(x)} \triangle_y u = \frac{1}{|B_1|} \int_{B_1(x)} (\triangle_y u)(x + r\,z)\,\mathrm{d}z, \tag{23}$$

It is clear that the regularity of the LHS is the same as the regularity of $\triangle_y u$. Similar argument shows that the same holds for the term

$$\frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \triangle_y u. \tag{24}$$

Therefore U_{rr} as the same regularity as $\triangle_y u$, which shows $U \in C^m$ when $u \in C^m$.

We further have

$$U_{tt}(x; r, t) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u_{tt} = -\frac{1}{|\partial B_r|} \int_{\partial B_r(x)} \triangle_y u \tag{25}$$

using the equation. □

3. $n = 3$, Kirchhoff's formula.

Let U, G, H be the spherical means. We set

$$\tilde{U} = r\,U, \quad \tilde{G} = r\,G, \quad \tilde{H} = r\,H. \tag{26}$$

Some calculation yields

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad \mathbb{R}_+ \times (0, \infty); \qquad \tilde{U} = \tilde{G}, \quad \tilde{U}_t = \tilde{H}. \tag{27}$$

Remark 6. Note that here we used the fact that $n = 3$.

Thus we need to solve the wave equation in the first quadrant.

Example 7. Consider the wave equation in the first quadrant:

$$u_{tt} - u_{xx} = 0, \quad x > 0, t > 0; \qquad u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad u = 0 \text{ for } x = 0, t > 0. \tag{28}$$

Let

$$\tilde{u}(x, t) = \begin{cases} u(x, t) & x > 0 \\ -u(-x, t) & x < 0 \end{cases} \tag{29}$$

and define similarly $\tilde{g}, \tilde{h}$. Then it is clear that $\tilde{u}$ solves the wave equation with initial values $\tilde{g}, \tilde{h}$. Thus we have

$$\tilde{u}(x, t) = \frac{1}{2} [\tilde{g}(x + t) + \tilde{g}(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y)\, \mathrm{d}y. \tag{30}$$

Therefore the solution to the original problem is

$$u(x, t) = \begin{cases} \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, \mathrm{d}y & x \geqslant t \geqslant 0 \\ \frac{1}{2} [g(x + t) - g(t - x)] + \frac{1}{2} \int_{t-x}^{t+x} h(y)\, \mathrm{d}y & t \geqslant x \geqslant 0 \end{cases} \tag{31}$$

Now for our purpose, we only need the case $t \geqslant r$ (remember that finally we will let $r \searrow 0$ and recover u from U). In this case

$$\tilde{U}(x; r, t) = \frac{1}{2} \left[\tilde{G}(r + t) - \tilde{G}(t - r) \right] + \frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(y)\, \mathrm{d}y. \tag{32}$$

We have

$$\begin{aligned} u(x, t) &= \lim_{r \searrow 0} \frac{\tilde{U}(x; r, t)}{r} \\ &= \tilde{G}'(t) + \tilde{H}(t) \\ &= \frac{\partial}{\partial t} \left(\frac{t}{|\partial B_t|} \int_{\partial B_t(x)} g(w)\, \mathrm{d}S_w \right) + \frac{t}{|\partial B_t|} \int_{\partial B_t(x)} h(w)\, \mathrm{d}S_w. \end{aligned} \tag{33}$$

Further computation yields

$$u(x, t) = \frac{1}{|\partial B_t|} \int_{\partial B_t(x)} [t\, h(w) + g(w) + \nabla g(w) \cdot (w - x)]\, \mathrm{d}S_w \tag{34}$$

which is Kirchhoff's formula.

4. n=2, Method of descent and Poisson's formula.

It is not possible to simplify as we did in the $n = 3$ case. Instead, we use the so-called "method of descent", which treats the solution $u(x, t)$ of the 2D wave equation as a solution to the 3D equation. We set

$$\bar{u}\,(x_1, x_2, x_3, t) \equiv u(x_1, x_2, t). \tag{35}$$

and define $\bar{g}\,, \bar{h}\,$ similarly.

Using the Kirchhoff's formula we have

$$\begin{aligned} u(x,t) &= \bar{u}\,(\bar{x}\,,t) \\ &= \frac{1}{|\partial B_t(\bar{x}\,)} \int_{\partial B_t(\bar{x})} t\,\bar{h}\,(\bar{w}\,) + \bar{g}\,(\bar{w}\,) + \nabla_{\bar{x}} \bar{g}\,(\bar{w}\,) \cdot (\bar{w} - \bar{x}\,)\, \mathrm{d}S_{\bar{w}}. \end{aligned} \tag{36}$$

where $\bar{x} = (x, x_3)$ and $B_t(\bar{x}\,)$ is the ball in $\mathbb{R}^3$.

From definitions of the variaous bar-ed functions, we have

$$u(x,t) = \frac{1}{4\,\pi\,t^2} \int_{\partial B_t(\bar{x})} t\,h(y) + g(y) + \nabla_y g(y) \cdot (y - x)\, \mathrm{d}S_{\bar{w}} \tag{37}$$

where $\bar{w} = \left(y, \pm \sqrt{t^2 - |y|^2} \right)$.

Finally, let $D_t(x)$ denote the ball in $\mathbb{R}^2$ centered at x with radius t, we have

$$\begin{aligned} u(x,t) &= \frac{2}{4\,\pi\,t^2} \int_{D_t(x)} \frac{t\,h(y) + g(y) + \nabla_y g(y) \cdot (y - x)}{\left(1 - \frac{|y - x|^2}{t^2} \right)^{1/2}}\, \mathrm{d}y \\ &= \frac{1}{2} \frac{1}{|D_t|} \int_{D_t(x)} \frac{t\,g(y) + t^2\,h(y) + t\,\nabla g \cdot (y - x)}{\left(t^2 - |y - x|^2 \right)^{1/2}}\, \mathrm{d}y. \end{aligned} \tag{38}$$

This is the Poisson's formula.

Remark 8. (Huygens' Principle) We notice that the behavior of the solutions for the 2D and 3D wave equations are drastically different. In 2D, $u(x,\,t)$ depends on initial data in the whole ball $D_t(x)$ while in 3D it only depends on the data on the boundary of the ball $B_t(x)$. Or equivalently, in 3D the effect of a vibration is only felt at the front of its propagation while in 2D it is felt forever after the front passed.[1] This is the so-called Huygens' principle.

Remark 9. For general n, we define

$$\tilde{U}\,(r, t) = \left(\frac{1}{r}\,\partial_r \right)^{k-1} \left(r^{2k-1}\, U(x; r, t) \right) \tag{39}$$

and define $\tilde{G}\,, \tilde{H}\,$ accordingly. Some calculation yields the solution

$$u(x,t) = \frac{1}{\gamma_n} \left[\left(\frac{\partial}{\partial t} \right) \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \frac{1}{|\partial B_t|} \int_{\partial B_t(x)} g\, \mathrm{d}S \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \frac{1}{|\partial B_t|} \int_{\partial B_t(x)} h\, \mathrm{d}S \right) \right] \tag{40}$$

for n odd, where $\gamma_n = 1 \cdot 3 \cdot \cdots \cdot (n-2)$. Then the method of descent yields

$$u(x,t) = \frac{1}{\gamma_n} \left[\left(\frac{\partial}{\partial t} \right) \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{|B_t|} \int_{B_t} \frac{g(y)\, \mathrm{d}y}{\left(t^2 - |y - x|^2 \right)^{1/2}} \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{|B_t|} \int_{B_t} \frac{h(y)\, \mathrm{d}y}{\left(t^2 - |y - x|^2 \right)^{1/2}} \right) \right].$$

for n even, where $\gamma_n = 2 \cdot 4 \cdot \cdots \cdot (n-2) \cdot n$.

See Evans pp. 75–80 for details.

Remark 10. (Nonhomogeneous problem) For the nonhomogeneous problem

$$\Box u = f, \qquad u = 0, \qquad u_t = 0, \tag{41}$$

1. If $u(x, t)$ also depends on data in the whole ball $B_t(x)$ in 3D, we would not be able to clearly hear anything!

we use the Duhamel's principle, obtaining

$$u(x,t) = \int_0^t u(x,t;s)\,\mathrm{d}s \tag{42}$$

where $u(x,t;s)$ solves

$$u_{tt} - \triangle u = 0, \qquad u(x,s;s) = 0, \quad u_t(x,s;s) = f(\,\cdot\,,s). \tag{43}$$

In particular, we have

- $n = 1$:
$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{44}$$
- $n = 3$:
$$u(x,t) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(y,t-|y-x|)}{|y-x|}\,\mathrm{d}y. \tag{45}$$

  Here the integrand is called the "retarded potential".