HEAT EQUATION: MAXIMUM PRINCIPLES AND ENERGY METHOD

We continue the discussion of the heat equation

$$u_t - \triangle u = f \qquad \text{in } U_T; \qquad u = g \qquad \text{on } \partial^* U_T \tag{1}$$

where

$$U_T := U \times [0, T); \qquad \partial^* U_T := \big(\bar{U} \times \{0\}\big) \cup (\partial U \times [0, T]), \tag{2}$$

with $U \subseteq \mathbb{R}^n$.

1. Maximum principles.

Recall that the motivation for deriving maximum principles is to show uniqueness of the solution. In our case, to show the uniqueness of the heat equation, all we need to do is to show that

$$u_t - \triangle u = 0 \qquad \text{in } U_T; \qquad u = 0 \qquad \text{on } \partial^* U_T \tag{3}$$

implies

$$u \equiv 0 \qquad \text{in } U_T. \tag{4}$$

Further recall that, for harmonic functions (satisfying $\triangle u = 0$), there are two versions of maximum principles:

- Weak maximum principle:
$$\max_{\bar{U}} u = \max_{\partial U} u. \tag{5}$$
- Strong maximum principle: If U is connected and there is $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then $u \equiv u(x_0)$ in U.

We will show that similar principles hold for the heat equation too.

1.1. Weak maximum principle.

We try to prove the following:

Theorem 1. (Weak maximum principle for bounded U) *Let U be a bounded domain in $\mathbb{R}^n$. $u \in C_1^2(U_T) \cap C(U_T \cup \partial^* U_T)$. Then*

$$\max_{\bar{U}_T} u = \max_{\partial^* U_T} u. \tag{6}$$

Proof. The basic idea is the following. As $\bar{U}_T$ is a bounded closed set in $\mathbb{R}^{n+1}$, it is compact. As a consequence there is $(x_0, t_0) \in \bar{U}_T \backslash \partial^* U_T$ such that

$$u(x_0, t_0) = \max_{\bar{U}_T} u. \tag{7}$$

Now at this particular point, since $x_0 \in U$ is the maximizer of the function $u(\,\cdot\,, t_0)$, we have

$$\triangle u(x_0, t_0) \leqslant 0. \tag{8}$$

On the other hand, as $u(x_0, t_0) \geqslant u(x_0, t)$ for all $t < t_0$, we have

$$u_t(x_0, t_0) \geqslant 0. \tag{9}$$

As a consequence we have

$$u_t - \triangle u \geqslant 0 \text{ at } (x_0, t_0). \tag{10}$$

Unfortunately we can conclude nothing here – we would have obtained contradiction if $\geqslant$ is replaced by $>$. Furthermore there is another problem in the above argument. As $u \in C_1^2(U_T)$, $u_t(x_0, t_0)$ may not exist if $t_0 = T$.

In the following we overcome these difficulties through a few tricks.

First, instead of trying to prove

$$\max_{\bar{U}_T} u = \max_{\partial^* U_T} u \tag{11}$$

directly, we try to prove

$$\max_{\bar{U}_{T-\delta}} u = \max_{\partial^* U_{T-\delta}} u \tag{12}$$

for any $\delta > 0$. This guarantees that u_t exists for any point under consideration. It is clear that once this is established, we can let $\delta \searrow 0$ to get the original claim.

Next, instead of considering u, we consider $v(x,t) \equiv u(x,t) - \varepsilon\, t$ for some $\varepsilon > 0$, and show

$$\max_{\bar{U}_{T-\delta}} v = \max_{\partial^* U_{T-\delta}} v. \tag{13}$$

It is easy to see that once this is shown, we can let $\varepsilon \searrow 0$ to get the corresponding relation for u.

For v, we have

$$v_t - \triangle v = u_t - \varepsilon - \triangle u = -\varepsilon < 0. \tag{14}$$

Now assume (x_0, t_0) is a maximizer for v. Then we have

$$v_t(x_0, t_0) \geqslant 0, \qquad \triangle v(x_0, t_0) \leqslant 0 \implies v_t - \triangle v \geqslant 0 \text{ at } (x_0, t_0) \tag{15}$$

which gives the desired contradiction! □

Remark 2. It is clear that $u_t - \triangle u = 0$ can be replaced by $u_t - \triangle u \leqslant 0$, thus obtaining weak maximum principle for subsolutions of the heat equation.

Remark 3. It is easy to see that the above proof breaks down when U is not bounded. One can try to fix as follows.

Consider $U^R \equiv B_R \cap U$. Then apply the weak maximum principle to U_T^R and finally let $R \nearrow \infty$. It is clear from this approach that the limiting values of u at ∞ matters. In fact, examples can be constructed to show that non-zero solutions exist for

$$u_t - \triangle u = 0 \qquad \text{in } \mathbb{R}^n \times [0, \infty), \qquad u = 0 \qquad \text{on } \mathbb{R}^n \times \{0\} \tag{16}$$

if no restriction is put on how u grows at ∞. See Lecture 13 of Fall 2008 Math 527. Also see Chapter 7 of F. John's **Partial Differential Equations** for more discussion on this issue.

On the other hand, once we put some "growth condition" onto u, we have weak maximum principle for the case $U = \mathbb{R}^n$, as the following theorem illustrates.

Theorem 4. (Weak maximum principle for $U = \mathbb{R}^n$) *Suppose*

$$u_t - \triangle u \leqslant 0 \text{ in } U_T; \qquad u(x,t) \leqslant M e^{\lambda |x|^2} \text{ in } U_T \text{ for } M, \lambda > 0; \qquad u(x,0) = g(x), \tag{17}$$

then

$$\sup_{\bar{\Omega}_T} u \leqslant \sup_{\mathbb{R}^n} g. \tag{18}$$

Proof. First we divide $(0, T)$ into subintervals with size $r < \frac{1}{4\lambda}$. It suffices to prove the claim on a subinterval. From now on we assume $T < \frac{1}{4\lambda}$.

Consider the auxiliary function

$$v(x,t) \equiv u(x,t) - \delta\, \frac{1}{(4\pi(T + \varepsilon - t))^{n/2}} e^{\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)}. \tag{19}$$

where ε is chosen such that $T + \varepsilon < \frac{1}{4\lambda}$.

One can show that the perturbation term satisfies the heat equation. Thus we have

$$v_t - \triangle v \leqslant 0. \tag{20}$$

The strategy is the show first

$$v(x,t) \leqslant \sup_{\mathbb{R}^n} g \tag{21}$$

for any (x,t) and then letting $\delta \searrow 0$.

We first notice that, on the sphere $|x-y|=R$, we have

$$v(x,t)\leqslant M e^{\lambda(|x|+R)^2}-\delta\frac{1}{(4\pi(T+\varepsilon-t))^{n/2}}e^{\left(\frac{R^2}{4(T+\varepsilon-t)}\right)}\leqslant\sup_{\mathbb{R}^n} g \tag{22}$$

once R is big enough. Now apply the weak maximum principle on the domain $B_R\times(0,T)$, we obtain $v(x, t)\leqslant\sup g$. □

Corollary 5. (Uniqueness) *The solution to the heat equation*

$$u_t-\triangle u=f \qquad \text{in } U_T; \qquad u=g \qquad \text{on } \partial^* U_T \tag{23}$$

is unique when either U is bounded, or $U=\mathbb{R}^n$ with u satisfying the growth estimate

$$|u(x,t)|\leqslant M e^{\lambda|x|^2} \tag{24}$$

for constants $M,\lambda>0$.

1.2. Strong maximum principle.

As in the case of harmonic functions, to establish strong maximum principle, we have to obtain first some kind of mean value property. It turns out, the mean value property for the heat equation looks very weird.

Theorem 6. (Mean value property for the heat equation) *Let $u\in C_1^2(U_T)$ solve the heat equation, then*

$$u(x,t)=\frac{1}{4r^n}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,\mathrm{d}y\,\mathrm{d}s. \tag{25}$$

for each $E(x,t;r)\subset U_T$. Here the "heat ball" $E(x,t;r)$ is defined as

$$E(x,t;r)\equiv\left\{(y,s)\in\mathbb{R}^{n+1}\,|\,s\leqslant t,\ \Phi(x-y,t-s)\geqslant\frac{1}{r^n}\right\}. \tag{26}$$

Proof. The proof is quite technical. See pp. 53 – 54 of Evans. Also see p.4 of Lecture 13 of Fall 2008 Math 527 for the "details" omitted in Evans. □

Remark 7. Recall that for harmonic functions, we have not only a "ball" version of mean value formula

$$u(x)=\frac{1}{|B_r|}\int_{B_r(x)} u(y)\,\mathrm{d}y, \tag{27}$$

but also a "sphere" version

$$u(x)=\frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y). \tag{28}$$

The situation is similar for the heat equation. The "sphere" version is as follows:

$$u(x,t)=\frac{1}{r^n}\int_{\partial E(x,t;r)} u(y,s)\frac{|x-y|}{2(t-s)}\,\mathrm{d}S. \tag{29}$$

See J. Jost **Partial Differential Equations**, pp. 81 – 83 for details.

Now we can prove the strong maximum principle for the heat equation.

Theorem 8. (Strong maximum principle) *If U is connected and there is $x_0\in U$ such that $u(x_0)=\max_{\overline{U_T}} u$, then $u\equiv u(x_0)$ in U_T.*

Proof. Suppose there is $(x_0, t_0)\in U_T$ such that $u(x_0, t_0)=\max_{\bar{U}_T} u$, then by picking r small enough so that $E(x_0, t_0; r)\subset U_T$, and using the mean value property, we conclude that u is constant inside $E(x_0, t_0; r)$.

Next for any $(y_0, s_0) \in U_T$ such that the line segment connecting x_0, y_0 is in U, we can show that $u(y_0, s_0) = u(x_0, t_0)$ whenever $s_0 < t_0$ by covering the line segment connecting (y_0, s_0) and (x_0, t_0) with the heat balls.

Finally, since U is connected, any y_0 can be connected from x_0 via finitely many line segments. And therefore $u(y, s) = u(x_0, t_0)$ for all $y \in \Omega$, $s < t_0$. □

2. Energy method.

A particularly effective approach to evolution equations (such as the heat equation) is to estimate a certain quantity called "energy". For example, the uniqueness of solutions to the heat equation can be shown easily as follows.

Consider the initial/boundary-value problem

$$u_t - \triangle u = f \text{ in } U_T; \qquad u = g \text{ on } \partial^* U_T. \tag{30}$$

We assume $U \subset \mathbb{R}^n$ is open, bounded, and that ∂U is C^1. Let $T > 0$ be fixed.

Theorem 9. (Uniqueness) *There exists at most one solution in $C_1^2(\bar{U}_T)$.*

Proof. If $\tilde{u}, u$ are two different solutions, we set $w = u - \tilde{u}$. Then w solves

$$w_t - \triangle w = 0 \text{ in } U_T; \qquad w = 0 \text{ on } \partial^* U_T. \tag{31}$$

Now set

$$e(t) \equiv \int_U w^2(x, t)\, \mathrm{d}x. \tag{32}$$

It is clear that it suffices to show $e(t) \equiv 0$ for $0 \leqslant t \leqslant T$. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_U w^2(x, t)\, \mathrm{d}x \\ &= \int_U \frac{\mathrm{d}}{\mathrm{d}t} \big(w^2(x, t)\big)\, \mathrm{d}x \\ &= \int_U 2\, w(x, t)\, w_t(x, t)\, \mathrm{d}x \\ &= 2 \int_U w(x, t) \triangle w(x, t)\, \mathrm{d}x \qquad \text{(We have used the equation here)} \\ &= -2 \int_U |\nabla w(x, t)|^2\, \mathrm{d}x \leqslant 0. \end{aligned} \tag{33}$$

Combined with $e(0) = 0$, we conclude that

$$e(t) \equiv 0 \tag{34}$$

for all $0 \leqslant t \leqslant T$ and ends the proof. □

Remark 10. The above argument can also be applied to the unbounded case. However we need to require the solutions to decay at ∞, or more precisely $u(\,\cdot\,, t) \in L^2$. This is much more restrictive than the condition in the uniqueness theorem following from the weak maximum principle.